THE CALCULUS RELATION DETERMINATION, WITH WHATEVER PRECISION, OF COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

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0. ABSTRACT.

These papers show a calculus relation (50) of complete elliptic integral K(k) with minimum **15** precise decimals and the possibility to obtain a more precisely relation.. It results by application Landen's method, of geometrical-arithmetical average, not for obtain a numerical value but to obtain a compute algebraically relation after 5 steps of a geometrical transformation, called "CENTERED PROCESS".

The frequency is the is the physical quantity which can be measured today with maximum precision. Therefore, length unity definition (standard metre from Sèvres-Paris) was replaced, in 1983, by multiples of an oscilation wavelength (krypton 86 radiation), abd the time unit was redefined by multiples of periods of a certain oscillation. The calculus of the frequencies of different technical systems, especially non-linears, could not be raised, unfortunately, at the same desired precision level.

The complete elliptical integral of the first kind **K(k)** could offer the solution for precise determination of the frequencies of some non-linear systems, but the ascending power series trough it is expressed is low convergent. Therefore, some numerical methods appeared, such as Landen method, or geometrical-aritmetical average, which offers a precise (numerical) value of **K** for a given **k**, values that are tabulated, with various number of exact decimals- 9 in Abramovitz [20] s.a.[19].

The Autor's idea was to obtain not the numerical value of K(k), but an algebrical expression (computing relation) from which can result with an imposed precison (whatever high wanted), the integral value for any k value, and not only for those existing in tables, avoiding this way the sometimes necessary interpolations. For unlimited precisions, this computing relation is **K(k) =** π **/2R(k) ,** and, for minimum **15** exact decimals, is shown that only 5 steps are needed, so **R_5(k)** is the perfect square.

$$\mathbf{R_5(k)} = \frac{1}{4}\left[\frac{A+G}{2} + \sqrt[4]{\frac{A^2+G^2}{2}AG}\right]^2 = \frac{1}{4}\left[\sqrt{A_2(R_2,p_2)} + \sqrt{G_2(R_2,p_2)}\right]^2 \text{ noting}$$

$$\mathbf{G} = \sqrt[8]{1-k^2} = \sqrt[4]{k'} = \sqrt[4]{\sqrt{1-k^2}} = \sqrt{p_1} \quad \text{and} \quad \mathbf{A} = \sqrt{\frac{1+\sqrt{1-k^2}}{2}} = \sqrt{\frac{1+G^4}{2}} = \sqrt{R_1}$$

The calculation algorithm shown in this work, which is at the same time, a new geometrical transformation, named "**centering**" - because for N$\rightarrow\infty$ the eccentric trigonometrical circle , with numerical excentricity $k \neq 0$, is transforming in the circle with zero numerical eccentricity ($k_N = 0$), therefore centric - establish the step-by-step transformations, over two, three or four steps and can establish, furthermore, the transformations over more steps. For example, the previous relation $\mathbf{R_5}$ give the dependence between the step 5 with those obtained after step 1, namely over 4 steps. Similarly, $\mathbf{R_9}$ can be obtained, where $\sqrt{R_5} \rightarrow A$ and $\sqrt{p_5} \rightarrow G$, but the precisions obtained this way by far exceed practical demands.

1. Introduction

The integrals with $\int R(z,w)dz$, form where R is a rational function of two arguments and w^2 = P(z) is a third or fourth degree polynom, are named **elliptical**. Any eliptical integral can be arranged in one of the three forms named of **first kind,** of **second kind** or of **third kind**.

The real elliptical integrals of the first kind, noted $\boldsymbol{F}$(k, φ) and of second kind, noted $\boldsymbol{E}$ (k, φ), are the integrals defined, in normal trigonometrical for, respectively, normal (standard) form Legendre form by the expressions:

(**1**) $\boldsymbol{F(\varphi, k)} \equiv \int_0^{\varphi} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ = u, of first kind and

(**2**) $\boldsymbol{E(\varphi, k)} \equiv \int_0^{\varphi} \sqrt{1-k^2\sin^2\psi}\, d\psi = \int_0^{\sin\varphi} \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx$, of second kind

The trigonometric form results from the standard one by variable changing:

(**3**) $x = \sin\psi$

The **k** parameter, underunitary in abolute value, is named **the modulus** of these integrals, as, on the other hand, of the elliptical **Jacobi** functions in **Gudermann**'s notation.

(**4**) sn (u, k) = $\sin\varphi$, cn (u, k) = $\cos\varphi$ si dn (u, k) = $\sqrt{1-k^2\sin^2\varphi}$ and represents, at the same time, the numerical excentricity (e ≡ k) of the excentric circular supermathematical functions. [1 ,... ,14], and the expression

(**5**) **k**' = $\sqrt{1-k^2}$ = p is named ***<u>the complementary modulus</u>*** noted in this work also with p

For the superior lomits of the real integrals $\varphi = \pi/2$ respectiveley, $\sin\varphi = 1$ we obtain the ***complete*** ellipticalintegrals of the first kind K(k) (1') or of second kinde E(k) .

