Exact solution of viscous-plastic flow equations for Glacier dynamics in 2-dimensional case.

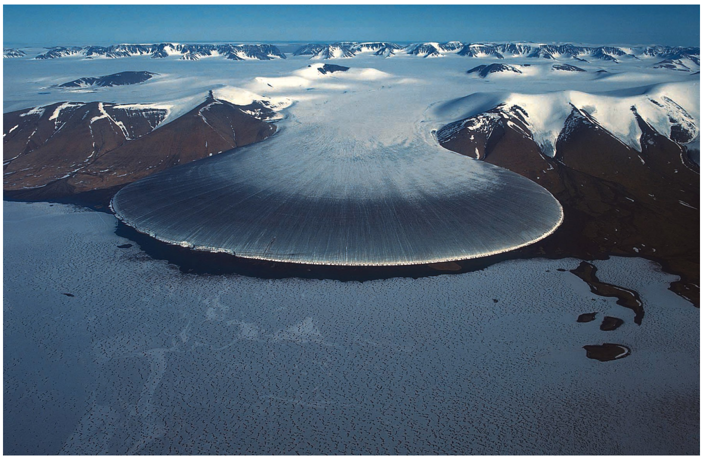
Sergey V. Ershkov
Institute for Time Nature Explorations,
M.V. Lomonosov's Moscow State University,
Leninskie gory, 1-12, Moscow 119991, Russia.

Keywords: Glacier dynamics, exact solution; basal slip, viscous fluid, plastic flow; glacial ice, dynamic viscosity; critical maximal level of stress; shared layer of glacial ice; *Riccati's type* of ODE, *Bernoulli's type* of ODE; surging glaciers.

Here is presented a new exact solution of *Ice dynamics* in Glaciers in terms of viscous-plastic theory of movements, for 2-dimensional case. In general case, 2-D solution of *Ice dynamics* could be classified as *Riccati's type*. Due to a very special character of *Riccati's type* equation, it's general solution is proved to have *a proper gap* of components of such a solution.

It means a possibility of *sudden gradient catastrophe* at definite moment of time-parameter, in regard to the components of solution (*2-D profile of Glacier, 2-D components of ice velocity moving*).

That's why surging glacier seems to be *accelerating* from time to time: it's velocity of moving is *suddenly* rising from few meters to hundreds meters /per day.



A glacier is a massive, slowly moving mass of compacted snow and ice. The action of gravity moves the mass of ice down the slope side: glaciers are being moved from a millimeter to hundreds meters a day. There are two kinds of motion: 1) a slow sliding motion and an avalanche like flow; 2) the internal movement of glacial ice, is a flow similar to plastic flow and viscous flow.

Glaciers move by two mechanisms: basal slip and viscous-plastic flow. In basal slip, the entire glacier slides over bedrock. A glacier also moves by plastic flow, in which it flows as a viscous fluid.

In accordance with [1], 2-dimensional case of glacial ice viscous-plastic flow should be represented in the Cartesian system of coordinates as below (*axis* ***Ox*** *coincides to initial direction of glacial ice flow, which is assumed to be a plane-parallel flow*, $z = const$):

$$\rho\cdot\left(\frac{\partial v_x}{\partial t}+v_x\cdot\frac{\partial v_x}{\partial x}+v_y\cdot\frac{\partial v_x}{\partial y}\right)=-\frac{\partial p}{\partial x}+G_x+\frac{\partial s_{xx}}{\partial x}+\frac{\partial s_{xy}}{\partial y},$$

$$\rho\cdot\left(\frac{\partial v_y}{\partial t}+v_x\cdot\frac{\partial v_y}{\partial x}+v_y\cdot\frac{\partial v_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+G_y+\frac{\partial s_{xy}}{\partial x}-\frac{\partial s_{xx}}{\partial y}, \qquad (1.1)$$

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0, \quad U=\sqrt{4\left(\frac{\partial v_x}{\partial x}\right)^2+\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right)^2},$$

$$s_{xx}=2\left(\mu+\frac{\tau_s}{U}\right)\cdot\frac{\partial v_x}{\partial x}, \quad s_{xy}=2\left(\mu+\frac{\tau_s}{U}\right)\cdot\left(\frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x}\right).$$

- where v_x – is the component of ice velocity in the direction x of Cartesian system x, y; v_y – the component of ice velocity in the direction y; p – is an internal pressure in glacial ice; G_x, G_y – are the appropriate *projections* of gravity (central force) to the chosen initial direction x, y of glacial ice plane-parallel flow; S_{xx}, S_{xy} – are the appropriate components of stress tensor; μ – is a coefficient of glacial ice dynamic viscosity; τ_s – is a critical maximal level of stress in shared layer of glacial ice when it starts to move as viscous flow (*stage of plastic flow*: *if an absolute meaning of stress tensor less than a critical maximal level of stress in shared layer* < τ_s, → *glacial ice does not move*).

