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A New Approach on Smarandache $\mathbf{tn}_1$ Curves in terms of Spacelike Biharmonic Curves with a Timelike Binormal in the Lorentzian Heisenberg Group Heis^3

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***ABSTRACT*:** In this paper, we study spacelike biharmonic curve with a timelike binormal in the Lorentzian Heisenberg group Heis^3. We define a special case of such curves and call it Smarandache $\mathbf{tn}_1$ curves in the Lorentzian Heisenberg group Heis^3. We construct parametric equations of Smarandache $\mathbf{tn}_1$ curves in terms of spacelike biharmonic curves with a timelike binormal in the Lorentzian Heisenberg group Heis^3.

***KEYWORDS*:** Heisenberg group, biharmonic curve, Smarandache curve.

I. INTRODUCTION

It is safe to report that the many important results in the theory of the curves in E^3 were initiated by G. Monge and G. Darboux pionnered the moving frame idea. Thereafter, Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry. At the begining of the twentieth century, A. Einstein's theory opened a door of use of new geometries. One of them, Minkowski space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold, was introduced and some of classical differential geometry topics have been treated by researchers.

Let (N,h) and (M,g) be Riemannian manifolds. Denote by R^N and R the Riemannian curvature tensors of N and M, respectively. We use the sign convention:

$$R^N(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]},\ X,Y\in\Gamma(TN).$$

For a smooth map $\phi:N\to M$, the Levi-Civita connection ∇ of (N,h) induces a conncetion ∇^ϕ on the pull-back bundle

$$\phi^*TM =_{p\in N} T_{\phi(p)}M.$$

The section $\mathsf{T}(\phi):=\mathrm{tr}\nabla^\phi d\phi$ is called the tension field of ϕ. A map ϕ is said to be harmonic if its tension field vanishes identically.

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A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathrm{T}(\phi) \right|^2 dv_h.$$

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_\phi \mathrm{T}(\phi) + \mathrm{tr} R(\mathrm{T}(\phi), d\phi) d\phi, \tag{1.1}$$

and called the bitension field of ϕ. The operator Δ_ϕ is the rough Laplacian acting on $\Gamma(\phi^* TM)$ defined by

$$\Delta_\phi := -\sum_{i=1}^{n} \left(\nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i}^N e_i}^\phi \right),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study spacelike biharmonic curve with a timelike binormal in the Lorentzian Heisenberg group Heis^3. We define a special case of such curves and call it Smarandache $\mathbf{tn}_1$ curves in the Lorentzian Heisenberg group Heis^3. We construct parametric equations of Smarandache $\mathbf{tn}_1$ curves in terms of spacelike biharmonic curves with a timelike binormal in the Lorentzian Heisenberg group Heis^3.

II. PRELIMINARIES

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{x} y + x \overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g = -dx^2 + dy^2 + (x dy + dz)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2, \mathbf{e}_3] = 2\mathbf{e}_1, [\mathbf{e}_3, \mathbf{e}_1] = 0, [\mathbf{e}_2, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \ g(\mathbf{e}_3, \mathbf{e}_3) = -1.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric* g *, defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j) *-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Moreover, we put

$$R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d),$$

where the indices a, b, c and d take the values $1,2$ and 3.

$$R_{1212} = -1, R_{1313} = 1, R_{2323} = -3. \tag{2.3}$$

III. SPACELIKE BIHARMONIC CURVES IN THE LORENTZIAN HEISENBERG GROUP HEIS^3

Let $\gamma : I \to Heis^3$ be a spacelike biharmonic curve with a timelike binormal on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}_1$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ), and $\mathbf{n}_2$ is chosen so that $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} &\nabla_{\mathbf{t}}\mathbf{t} = \kappa\mathbf{n}_1, \\ &\nabla_{\mathbf{t}}\mathbf{n}_1 = -\kappa\mathbf{t} + \tau\mathbf{n}_2, \end{aligned} \tag{3.1}$$

$$\nabla_{\mathbf{t}}\mathbf{n}_2 = \tau\mathbf{n}_1,$$

where κ is the curvature of γ and τ is its torsion

$$g(\mathbf{t},\mathbf{t}) = g(\mathbf{n}_1,\mathbf{n}_1) = 1,\ g(\mathbf{n}_2,\mathbf{n}_2) = -1,$$
$$g(\mathbf{t},\mathbf{n}_1) = g(\mathbf{n}_1,\mathbf{n}_2) = (\mathbf{t},\mathbf{n}_2) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3,$$
$$\mathbf{n}_1 = n_1^1\mathbf{e}_1 + n_1^2\mathbf{e}_2 + n_1^3\mathbf{e}_3,$$
$$\mathbf{n}_2 = \mathbf{t}\times\mathbf{n}_1 = n_2^1\mathbf{e}_1 + n_2^2\mathbf{e}_2 + n_2^3\mathbf{e}_3.$$