(**1'**) **F(** π/2, k) ≡ K(k) ≡ $\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi}}$

The development in acending power series of the function K(k) is

(**6**) **K(k)** = $\frac{\pi}{2}\left\{1+(\frac{1}{2})^2k^2+(\frac{1.3}{2.4})^2k^4+(\frac{1.3.5.}{2.4.6.})^2k^6+...+[\frac{(2n-1)!!}{(2n)!!}]^2k^{2n}+...\right\} = \frac{\pi}{2}\boldsymbol{F}(\frac{1}{2},\frac{1}{2};1;k^2)$

and presents a very low convergence for $|k| < 1$. In the second experssion $\boldsymbol{F}(\alpha,\beta;\gamma;z)$ is the function, or the hypergeometric series, with **Gauss** notation, where $\boldsymbol{Re}(\alpha+\beta-\gamma) = 0$ The series is convergent in the whole unitary radius circle (trigonometrical TC), except the $z = k^2 = 1$ [21] point, where neither the following calculus relation is not available

In 1826 , **Adrien Marie Legendre** (1752-1833) in "Traite des functions elliptiques et des integrales Euleriennes" which represented the synthesis of 40 years of researches in the elliptic and eulerian integrals field, presented the tables of the F (φ, k) si E (φ, k) integrals values, given for all for all the values of the angle φ from degree to degree and for 90 values of k, corresponding for the angle.

(**7**) β_M = arcsin k , (angle noted in literature as $\alpha \equiv \beta_M$) also from degree to degree, therefore 16.200 results with ten exact decimals for $\varphi \in [0, \pi/4]$ and with nine exact decimals for $\varphi \in [\pi/4, \pi/2]$; the calculations being done by himself with 14, respectiveley 12 exact decimals. [23].

The numerical calculus problems regarding the elliptical integrals and functions are easier treated with the **theta-elliptical** functions[28]. They are defined as sums of the q parameter of **Jacobi** [20]- for $|q| < 1$-

For example,

(**8**) ϑ_3(u) = 1 + 2 $\sum_{n=1}^{\infty} q^{n^2}\cos 2nu$ and the link with K (k) is

(**9**) K (k) = $\frac{\pi.\vartheta_3^2}{2} = \frac{\pi/2}{R_N}$

At the monthly communications ofthe Academy of Berlin in 1883, **Weierstrass** presented the possibility to increase the precision for calculating the q parameter by **Landen** method, by stopping to a single iteration, and he obtained what in this work is called the numerical excentricity k after the first step, from the centering transformation which will be presented further. With actual notation, from the present work, which refers to circle radius R_1 and the real excentricity e_1, all of them after one step of centering transformation, the numerical excentricity k_1 is

(**10**) $k_1 = \frac{e_1}{R_1} = \frac{1-k'}{1+k'} = \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} = \frac{2q+2q^9+2q^{25}+...}{1+2q^4+2q^{16}+...}$

and by reversing of this series (10) the q parameter is obtained, which is the infinite series [22]

(**11**) $q = \frac{k_1}{2} + 2\left(\frac{k_1}{2}\right)^5 + 15\left(\frac{k_1}{2}\right)^9 + 150\left(\frac{k_1}{2}\right)^{13} + 1701\left(\frac{k_1}{2}\right)^{17} + ...$

Weierstrass foresaw the possibility to furthermore increase the precision of q, by continuing the calculus algorithm, but he didn't continued this way, preferring other ways.

2. The radial-excentric function rexθ

Is the supermathematical excentrical function (SMF) [see .8 and 9] defined in the work[1] as a function of variable at excenter θ and re-defined in the work [2] as an adimmensional and multiple function with important applications [7], [14],also in the works [12] and [13] as function of variable at center α.

It is an excentric trigonometrical function or elementary excentric circular function (**ECF**) which can represent the equations of all known plane curves and of other new ones, resulted after the introduction in mathematics of **SMF**; These new curves are named excentrics[3], [4], [5], [6], to differentiate them from the old ones, named from now, centrics. In this way, to each knwn centric form, as circle, ellipse, hyperbola, etc corresponds an infinity of plane excentric curves. The situation is similar in the spaces with more dimmensions. The extremely large possibilities to express this function derive from its trigonometrical expressions, which represents, at the same time, the distnce between two points in the plane, in polar coordinates: one being situated on the trigonometric circle (**TC**), of centric polar coordinates W (α, R=1) or of the excentric polar coordinates W(θ, ρ= **rex**(θ,E)) - depending on how the origin **O** of the coordinate axis system is chosen in the center of the circle TC or in the excenter E- and the second, named excenter, of the polar coordinates
E (ε, e = k), in the circle's plane. If the **ECF** are defined on circele with radiuses R $\neq$ 1, the points will be noted with M_i, and on the **CT** with W_i.