From (1.1) we obtain the appropriate equalities below:

$$U = \frac{1}{\mu} \cdot \left(\sqrt{ s_{xx}^{\;2} + \frac{s_{xy}^{\;2}}{4} } - \tau_s \right),$$

$$\frac{\partial v_x}{\partial x} = s_{xx} \Big/ 2(\mu + \frac{\tau_s}{U}) \; .$$

Let's assume in our modeling that the left part of (1.1) *equals to zero* due to negligible terms for the case of *slowly moving* glacial ice. But for the case of *slow* glacial ice flow system (1.1) could be reduced as below

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + G_x + \frac{\partial s_{xx}}{\partial x} + \frac{\partial s_{xy}}{\partial y}, \\ 0 &= -\frac{\partial p}{\partial y} + G_y + \frac{\partial s_{xy}}{\partial x} - \frac{\partial s_{xx}}{\partial y}, \end{aligned} \qquad (1.2)$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad \frac{\partial v_x}{\partial x} = s_{xx} \Big/ 2(\mu + \frac{\tau_s}{U}),$$

$$U = \frac{1}{\mu} \cdot \left(\sqrt{ s_{xx}^{\;2} + \frac{s_{xy}^{\;2}}{4} } - \tau_s \right).$$

Then for finding a solution, we should cross-differentiate 1-st & 2-nd equation (1.2) in regard to x & y, as well as we should combine it by a proper linear way (*besides, on open air* $p(x, y) = const$); in result, we obtain:

$$\frac{\partial^2 s_{xy}}{\partial x^2} + \frac{\partial^2 s_{xy}}{\partial y^2} = 0 ,$$

- it means that S_{xy} – is the *harmonic function* [2].

According to *Liouville's theorem:* "if f is a harmonic function defined on all of $\mathbf{R}^n$ which is bounded above or bounded below, then f is constant" [2].

It is evident that S_{xy}, being the component of stress tensor, is bounded above - *in regard to it's absolute meanings* - due to general physical sense.

So, we have: 1) *S_{xy} is a harmonic function*, 2) *S_{xy} is bounded above*. Thus, in accordance with *Liouville's theorem*, S_{xy} is a constant: S_{xy} = const = 2C. Then from (1.2) we obtain $S_{xx} = - G_x \cdot x + G_y \cdot y + C_0$ (C_0 = const $\neq 0$), but:

$$U = \frac{1}{\mu} \cdot \left(\sqrt{(-G_x \cdot x + G_y \cdot y + C_0)^2 + C^2} - \tau_s \right) ,$$

$$\frac{\partial v_x}{\partial x} = \frac{s_{xx}}{2\mu} (1 - \frac{\tau_s}{\sqrt{s_{xx}^2 + \frac{s_{xy}^2}{4}}}) ,$$

- hence, we obtain in result:

Let's choose $C = 0$, then above equality could be simplified to the form below

$$\frac{\partial v_x}{\partial x} = \frac{1}{2\mu} \left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} ,$$

If we take also into consideration *the continuity equation* (*see* (1.2)):

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \ ,$$

- we obtain that initial system (1.1) is reduced to representation below

$$\frac{\partial v_x}{\partial x} = \frac{1}{2\mu}\left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} ,$$

$$\frac{\partial v_y}{\partial y} = -\frac{1}{2\mu}\left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} . \qquad (1.3)$$

The system above could be easily solved if $G_x = 0$ or $G_y = 0$. Indeed, let's choose for example $G_y = 0$, $G_x \neq 0$ in (1.3), then we obtain below ($C_1 = \text{const} \neq 0$):

$$v_x \equiv \frac{\partial x}{\partial t} = \frac{1}{2\mu}\left\{ -G_x \cdot \frac{x^2}{2} + (C_0 - \tau_s)\cdot x + C_1 \right\} , \quad \Rightarrow$$

$$\Rightarrow \int \frac{d\,x}{\left\{ -G_x \cdot \frac{x^2}{2} + (C_0 - \tau_s)\cdot x + C_1 \right\}} = \frac{t}{2\mu} \ ,$$

