Hamiltonian Dynamics In The Theory of Abstraction

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Abstract:

This paper deals with fluid flow dynamics which may be Hamiltonian in nature and yet chaotic. Here we deal with sympletic invariance, canonical transformations and stability of such Hamiltonian flows. As a collection of points move along, it carries along and distorts its own neighbourhood. This in turn affects the stability of such flows.

Introduction:

There may be many settings of physical interest, where dynamics is reversible, such as finite-dimensional Hamiltonian flows. In such transactions, the family of evolution maps f^t form a group. The evolution rule f^t is a family of mappings of strips of transport B, that we may consider, such that,

1) $f^0(x) = x$
2) $f^t[f^{t'}(x)] = f^{t+t'}(x)$
3) $(x,t) \to f^t(x)$ from $B \times R$ into B is continuous;

where t represents a time interval and $t \in R$.

For infinitesimal times, we may write the trajectory of a given transaction as,

$$x(t+\tau) = f^{t+\tau}(x_0)$$
$$= f[f(x_0,t),\tau]$$

The time derivative of this trajectory at point $x(t)$ is,

$$\left.\frac{dx}{d\tau}\right|_{\tau=0} = \partial_\tau f[f(x_0,t),\tau]|_{\tau=0} = \dot{x}(t)$$

The vector field is a generalized velocity field,

$$\dot{x}(t) = v(x) \qquad \dots(1)$$

If x_q represents an equilibrium point, the trajectory remains stuck at x_q forever. Otherwise, the trajectory passing through x_0 at time $t = 0$ may be obtained by,

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v[x(\tau)], x(0) = x_0 \qquad \dots(2)$$

The Euler integrator, which advances the trajectory by $\delta\tau$ ×velocity at each time step is,

$$x_i = x_i + v_i(x)\delta\tau.$$

This may be used to integrate the equations of the dynamics concerned.

Hamiltonian Chaotic Dynamics:

For a Hamiltonian $H(q,p)$and equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = \frac{\partial H}{\partial q_i}$$

With D degrees of freedom,

$$x = (q,p),$$

$$q = (q_1, q_2, q_3, \dots, q_D),$$

$$p = (p_1, p_2, p_3, \dots, p_D).$$

The value of the Hamiltonian function at the state space point $x = (q,p)$ is constant along the trajectory $x(t)$. Thus the energy along the trajectory $x(t)$ is constant,

$$\frac{d}{dt}H[q(t),p(t)] = \frac{\partial H}{\partial q_i}\dot{q}_i(t) + \frac{\partial H}{\partial p_i}\dot{p}_i(t) = \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = 0$$

The trajectories therefore lie on surfaces of constant energy or level sets of the Hamiltonian $[(q,p): H(q,p) = E]$.

Given a smooth function $g(x)$, the standard map is,

$$x_{n+1} = x_n + y_{n+1}$$

$$y_{n+1} = y_n + g(x_n).$$

This is an area-preserving map. The corresponding n^{th} iterate Jacobian matrix is,

$$M^n(x_0, y_0) = \prod_{K=n}^{1} \begin{pmatrix} 1 + g'(x_K) & 1 \\ g'(x_K) & 1 \end{pmatrix}$$

Let $M = 1$ as the map preserves areas and also B is symplectic.

The standard map corresponds to the choice $g(x) = k/(2\pi \sin(2\pi x))$. When $k = 0, y_{n+1} = y_n = y_0$, so that angular momentum is conserved, and the angle x rotates with uniform velocity,

$$x_{n+1} = x_n + y_0 = x_0 + (n+1)y_0 \qquad \dots (3)$$

The standard map provides a stroboscopic view of the flow generated by a time dependent Hamiltonian,

$$H(x, y, t) = \frac{1}{2} y^2 + G(x)\delta_1(t);$$

where δ_1 denotes the periodic delta function,

$$\delta_1(t) = \sum_{m=-\infty}^{\infty} \delta\,(t - m)$$

$$and \qquad G'(x) = -g(x) \qquad \ldots (4)$$

A complete description of the dynamics for arbitrary values of the nonlinear parameter k is fairly complex. When K is sufficiently large, single trajectories wander erratically on a large fraction of the phase space.

Stability In Hamiltonian Flows:

The equations of motion for a time-independent D-degrees of freedom Hamiltonian can be written as,

$$\dot{x}_i = \omega_{ij} H_j(x), \omega = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, H_j(x) = \frac{\partial}{\partial x_j} H(x);$$

where $x = (q, p) \in B$ is a phase space point. $H_K = \partial_K H$ is the column vector of partial derivatives of H, I is the $[D \times D]$ unit matrix and ω the $[2D \times 2D]$ symplectic form.

$$\omega^{\mathrm{T}} = -\omega, \omega^2 = -1$$

The evolution of J^t is determined by the stability matrix A,

$$\frac{d}{dt} J^t(x) = A(x) J^t(x),$$

$$A_{ij}(x) = \omega_{ik} H_{kj}(x);$$

where the matrix of second derivations $H_{kn} = \partial_k \partial_n H$ is the Hessian matrix. For symmetry of H_{kn} ,

$$A^T \omega + \omega A = 0$$

This is the defining property for infinitesimal generators of symplectic or canonical transformations, which leave the symplectic form ω invariant.