The two algebrical (trigonometrical) experssions of the rex function each of them with two determinations (1- the main-one, corresponding to the sign + in the front of rhe square root and 2- the second-one) are:

(12) $\mathbf{rex_{1,2}}\,\theta = R\,[-e.\cos(\theta-\varepsilon) \pm \sqrt{1-e^2\sin^2\theta}\,]$, as a function of the excentrical variable θ and

(12') $\mathbf{rex}\,\alpha_{1,2} = R\,[\pm\sqrt{1+e^2-2e\cos(\alpha_{1,2}-\varepsilon)}\,]$, as a function of the excentrical variable $\alpha_{1,2}$.

The **rex** function has the property of first degree homogenity because, being o function of the circle's radius (R = 1 of **TC** and R_i of any other circle) and his excentricity (e = k on **TC** and R_i on other circles, but with the same numerical excentricity k) - for ε = 0 - f (R = 1, e), by multiplying the variables with the scalar $R_i > 0$ we obtain
(13) f (R_i . R, e . R_i) = R_i f (R = 1,e) as results also from (12) and (12').
In this worl, only the main determinations will be used, abandoning these indices. The indices that will appear will refer to the number of the step (i = 1...N) of the centering geometrical transformation.

For θ= 0, π/2 , π and e = k we obtain the real values: minimal (**m**) , weighted (**p**) and maximal (**M**) and the numerical ones (in respect to the radius) k and k' of rex θ. In the initial stage, on **TC**, because the radius R = 1, all the real values are equal with the numeric ones.

(14) **m** = 1 - e = 1 - k , $\mathbf{p} = \sqrt{1-e^2} = \sqrt{m.M} = \sqrt{1-k^2}$ = **k'** and, respectiveley, **M** = 1 + e = 1 + k

For a circle with a certain radius R_i, the real excentricity e_i, the maximum M_i, the weight p_i and the minimum m_i are real quantities and the numarical excentricity k_i,

(15) $k_i = \frac{e_i}{R_i} = \frac{M_i - m_i}{2R_i} = \frac{M_i - m_i}{M_i + m_i}$, as its complementary

(15') $k'_i = \frac{p_i}{R_i} = \frac{\sqrt{M_i . m_i}}{M_i + m_i}$ are real quatities. The real quantities are

(16) $m_i = R_i - e_i = R_i\,(1 - k_i)$, $p_i = \sqrt{m_i . M_i} = \sqrt{R_i^2 - e_i^2} = R_i\sqrt{1-k_i^2}$, $M_i = R_i + e_i = R_i\,(1 + k_i)$

We can easily observe that M = $\sup_{\theta \in I}$ rexθ si m = $\inf_{\theta \in I}$ rexθ , so rex θ belongs to the class of the functions with a variation bordered by a fixed number V_{Δ} = M - m = 2 e and, by consequence, o such of function is the differnce of two non-descending function and vice-versa.[24, pag. 19].

3. Expressing some means by rex θ functions

Noting with $\mathbf{A}^+$ the arithmetical mean (semisum) of two positive numbers, with $\mathbf{A}^-$ (semidifference) or the arithmetical mean where minimum m change the sign (m $\to$ - m) and with G their geometrical mean

(17) $\mathbf{A}^+$(m, M) = R = 1, A^- (-m, M) = e = k and G (m, M) = p = k', in the initial moment, on the trigonometrical circle TC of R=1 and for any other circle with parameters R_i , e_i si p_i , they are

(18) A_i^+ (m_i , M_i) = R_i, A_i^- (-m_i, M_i) = e_i , and G_i (m_i ,M_i) = p_i = $\sqrt{m_i M_i}$

A perpendicular raised in the excenter E (e, ε) $\equiv$ K (k, 0) cross the trigonometric circle **TC** in the point W and

(19) $\|EW\|$ = rex (π/2, e = k) = ρ(π/2, k) = $\sqrt{(1-k)(1+k)}$ = $\sqrt{m.M}$ = $\sqrt{1-k^2}$ = k' = p ,

and from the points K_i cross the TC in the points W_i for

(19') $\|K_i W_i\|$ = rex (π/2, k_i) = $\sqrt{(1-k_i)(1+k_i)}$ = $\sqrt{1-k_i^2}$ = k'_i = p_i / R_i and from the excenters E_i cross the inner circles in the M_i for

(20) $\|E_i M_i\|$ = R_i rex (π/2, k_i) = $\sqrt{(R_i - e_i)(R_i + e_i)}$ = R_i $\sqrt{1-k_i^2}$ = R_i k'_i = p_i

p being named ***weight*** , or weighted geometrical mean value, of weight 1, of the rexθ, function, because it represents the geometrical mean of the extreme values than takes the excentrical radial function rexθ.