- where [4]:

$$\int \frac{d\,x}{\left\{ -G_x \cdot \frac{x^2}{2} + (C_0 - \tau_s)\cdot x + C_1 \right\}} \equiv$$

$$1)\ \frac{2}{\sqrt{\Delta}}\, arctg\, \frac{-G_x \cdot x + (C_0 - \tau_s)}{\sqrt{\Delta}}$$

$$(\ \Delta > 0,\ \Delta = -2\,G_x \cdot C_1 - (C_0 - \tau_s)^2\ )$$

$$2)\ \frac{1}{\sqrt{-\Delta}} \ln \frac{-G_x \cdot x + (C_0 - \tau_s) - \sqrt{-\Delta}}{-G_x \cdot x + (C_0 - \tau_s) + \sqrt{-\Delta}} .$$

$$(\ \Delta < 0\ )$$

Let's choose in above equalities $C_0 = \tau_s$ (*for the aim of clear presentation of final solution*); in such a case the equalities above are simplified then we could obtain a final solution:

$$1)\quad x \;=\; -\frac{\sqrt{\Delta}}{G_x}\cdot tg\,\frac{\sqrt{\Delta}}{4\mu}\,t\,; \qquad 2)\quad x = \frac{\sqrt{-\Delta}}{G_x}\cdot\frac{1+\exp(-\frac{\sqrt{-\Delta}}{2\mu}t)}{1-\exp(-\frac{\sqrt{-\Delta}}{2\mu}t)} \qquad (1.4)$$

$$(\Delta = -2\,G_x\cdot C_1\,,\Rightarrow\; C_1 < 0\,) \qquad\qquad (\,\Delta < 0,\;\Rightarrow\; C_1 > 0\,)$$

First type of solutions (1.4) could be associated with *pulsating glaciers* or *surging glaciers*, which are characterized by periodic movements of glacial ice.

As for coordinate $y = y\,(t)$, we could obtain from (1.3):

$$\frac{\partial v_y}{\partial y} \equiv \frac{\partial v_y}{\partial t}\cdot\frac{\partial t}{\partial y} \equiv \ddot{y}\cdot(\dot{y})^{-1}, \quad \Rightarrow$$

$$\Rightarrow\; \ddot{y} \;-\; \left(\frac{G_x\cdot x}{2\mu}\right)\cdot\dot{y} \;=\; 0\;\;,$$

- *Bernoulli's type* ordinary differential equation, which has a proper regular solution [4].

But in general case, if G_x, $G_y \neq 0$, equations (1.3) could be classified as *Riccati's type*. Due to a very special character of *Riccati's type* equation, it's general solution is proved to have *a proper gap* of components of such a solution [3-4].

It means a possibility of *sudden gradient catastrophe* [5] at definite moment of time-parameter, in regard to the components of solution (*2-D profile of Glacier, 2-D components of ice velocity moving*). That's why Glacier seems to be *accelerating* from time to time: it's velocity of moving is *suddenly* rising from few meters to hundreds meters /per day.

Let's also explore the case $C_0 = \tau_s$, $C_1 = 0$ (*we choose all new constants below are equal to zero*):

$$\frac{\partial v_x}{\partial x} = -\frac{1}{2\mu} G_x \cdot x\,, \;\Rightarrow\; v_x = \dot{x} = -\frac{1}{4\mu} G_x \cdot x^2\,, \;\Rightarrow\; x = \left(\frac{4\mu}{G_x}\right) \cdot t^{-1}\,,$$

$$\frac{\partial v_y}{\partial y} \equiv \frac{\partial v_y}{\partial t} \cdot \frac{\partial t}{\partial y} \equiv \ddot{y} \cdot (\dot{y})^{-1}, \;\Rightarrow\; \ddot{y} \cdot (\dot{y})^{-1} = 2t^{-1}, \;\Rightarrow\; \ddot{y} - 2t^{-1} \cdot \dot{y} = 0\,,$$

- here the last equation is also the *Bernoulli's type* of ODE in regard to component *y (t)*, which has a proper regular solution [4]:

$$\ddot{y} - 2t^{-1} \cdot \dot{y} = 0, \;\Rightarrow\; (\dot{y} \cdot t^{-2})' = 0\,,$$

$$d\,y = C_2 \cdot t^2\, dt\,, \;\Rightarrow\; y = \frac{C_2}{3} \cdot t^3 + C_3,$$

- it means that *due to general physical sense*: $C_1 = \text{const} \neq 0$ (*never ever*).