Theorem 3.1. *(see [12]) Let $\gamma: I \to Heis^3$ be a non-geodesic spacelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is a non-geodesic biharmonic curve if and only if*

$$\kappa = \text{constant} \neq 0,$$
$$\kappa^2 - \tau^2 = 1 + 4\left(n_2^1\right)^2, \quad (3.2)$$
$$\tau' = -2n_1^1 n_2^1.$$

Corollary 3.2. *(see [12]) Let $\gamma: I \to Heis^3$ be a non-geodesic spacelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\kappa = \text{constant} \neq 0,$$
$$\tau = \text{constant}, \quad (3.3)$$
$$n_1^1 n_2^1 = 0,$$
$$\kappa^2 - \tau^2 = 1 + 4\left(n_2^1\right)^2.$$

Theorem 3.3. *(see [12]) Let $\gamma: I \to Heis^3$ be a non-geodesic spacelike biharmonic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Then*

$$\mathbf{t}(s) = \cosh\sigma\mathbf{e}_1 + \sinh\sigma\sinh\psi(s)\mathbf{e}_2 + \sinh\sigma\cosh\psi(s)\mathbf{e}_3, \quad (3.4)$$

where $\sigma \in \mathbb{R}$.

IV. SMARANDACHE $\mathbf{tn}_1$ CURVE OF SPACELIKE BIHARMONIC CURVE IN THE LORENTZIAN HEISENBERG GROUP HEIS^3

Definition 4.1. *Let $\gamma : I \to Heis^3$ be a unit speed regular curve in the Lorentzian Heisenberg group $Heis^3$, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.*

Now, let us define a special form of Definition 4.1.

Definition 4.2. *Let $\gamma : I \to Heis^3$ be a unit speed regular curve in the Lorentzian Heisenberg group $Heis^3$ and $\{\mathbf{t},\mathbf{n}_1,\mathbf{n}_2\}$ be its moving Frenet-Serret frame. Smarandache $\mathbf{tn}_1$ curves are defined by*

$$\varphi = \frac{1}{\sqrt{2\kappa^2 - \tau^2}}\left(\mathbf{t} + \mathbf{n}_1\right). \tag{4.1}$$

Theorem 4.2. *Let $\gamma : I \to Heis^3$ be a unit speed spacelike biharmonic curve and φ its Smarandache $\mathbf{tn}_1$ curve on $Heis^3$. Then, the parametric equations of φ are*

$$\begin{aligned}
x_\varphi(s) &= \frac{\sinh\sigma}{\sqrt{2\kappa^2 - \tau^2}}(\cosh(\beta s + \zeta) + \frac{1}{\kappa}(\beta + 2\cosh\sigma)\sinh(\beta s + \zeta)),\\
y_\varphi(s) &= \frac{\sinh\sigma}{\sqrt{2\kappa^2 - \tau^2}}(\sinh(\beta s + \zeta) + \frac{1}{\kappa}(\beta + 2\cosh\sigma)\cosh(\beta s + \zeta)),\\
z_\varphi(s) &= \frac{1}{\sqrt{2\kappa^2 - \tau^2}}(\cosh\sigma - \frac{1}{\beta}\sinh^2\sigma\sinh^2(\beta s + \zeta) - a_1\sinh\sigma\sinh(\beta s + \zeta)\\
&+ \frac{1}{\kappa}\sinh\sigma(\beta + 2\cosh\sigma)(-\frac{1}{\beta^2}\sinh(\beta s + \zeta)\cosh(\beta s + \zeta)\\
&- c_1 s\cosh(\beta s + \zeta) - c_2\cosh(\beta s + \zeta))),
\end{aligned} \tag{4.2}$$

where $\zeta,\ a_1,\ a_2,\ a_3,\ c_1,\ c_2$ are constants of integration and $\beta = \dfrac{\kappa - \sinh 2\sigma}{\sinh\sigma}$.

Proof. Equations (2.1), (3.1) and (3.4) imply

$$\mathbf{t} = \cosh\sigma\mathbf{e}_1 + \sinh\sigma\sinh(\beta s + \zeta)\mathbf{e}_2 + \sinh\sigma\cosh(\beta s + \zeta)\mathbf{e}_3, \tag{4.3}$$

where $\beta = \dfrac{\kappa - \sinh 2\sigma}{\sinh\sigma}$.

Using (2.1) in (4.7), we obtain

$$\begin{aligned}
\mathbf{t} = (&\sinh\sigma\cosh(\beta s + \zeta), \sinh\sigma\sinh(\beta s + \zeta), \cosh\sigma\\
&- \frac{1}{\beta}\sinh^2\sigma\sinh^2(\beta s + \zeta) - a_1\sinh\sigma\sinh(\beta s + \zeta)),
\end{aligned} \tag{4.4}$$

where a_1 is constant of integration.