The quantity which is obtained by forming the arithmetical and geometrical means of the values of two quantities, then forming the arithmetical and geometrical means of these means, and repeating the operations until the means obtained by this way become equal, is named arithmetic-geometric average (or mean) of the two values. In this case, such averages can be obtained by two ways: by chosing as initial quantities the extreme values m and M of the rex θ function we can obtain the caracteristical quatities **R** and **e** specific to SMF on a circle of radius R=1, or R_i , wich we will mane internal means - by example (17) and (18)- and the means of the same kind, wich permit the leap from a circle to another with other radius, or from an orbit to another, making the connection between two consecutive orbits, named external means and which are (see. Fig.1)

(21) $\mathbf{A}_1^+$ (R, p) = (1+$\sqrt{1-k^2}$)/2 = R_1 and A_1^- (R, -p) = (1-$\sqrt{1-k^2}$)/2 = e_1 , so

PE 1 : The radius of an orbit is equal with the semisum of the radius and the weight of external orbit (the biggest) and

PE 2 : The excentricity of an orbit is equal with the semidifference of the radius and the weight of the external orbit.

The two arithmetical means can be wrote concentrated as following:

(22) R_1 ,e_1 = $A_1^{\pm}$ (R, $\pm$ p) = (1 $\pm$ $\sqrt{1-k^2}$)/2. and give the two main quantities of a circular orbit, teh radius and excentricity, and serve to calculate the extremes of the orbit.

(23) m_1,M_1 = (R_1 $\mp$ e_1), $\Rightarrow$ m_1 = $\sqrt{1-k^2}$ = p si M_1 = R = 1 and of the weighti

(24) p_1 = G_1(m_1, M_1) = $\sqrt{m_1 . M_1}$ = $\sqrt{1.\sqrt{1-k^2}}$ -as internal mean after the first leap (step).

The relations (21) si (22) give the dependence between the quantities from the initial orbit (**TC** : R = 1, e = k si p = k'=$\sqrt{1-k^2}$) and those of the following orbit, of indice 1. The real radius and excentricity of the new orbit are:

(25) R_1 =A_1^+ (R, p) , e_1 = A_1^- (R, -p) , so, another property of the transformation is

PE 3:The sum of the real excentricity and the radius from a specific orbit is equal with the radius of the biggest circular orbit. .This radius belongs to the previous orbit at the leap from a bigger orbit to a smaller one, and to the next orbit when the leap is reversed, from small to high.. These are the two possible transformations: direct, or impanding to the center, respectiveley, reversed, or expanding.

(26) $M_1 = R_1 + e_1 = R = 1$, because $A^+ (A^+_1 , A^-_1) = A^+_1 + A^-_1 = R = 1$, a main property of **ECF**, which will be used further. But, the sum (26) express the value of M_1, so

PE 4 : when leaping from an orbit to another, the maximum of the orbit with the smaller radius is equal, in value, with the radius of the orbit with the bigger radius. This is also the property on horizontal, or on x axis, of the geometrical transformed of ECF rex, wheb leaping from an orbit to another. On the other hand, because

$A^- (A^+_1, A^-_1) = A^+_1 - A^-_1 = p = k'$ results

(27) $m_1 = R_1 - e_1 = p = k'$ and

PE 5 : the minimum of the orbit of smaller radius (m_{i+1}) is equal with the weight p_i of the orbit of bigger radius.. Because the weight is oriented in the vertical direction y ($\theta = \pi/2$) we will nwme this property as being on "the vertical" of the transformation.

We can easily observe that

(28) $M_1 - m_1 = 2 e_1 = 1 - p = 1 - k' = 1-\sqrt{1-k^2} = V_{\Delta 1}$ and the new weight will result as

(29) $p_1 = G_1 (m_1 , M_1) = G_1 (p, R) = \sqrt{1.\sqrt{1-k^2}} = \sqrt[4]{1-k^2} = \sqrt{p} = \sqrt{k'}$, so

$p_1^2 = p$ sau $(R_1k'_1)^2 = Rk'$ $\sqrt{R_1} = \sqrt{\dfrac{1+\sqrt{1-k^2}}{2}}$ and $\sqrt{p_1} = \sqrt[8]{1-k^2}$