Besides, let's obtain solution in general case G_x, $G_y \neq 0$ - for equations (1.3):

$$\frac{\partial v_x}{\partial x} \equiv \frac{\partial v_x}{\partial t} \cdot \frac{\partial t}{\partial x} \equiv \ddot{x} \cdot (\dot{x})^{-1} = \frac{1}{2\mu} \left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\}\,,$$

$$\frac{\partial v_y}{\partial y} \equiv \frac{\partial v_y}{\partial t} \cdot \frac{\partial t}{\partial y} \equiv \ddot{y} \cdot (\dot{y})^{-1} = -\frac{1}{2\mu} \left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\}\,.$$

If we designate: $p(y) = y'(t)$, $q(x) = x'(t)$, → (1.3) could be transformed to the form below

$$q'(x) \cdot q(x) = \frac{q(x)}{2\mu} \left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\}\,,$$

$$p'(y) \cdot p(y) = -\frac{p(y)}{2\mu} \left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\}\,,$$

- then we obtain:

$$q'(x) = \frac{1}{2\mu}\left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} \,,$$

$$\Rightarrow \quad q'(x) + p'(y) = 0$$

$$p'(y) = -\frac{1}{2\mu}\left\{ G_y \cdot y - G_x \cdot x + (C_0 - \tau_s) \right\} \,,$$

- but

$$q'(x) + p'(y) = 0, \;\Rightarrow\; q(x) + p(y) = C_{qp}, \;\Rightarrow\; x(t) + y(t) = C_{qp} \cdot t + (x_0 + y_0) \;.$$

Thus, we obtain in result ($C_{q,p}$ = const):

$$x''(t) = \frac{x'(t)}{2\mu}\left\{ G_y \cdot (-x + C_{qp} \cdot t + (x_0 + y_0)) - G_x \cdot x + (C_0 - \tau_s) \right\} , \quad \Rightarrow$$

$$x''(t) = \frac{x'(t)}{2\mu}\left\{ -(G_x + G_y) \cdot x + (G_y \cdot C_{qp}) \cdot t + \left[G_y \cdot (x_0 + y_0) + (C_0 - \tau_s) \right] \right\},$$

- but if $C_{q,p} = 0$:

$$d\left(x'(t) + \frac{(G_x + G_y)}{4\mu} \cdot x^2 \right) = \frac{1}{2\mu}\, d\left(\left[G_y \cdot (x_0 + y_0) + (C_0 - \tau_s) \right] \cdot x \right), \quad \Rightarrow$$

$$x'(t) + \frac{(G_x + G_y)}{4\mu} \cdot x^2 = \frac{\left[G_y \cdot (x_0 + y_0) + (C_0 - \tau_s) \right]}{2\mu} x + C_t \,,$$

- the last equation could be classified as *Riccati's type* [4], where C_t = const.

References:

1. Klimov D.M., Petrov A.G. Analytical solutions of the boundary- value problem of non stationary flow of viscous-plastic medium between two plates// Archive of Applied Mechanics. 2000, Vol. 70. P. 3-16.

   See also, in Russian (*branch 2.2.2.*): http://www.ipmnet.ru/~petrov/files/vpdchp.pdf.

2. Weisstein, Eric W., "Harmonic Function" from MathWorld.

   See also: http://en.wikipedia.org/wiki/Harmonic_function.

3. Ershkov S.V., Schennikov V.V. Self-Similar Solutions to the Complete System of Navier-Stokes Equations for Axially Symmetric Swirling Viscous Compressible Gas Flow // *Comput. Math. and Math. Phys. J.* (2001) Vol.**41**, № 7. P.1117-1124.

4. Dr. E.Kamke. Hand-book for ordinary differential equations // Moscow: “Science” (1971).

5. Arnold V.I. Catastrophe Theory, 3rd ed. Berlin: Springer-Verlag, 1992.