From Frenet formulas (3.1) and (4.3), we have

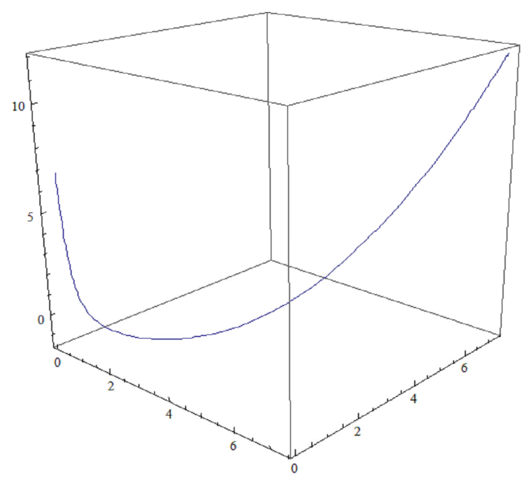
$$\mathbf{n}_1 = \frac{1}{\kappa}[\sinh\sigma(\beta+2\cosh\sigma)\cosh(\beta s+\zeta)\mathbf{e}_2 + \sinh\sigma(\beta+2\cosh\sigma)\sinh(\beta s+\zeta)\mathbf{e}_3]. \tag{4.5}$$

Smilarly, using (2.1) in (4.5), we obtain

$$\mathbf{n}_1 = \frac{1}{\kappa}\sinh\sigma(\beta+2\cosh\sigma)(\sinh(\beta s+\zeta),\cosh(\beta s+\zeta), -\frac{1}{\beta^2}\sinh(\beta s+\zeta)\cosh(\beta s+\zeta) - c_1 s\cosh(\beta s+\zeta) - c_2\cosh(\beta s+\zeta)). \tag{4.6}$$

Next, we substitute (4.4) and (4.6) into (4.1), we get (4.2). The proof is completed.

Using Mathematica in Theorem 4.2, yields



Corollary 4.3. *Let* $\gamma: I \to Heis^3$ *be a unit speed spacelike biharmonic curve. Then,*

$$\kappa = \sqrt{1+4(n_2^1)^2}\cosh\phi, \quad \tau = \sqrt{1+4(n_2^1)^2}\sinh\phi, \tag{4.7}$$

where ϕ *is arbitrary angle.*

Proof. Using Corollary 3.2 we have (4.7).

Theorem 4.4. *Let* $\gamma: I \to Heis^3$ *be a unit speed spacelike biharmonic curve and* φ *its Smarandache* $\mathbf{tn}_1$ *curve on* $Heis^3$. *Then, the parametric equations of* φ *are*

$$
\begin{aligned}
x_{\varphi}(s) &= \frac{\sinh\sigma}{\sqrt{1+4\left(n_2^1\right)^2\left(\cosh^2\phi+1\right)}}(\cosh(\beta s+\zeta) \\
&+ \frac{1}{\sqrt{1+4\left(n_2^1\right)^2}\cosh\phi}(\beta+2\cosh\sigma)\sinh(\beta s+\zeta)), \\
y_{\varphi}(s) &= \frac{\sinh\sigma}{\sqrt{1+4\left(n_2^1\right)^2\left(\cosh^2\phi+1\right)}}(\sinh(\beta s+\zeta) \\
&+ \frac{1}{\sqrt{1+4\left(n_2^1\right)^2}\cosh\phi}(\beta+2\cosh\sigma)\cosh(\beta s+\zeta)), \\
z_{\varphi}(s) &= \frac{1}{\sqrt{1+4\left(n_2^1\right)^2\left(\cosh^2\phi+1\right)}}(\cosh\sigma-\frac{1}{\beta}\sinh^2\sigma\sinh^2(\beta s+\zeta) \\
&+ \frac{1}{\sqrt{1+4\left(n_2^1\right)^2}\cosh\phi}\sinh\sigma(\beta+2\cosh\sigma)(-\frac{1}{\beta^2}\sinh(\beta s+\zeta)\cosh(\beta s+\zeta) \\
&- c_1 s\cosh(\beta s+\zeta)-c_2\cosh(\beta s+\zeta))-a_1\sinh\sigma\sinh(\beta s+\zeta)),
\end{aligned}
\tag{4.8}
$$

where $\zeta,\ a_1,\ a_2,\ a_3,\ c_1,\ c_2$ *are constants of integration and* $\beta=\dfrac{\kappa-\sinh 2\sigma}{\sinh\sigma}$.

V. CONCLUSION

In the last decade there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. Constructing the examples and **classi cation** results have become important from the differential geometric aspect. Also, it is the analytic aspect from the point of view of partial differential equations

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