The values of the function rex_i ($\pi/2$) , or the succesive weights, raise in variable ratio geometrical progression, property that derive also from the fact that $p^2_1 = R^2_1 - e^2_1 = (R_1 - e_1)(R_1 + e_1) = M_1 . m_1 = R p = p$, for the first step, because $R = 1$. For the next steps, considerring (24)

(30) $p_{i+1} = G_{i+1}(m_{i+1} , M_{i+1}) = \sqrt{p_i .R_i}$ si $(R_{i+1} k'_{i+1})^2 = k'_iR_i.R_i = k'_iR_i^2$ from where

(31) $k'_i = [\dfrac{R_{i+1}}{R_i} k'_{i+1}]^2$ sau $\sqrt{k'_i} = \sqrt{1-k_i^2} = \dfrac{p_{i+1}}{R_i}$ and the algorithm of leaping from an orbit to another becomes transparent.

4 . The excentric geometric transformation and the centering geometric transformation

The leaps from an orbit to another can take place in two senses. In direct transformation, the rotations of W_i points on **TC** take place in the left sense from W (k) -the initial point- to the final point W_N ($k_N = 0$) and through which the leaps from W in M_i take place from the initial orbit **TC** (of start, of R = 1) on the inner of this (with smaller radiuses, $R_i < 1$), from the point W (k), passing trough the points M_i (e_i) until the final point M_N ($e_N =0$; R_N), through which the excentricities of the orbits descends, by leaps, to the value $e_N = k_N = 0$ and which we name it, by this reason, CENTERING.

The centering is a circular, conformal transformation, composed from a homothety of ratio $h = R_{i+1}/ R_i$ combined with a rotation of angle $\Delta\alpha = \alpha_{i+1} - \alpha_i$ = arcsin(p_{i+1} / $R_{i=1}$) - arcsin (p_i / R_i) = arcsin k'_{i+1} - arcsin k'_i. = arcsin ($k'_{i=1}$ $\sqrt{1-k'^2_i}$ - k'_i $\sqrt{1-k'^2_{i=1}}$) The centrations set transfer the initial point W from **TC** in M_N from the circle with R_N radius, situated on y axis for $N \to \infty$. We will note the centration's final circular orbit with R_N , wich is, obviously, a constant, by one side -being the radius of a circle- and on the other side, a variable R (k) ,because it depends on the excentricity e = k (chosen as equal with the modulus of the eliptical integrals) and which from the transformation starts. Therefore,

(32) $R_N = R (k)$, for $N \to \infty$. independently from the initial position of W on TC, his transformed after the first leap, the M_1 point, will be placed on a parabola with the focal point in the origin O, the vortex on x axis in the point V (½, 0) and passing trough the point B (0, 1) $\equiv W_N \big|_{N\to\infty} \subset$ CT. It results that the centering transformation does not modify the position of W(k = 0) point, -which remains itself- and therefore R (k = 0) = 1 and K (0) = $\pi/2$. The initial point A (1, 0) $\equiv$ W (k = 1) after the first transformation arrives in V (0,5 ; 0) and by the next transformations it leaves not the x axis, so in the final it arrives in O (0, 0) , that means that the radius R(k =1) of the last circle of centration transformation will be zero (R(1) = 0) si K (1) = ∞.

The transformation in reverse sense, from **TC** to circular orbits with bigger and bigger radiuses ($R_i > 1$), when the excentricity of the orbits will also rise from k to $k_N = 1$, for $N \to \infty$, we will call it, from these reasons, EXCENTRICAL geometrical transformation.

In both these transformations, we start from **TC** with cu e = k, with values different from the values 0 or 1, previously discussed, we pass trough the real excentricities e_i, with diiferent values from the numerical ones k_i, and, in the final of each transformation, we will reach again the egalization of these ones with the and with the value 1, in the case of excentric transformation. In the centering case, the two final points W_N and M_N will be situated on the same vertical ($\alpha_N = \pi / 2$) ; the initial point W, corresponding to the angle at centre α = arccos k , suffering exclusively rotational transformations, in leaps, in sinistrorum sense on CT, trough the intermediary points W_i (α_i = arccos k_i = arcsin k'_i) , until in the final point W_N(de $\alpha_N = \pi/2$). The set of all the rotations being of angle β_M = arcsin k = arccos k'..

The points W_N and M_N has the same argument $\alpha_N = \pi / 2$ but the modulus (orbit radiuses) are R = 1 and, respectively, $R_N = R(k)$.