For an equilibrium point x_q the stability matrix A is constant. The characteristic polynomial of A for an equilibrium x_q satisfies,

$$\det(A - \lambda I) = \det[\omega^{-1}(A - \lambda I)\omega$$

$$= \det[-\omega A \omega - \lambda I]$$

$$= \det(A^T + \lambda I) = \det(A + \lambda I)$$

The symplectic invariance implies that if λ is an eigenvalues, then $-\lambda, \lambda^*$ and $-\lambda^*$ are also eigenvalues.

A Floquet multiplier $\Lambda = \Lambda(x_0, t)$ associated to a trajectory is an eigenvalues of the Jacobian matrix J and it satisfies

$$\det(J - \Lambda I) = \det(J^T - \Lambda I) = \det(-\omega J^T \omega - \Lambda I)$$

$$= \det(J^{-1}) \det(I - \Lambda J)$$

$$= \Lambda^{2D} \det(J - \Lambda^{-1} I) \qquad \dots (5)$$

This is because, $J^{-1} = -\omega J^T \omega$, J being symplectic. If Λ is an eigenvalue of J so are $\frac{1}{\Lambda}, \Lambda^*$ and $\frac{1}{\Lambda^*}$. Real eigenvalues always come paired as $\Lambda, \frac{1}{\Lambda}$. The complex eigenvalues come in pairs $\Lambda, \Lambda^*, |\Lambda| = 1$, or in loxodromic quartets $\Lambda, \frac{1}{\Lambda}, \Lambda^*$ and $\frac{1}{\Lambda^*}$.

For a trajectory originating near $x_0 = x(0)$ with an initial infinitesimal displacement $\delta x(0)$, the flow transports the displacement $\delta x(t)$ along the trajectory $x(x_0, t) = f^t(x_0)$.

This infinitesimal displacement is transported along the trajectory $x(x_0, t)$, with time variation given by,

$$\frac{d}{dt} \delta x_i(x_0, t) = \sum_j \frac{\partial v_i}{\partial x_j}(x) \Bigg|_{x = x(x_0, t)} \delta x_j(x_0, t) \qquad \dots (6)$$

The system of linear equations of variations for the displacement of the infinitesimally close neighbour $x + \delta x$ follows from the flow equations by Taylor expansion to linear order,

$$\dot{x}_i + \delta \dot{x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \qquad \dots (7)$$

Taking these together, the set of equations governing the dynamics in the tangent bundle $(x, \delta x) \epsilon\, TB$ obtained by adjoining the d-dimensional tangent space $\delta x \,\epsilon\, TB_x$ to every point $x \,\epsilon\, B$ in the d-dimensional state space $B \subset R^D$, may be written as,

$$\dot{x}_i = v_i(x), \delta \dot{x}_i = \sum_j A_{ij}(x) \delta x_j$$

and the stability matrix or the velocity gradients matrix may be written as,

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \qquad \dots (8)$$

Equation (8) describes the instantaneous rate of shearing of the infinitesimal neighbourhood of $x(t)$ by the flow.

Description of neighbourhoods of equilibria and periodic orbits is afforded by projection operators,

$$P_i = \prod_{j \neq i} \frac{M - \lambda^{(j)} I}{\lambda^{(i)} - \lambda^{(j)}} \qquad \dots (9);$$

where matrix $M$is either equilibrium stability matrix A, or periodic Jacobian matrix $\hat{J}$ restricted to a Poincare section. For each distinct eigenvalue $\lambda^{(i)}$ of M, the columns/rows of P_i are,

$$\left(M - \lambda^{(i)}\right)P_j = P_j\left(M - \lambda^{(j)} I\right) = 0$$

are the right/left eigenvectors $e^{(K)}, e_{(K)}$ of M which span the corresponding linearized subspace, provided M is not of Jordan type. For determining the eigenvalues of the finite time local deformation J^t for a general nonlinear smooth flow, the Jacobian matrix may be computed by integrating the equations of variations,

$$x(t) = f^t(x_0), \delta x(x_0, t) = J^t(x_0)\delta x(x_0, 0)$$

For equilibrium point x_q,

$$J^T\left(x_q\right) = e^{A_q t}, A_q = A\left(x_q\right) \qquad \dots (10)$$

Conclusion:

In Hamiltonian dynamics of real physical entities, the difference between support towards a flow and resistance against it not only causes the transactions to take place, but also distorts the neighbourhood of the flow. For sufficiently large values of K, a given Hamiltonian flow may turn into a highly chaotic one. In accordance with the Laws of Physical Transactions as per the Theory of Abstraction, such flows will generate disturbances in all parts of the vicinity of the transactions. These disturbances will tend to smooth out equally in every possible direction, causing secondary disturbances against stability.