# Is Zero to the Zero Power Equal to One?

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Abstract

Sometimes in physics we end up with a function that resembles $f(x)=0^0$, where for example we have a radius that goes to zero and an exponent goes to zero in $k/r^n$, where k is a constant. Is $0^0$ in such cases equal to unity?

It is tempting to say that, yes, zero to the zero power is equal to one, but alas it is not. The function $y=x^x=0^0$ in fact goes to infinity, and, therefore, $k/r^n$ in our example goes to zero value.

If we plot the function $y=x^x$ on our calculator for positive values of $x$ approaching zero, we will find that $y$ equals nines followed by a decimal point. The rule of thumb is that there are as many nines in the value of $y$ as there are decimal places in the value of $x$. For example, if $x=0.01$, we note that there are two decimal places in the value of $x$, which means the value of $y$ will be at least 0.99. A further example is if $x=0.000001$, $y$ will be at least 0.999999. We see from the graphing calculator that $y$ appears to approach unity as $x$ approaches zero.

The student will tend to say that $y$ is one at $x=0$, but that is a grave mistake. We show that the curve is in fact vertical at $x=0$.

We write the function

$$y = x^x \tag{1}$$

and we take the natural log of both sides,

$$\log_e y = \log_e x^x = x \log_e x \tag{2}$$

We now place (2) in differential form,

$$\frac{dy}{y} = dx + dx \log_e x \tag{3}$$

whose derivative then becomes

$$\frac{dy}{dx} = x^x \left[1 + \log_e x\right] = 0 \tag{4}$$

This provides

$$\log_e x = -1 \tag{5}$$

or

$$x = e^{-1} = \frac{1}{e} \tag{6}$$

The value of $y$ is then

$$y = \left(\frac{1}{e}\right)^{\frac{1}{e}} = \frac{1}{e^{\frac{1}{e}}} = 0.6922... \tag{7}$$

which indicates the relative minimum on the curve.

As $x$ approaches zero through positive values, the slope of the curve in the limit is

$$\frac{dy}{dx} = 1[1 - \infty] = -\infty \tag{8}$$

since the natural log of zero is a negative infinity. It follows that the curve approaches the point from below, but is vertical at $x=0$. Thus the function becomes discontinuous at $x=0$.

For the student who wishes to probe what happens when $x$ is negative, we can multiply by negative unity where $x$ itself still remains positive. Then we write

$$y = (-x)^{-x} = \frac{1}{(-x)^x} \tag{9}$$

From the student's knowledge that

$$-1 = e^{i\pi} = \cos \pi + i \sin \pi \quad \text{and} \quad 1 = e^{-i\pi} = \cos \pi - i \sin \pi \tag{10}$$

we can write (9) as

$$y = \frac{1}{\left(xe^{i\pi}\right)^x} = \frac{1}{x^x e^{i\pi x}} = \frac{e^{-i\pi x}}{x^x} = \frac{\cos \pi x - i \sin \pi x}{x^x} \tag{11}$$

This is a complex number except for the condition $\sin \pi x = 0$, $x$=0, 1, 2, 3, 4, ….∞. Since the cosine of zero is +1, we again have a point of discontinuity at $x$=0.  The curve remains discontinuous until the point $x$=-1 is reached, so that

$$y = \frac{-1}{1} = -1 \tag{12}$$

For negative real values of $x$ beyond this point, $y$ values alternate positive and negative, but are reduced in absolute value. The next integer value is

$$y = \left(-2\right)^{-2} = \frac{1}{4} \tag{13}$$

This is followed by

$$y = \left(-3\right)^{-3} = -\frac{1}{27} \tag{14}$$

What is particularly interesting about this curve in the negative-$x$ region is that there are no real zeros of the function in spite of the fact that $y$ values alternate in sign.

It is interesting to observe that a continuous curve can be plotted in the event that we define an axis of imaginaries at right angles to the $x,y$ plane. In this case a reducing spiral is generated around the negative-$x$ axis. The radius of the spiral at any point is

$$r = \frac{1}{x^x} \to 0 \text{ as } x \to -\infty \tag{15}$$

by the use of the absolute values of equation (11). The spiral penetrates the analytic plane at the points specified by the sequence (12), (13), (14),…. We find that the spiral rotates clockwise as we look in the negative-$x$ direction. It reaches a maximum radius of

$$r = e^{\frac{1}{e}} \text{ at } x = -\frac{1}{e} \tag{16}$$

and an angle of

$$\theta = \frac{\pi}{e} = \frac{180^{\circ}}{2.7182} = 66.22^{\circ} \tag{17}$$

measured clockwise as indicated. The radius is again unity at the point $x=-1$, and thereafter reduces as it spirals around the negative-$x$ axis.

In approaching the point $x=-1$, the slope of the curve is positive from the origin, which we prove here by using earlier arguments in the first part of this paper:

$$y = \frac{1}{x^x} = x^{-x} \tag{18}$$

$$\log_e y = -x \log_e x \tag{19}$$

$$\frac{dy}{y} = -dx\left[1 + \log_e x\right] \tag{20}$$

$$\frac{dy}{dx} = -\frac{1}{x^x}\left[1 + \log_e x\right] \tag{21}$$

This gives

$$\frac{dy}{dx} = -1\left[1 - \infty\right] = +\infty \tag{22}$$

Another interesting aspect of this analysis is that the $y$-axis serves as the tangent to the curve.

Thus as a result of this exercise, the student learns that although the graphing calculator indicates the curve will cross the $y$-axis at $x=0$, we find when we do the math that the curve in fact never crosses the $y$-axis at all. This is an important exercise for students so they can learn to be wary of appearances.