The **ECF** expressed on circles with radiuses $R_i \neq 1$, i = 1 ... N, has the definitory points which are noted with M_i and they are the transformed trough the homotety H_i (O, h_i) -of center of homotety in the origin O and homotety ratio h_i - of the points W_i from TC

(26) $h_i = R_i / R = e_i / e$ for a centering transformation, on the orbit i of radius R_i of the point M_i, to who is corresponding the W_i point from **CT** with R = 1. In the centerinfg transformation, the W_i points exclusively rotate, staying on **TC**, and meanwhile the M_i points rotate and are those who jump from an orbit to another, with different radiuses. Therefore, the transformation from W in M_1 take place trough a rotation $\Re_1(O,\Delta\alpha_1)$ (on TC from W in W_1) followed by a translation or an homotety H_1 (O, h_1) from W_1 in M_1 .All the rotations being of de same center O, the product of two rotations will be also a rotation, and the rotations set forms a commutative group in respect with the composing operation. The rotations product trough which W is transfered in W_N is

(33) $\Re_1 \circ \Re_2 \circ \Re_3 \circ\circ\circ \Re_N = \Re(O, \Delta\alpha_N = \pi/2 - \alpha) = \Re$ (O, β_M), for $N \to \infty$

The homoteties being of the same center O, their set forms an isomorph commutative group with the multiplicative group of non-zero real numbers. The product of two or more homoteties will be also an homotety.

(34) $H_1 \circ H_2 \circ H_3 \circ \circ\circ H_N = H(O, h = \prod h_i) = H[O, h = R(k)]$

By writing the property (26) of **ECF** beginning with the first orbit and finishing with the last one, in the first column, and in the second column the same normalized relations or adimmensional, by dividing with R_i radiuses results

(35)
$$\begin{aligned}
e_1 + R_1 &= R = 1, & 1 + k_1 &= R / R_1 = 1 / R_1\\
e_2 + R_2 &= R_1, & 1 + k_2 &= R_1 / R_2\\
e_3 + R_3 &= R_2, & 1 + k_3 &= R_2 / R_3\\
e_4 + R_4 &= R_3, & 1 + k_4 &= R_3 / R_4\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
e_i + R_i &= R_{i-1}, & 1 + k_i &= R_{i-1} / R_i\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
e_N + R_N &= R_{N-1}, & 1 + k_N &= R_{N-1} / R_N
\end{aligned}$$

By making the normalized relations product, from the second column, we obtain

(26) $\prod_{i=1}^{N}(1+k_i) = 1 / R_N$ or $R_N = 1 / \prod_{i=1}^{N}(1+k_i)$ and for $i \to \infty$ results R(k)

(27) $R(k) = 1 / \prod_{i=1}^{\infty}(1+k_i)$ from where, on the basis of relation (9) , we obtain one of the known forms [19], [20], [21] of the complete eliptic integral of the first kind

(38) $$K(k) = \frac{\pi}{2}\prod_{i=1}^{\infty}(1+k_i)$$

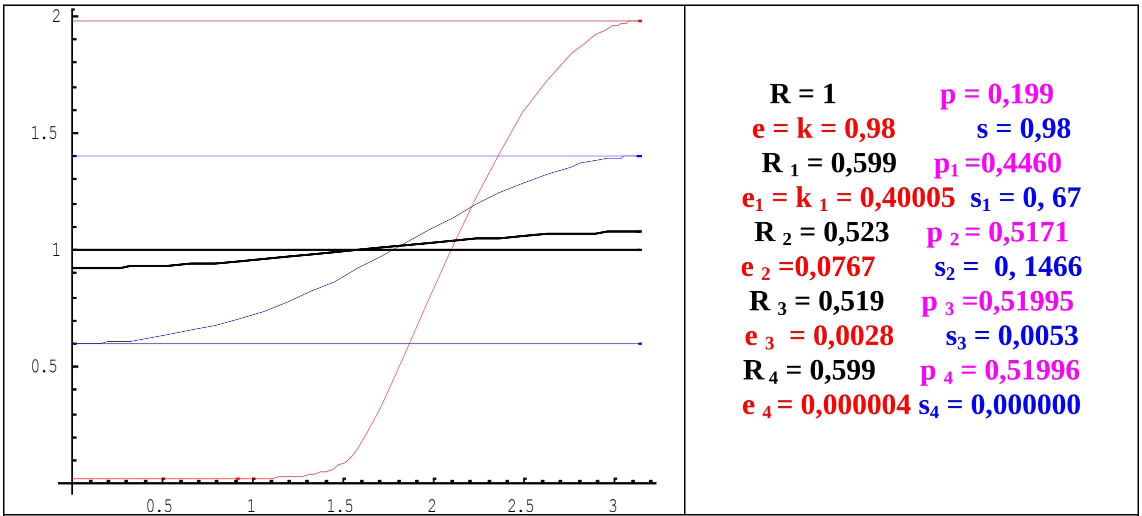

Fig. **1.** The **rexθ** function on 4 orbits

In these relations, for i = 1 results $k_0 = k$ and k_i will have the expression

(39) $k_i = e_i / R_i = \dfrac{1-\sqrt{1-k_{n-1}^2}}{1+\sqrt{1-k_{n-1}^2}}$, like we can deduce also from the relation (22) .

From this relation, far a large number of steps, we will obtain an algebrical expression much too large, being adequate only in the case when a computer programme is achieved, because it has a very simple algorithm.

On the base of **PE 1 ... PE 4** properties and considerring (30)

(40) $M_{i+1} = R_i$, $m_{i+1} = p_i$, so, from the sum and the difference of these relations, we obtain

(41) $2 R_{i+1} = R_i + p_i$, $2 e_{i+1} = R_i - p_i$ and $R_{i+1} = A^+_i = A^+ (R_i , p_i)$

$p_{i+1} = \sqrt{R_i p_i} = G_i = G (R_i , p_i)$, and for a double leap, as by example, from TC to the second orbit (for i = 0) or from the second orbit to the fourth, for i = 2 etc.

(42) $2 R_{i+2} = R_{i+1} + p_{i+1} = (R_i + p_i) / 2 + \sqrt{p_i R_i}$ or $4 R_{i+2} = R_i + p_i + 2 \sqrt{R_i p_i}$

$2 e_{i+2} = R_{i+1} - p_{i+1} = (R_i + p_i) / 2 - \sqrt{R_i p_i}$ or $4 e_{i+2} = R_i + p_i - 2 \sqrt{R_i p_i}$

From (42) we obtain

(43) $R_{i+2} = \left(\dfrac{\sqrt{R_i}+\sqrt{p_i}}{2}\right)^2 = \left(\dfrac{\sqrt{A_i^+}+\sqrt{G_i}}{2}\right)^2$ and $e_{i+2} = \left(\dfrac{\sqrt{R_i}-\sqrt{p_i}}{2}\right)^2 = \left(\dfrac{\sqrt{A_i^+}-\sqrt{G_i}}{2}\right)^2$

(44) $(e, R)_{i+2} = \left(\dfrac{\sqrt{R_i}\mp\sqrt{p_i}}{2}\right)^2 = \left(\dfrac{\sqrt{A_i^+}\mp\sqrt{G_i}}{2}\right)^2$ also shortly written

(45) $p_{i+2} = \sqrt{R_{i+1} . p_{i+1}} = \sqrt[4]{R_i p_i}\sqrt{\dfrac{R_i+p_i}{2}} = \sqrt{\dfrac{R_i+p_i}{2}\sqrt{R_i p_i}} = \sqrt{G_i}\sqrt{A_i^+} = \sqrt{A_i^+ . G_i}$

If in (44) we make $i \rightarrow i+2 \Rightarrow i+4$ we obtain

(46) $(e, R)_{i+4} = \left(\dfrac{\sqrt{R_{i+2}}\mp\sqrt{p_{i+2}}}{2}\right)^2 = \left(\dfrac{\sqrt{A_{i+2}^+}\mp\sqrt{G_{i+2}}}{2}\right)^2$ and $p_{i+4} = \sqrt{A_{i+2}^+ . G_{i+2}}$

For i = 1 in (44) respectively in (45) we obtain the square roots of the sets from the third orbit

(47) $\sqrt{R_3} = (\sqrt{A_1^+} + \sqrt{G_1}) / 2 = (\sqrt{R_1} + \sqrt{p_1}) / 2 = \left(\sqrt{\dfrac{1+\sqrt{1-k^2}}{2}} + \sqrt{\sqrt{1-k^2}} \right) / 2 =$

(48) $\sqrt{p_3} = \sqrt{\dfrac{R_1+p_1}{2}\sqrt{R_1}\sqrt{p_1}} =$

and for i = 1 in (46) results the radius of the **5**-th orbit

(49) $R_5 = \frac{1}{4}\left(\sqrt{R_3}+\sqrt{p_3}\right)^2 = \frac{1}{4}\left(\frac{A+G}{2}+\sqrt[4]{\frac{A^2+G^2}{2}AG}\right)^2$ so

(50) $K(k) \cong \frac{\pi}{2R_5}$ where is noted

(49’) $G = \sqrt{p_1} = \sqrt[8]{1-k^2}$ and $A = \sqrt{R_1} = \sqrt{\frac{1+\sqrt{1-k^2}}{2}} = \sqrt{\frac{1+G^4}{2}}$

5. Conclusions

The relations (49 and 50) here obtained, with the graphs presented in figure 2 respectively, 3, are much simpler then other similar relations, which doesn’t ensures the minimum**15** exact decimals precision, a precision that is practically observed by comparising the results with those of tables. From a practical point of view, in the mechanical engineering domain, it can be assimilated with an exact relation for determining the natural frequency of some non-linear dynamical systems. It can be memorized, in the memory of the computers, instead of the K(k) values tables, having the great advantage that the space allocated in the memory is much reduced.

The presented transformation can be considerred also as a linear transformation Transformarea prezentata poate fi considerata si ca o transformare liniara (fuchsiene), after **H. Poincare**, with double fixed point, which is the point C (-1, 0) from TC, named parabolical transformation. From this point of view, the transformation is achieved by succesive change of the center as excenter for the next orbit: O $\to E_1$, $O_1 \to E_2$, etc, until $O_N \equiv E_N$, for $e_N \to 0$.

In this case, a new property of the transformation occurs:

(51) $R_j + \sum_1^j e_j = 1$ from where, at limit, results $R_N = 1 - \sum_{i=1}^{N} e_i$

Also, the **Landen** transformation is valid:

(52) $(1 + k'_i)(1 + k_{i+1}) = 2$ sau $(1 + \cos\beta_{Mi})(1 + \sin\beta_{Mi+1}) = 2$

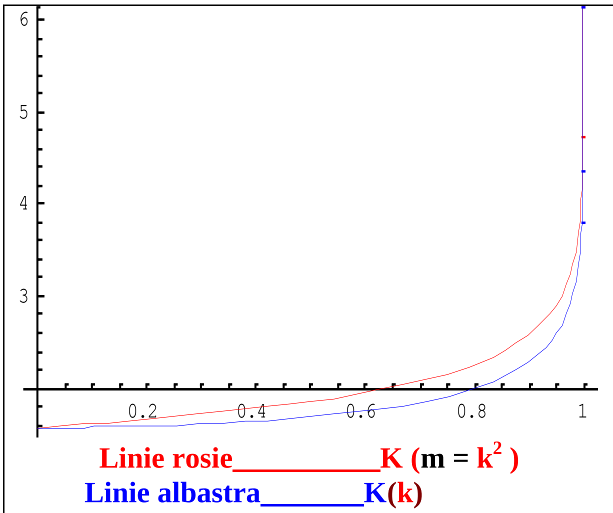

Fig. 3 The complete eliptical integral of the first kind K (k)

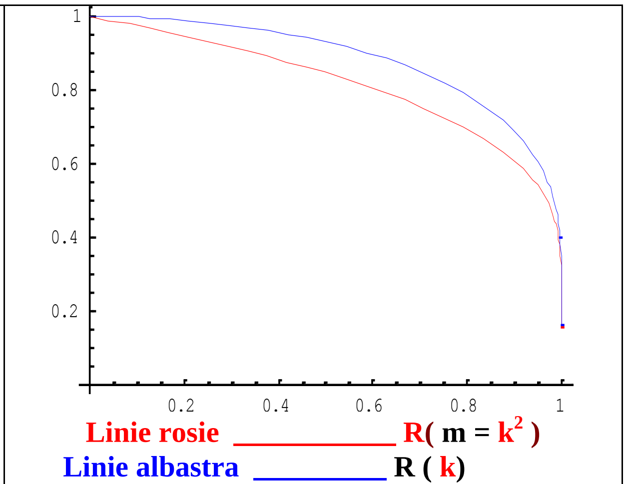

Fig. 2 The radius of the final centering geometrical transformation R_N (k) for k = 5

We can see in figure 1 that R_N is the value that is taken by the function rex in the point ξ. By the existance of a subinterval, named contraction interval , by the existance of this point, from Lagrange’s theorem of average, or from the finite growings theorem, was occupied **D. Pompeiu**, and some important examples offunctions and their contractions intervals were presented by **Miron Nicolescu**. In this domain, studies regarding the generalization of of the notion of divided difference of a function and the properties of average of these ones were studied by **Tiberiu Popoviciu**.

A numerical example, for a very large modulus (k = 0, 98), chosen intentionally for that a larger number of transformations being distinctives, is presented further. It corresponds to the curves from fig 1.

The Orbit i	RADIUS OF THE ORBIT $R_{1+1} = (R_I + p_i) / 2$	W E I G H T		E X C E N T R I C I T Y	
		R E A L	N U M E R I C A L	R E A L	N U M E R I C A L
		$p_i = R_i . k'_i$		$e_{i+1} = (R_i - p_i) / 2$	$k_i = e_i / R_i$
0	1	0, 1989974	0, 1989974	0, 98	0, 98
1	0, 59949870	0, 4460898	0, 7441047	0, 4005013	0, 66806030
2	0, 52279420	0, 5171365	0, 9891780	0, 0767044	0, 14672000
3	0, 51996530	0, 5199576	0, 9999852	0, 0028288	0, 00544403
4	0, 51996114	0, 5199614	1	0, 0000038	0, 00000730
N=5	0, 51996140	0, 5199614	1	0	0

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