PATTERNS RELATED TO THE SMARANDACHE CIRCULAR SEQUENCE PRIMALITY PROBLEM

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Abstract In this paper, we show the internal relations among the elements of the circular sequence (1,12,21,123,231,312,1234,2341,…). We illustrate one method to minimize the number of the “candidate prime numbers” up to a given term of the sequence. So, having chosen a particular prime divisor, it is possible to analyze only a fixed number of the smallest terms belonging to a given range, thus providing the distribution of that prime factor in a larger set of elements. Finally, we combine these results with another one, also expanding the study to a few new integer sequences related to the circular one.

Keywords Recurrence relations, factorization, patterns, integer sequences, permutations, primes.

§1. Candidate primes in the canonical circular sequence

We introduce a few notations to indicate (synthetically) some special groups of terms of the sequence **A001292** of Sloane's *On-line Encyclopedia* [8]. First of all, we have to explain what the circular sequence is [9-12].

1, **12**,21, **123**,231,312, **1234**,2341,3412,4123, **12345**,23451,34512,45123,51234, **123456**,234561,345612,456123,561234,612345,
M1 M2 M3 M4 M5 M6

1234567,2345671,3456712,4567123,5671234,6712345,7123456, **12345678**,23456781,34567812,45678123,56781234,67812345,
M7 M8

78123456,81234567, **123456789**,234567891,345678912,456789123,567891234,678912345,789123456,891234567,912345678,
M9

12345678910,23456789101,34567891012,45678910123,56789101234,67891012345,78910123456,89101234567,91012345678,
M10

10123456789, **1234567891011**,2345678910111,3456789101112,4567891011123,5678910111234,6789101112345,7891011123456,...
M11

The first terms of the Smarandache circular sequence and a few of the consecutive one [7].

Definition 1.1. Given the n-th term of the Smarandache consecutive sequence **A007908**, constructed through the juxtaposition of the first *n* integers, we define as "circular permutations" the *n* elements that constitute the subset of the permutations of the form:

a_{j-i+1}:=2_3_4_..._(n-1)_n_1 $\Rightarrow p_1=1$,
a_{j-i+2}:= 3_4_..._(n-1)_n_1_2 $\Rightarrow p_2=2$,
…
a_j:= (i+1)_(i+2)_..._(n-1)_n_1_2_..._(i-1)_i $\Rightarrow p_i=i$,
…
a_{j+n-i}:=1_2_3_..._(n-1)_n $\Rightarrow p_n=n$.
Now we have to clarify the two parameters necessary to isolate a given class of elements from the rest [7].

Definition 1.2. Let r:=n be the n-th term of the consecutive sequence (1_2_3_..._(n-1)_n) and let M(r) denote the whole set of the circular permutations of that element (Definition 1.1). Given *r*, we single out one particular element of the circular sequence simply by assigning a value to the parameter *p*.

For example, (r=n, p=i) denote a_j:=(i+1)_(i+2)_..._n_1_2_..._(i-1)_i ∈ M(n). Obviously $\overline{123\ldots(n-1)n}$, $n\geq 1$ [5], the generic element of the consecutive sequence, is characterized by $p\equiv r=n$.

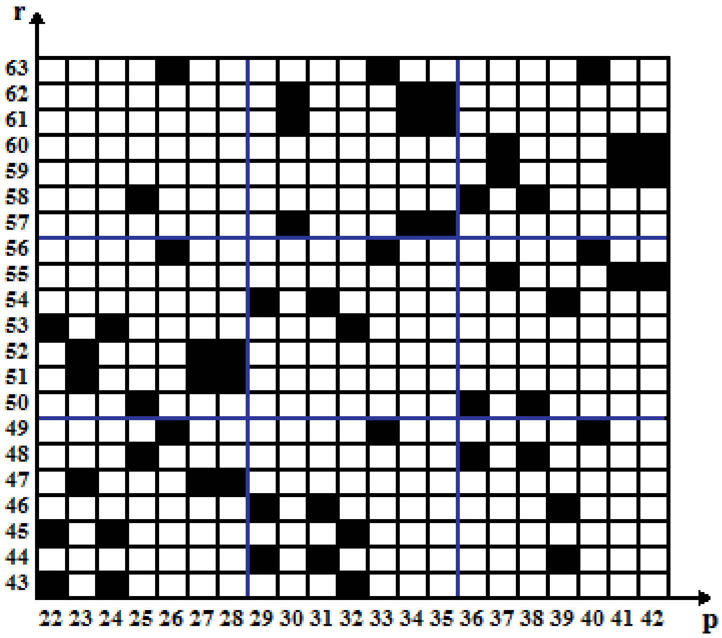
We know that only $13.\overline{3}\%$ of the terms are not divisible by 2, 3 or 5 (numbers that are relatively prime to 30) and Ripà [7] has spelt out their form.
It is possible to show that the terms divisible by a lot of larger prime factors outline well-defined patterns inside the sequence, and it is quite simple to detect their size.

Theorem 1.3. Given a specific prime factor $7\leq pr<\sqrt{a_j}$, where a_j:=(r=n, p=i), within the whole set of numbers characterized by *p* and *r* formed by a fixed amount of digits, if a_j | pr, $a_{j'}$:=(r=n+q, p=i+k) | pr. In particular, if the length in digits of *r* and *p* is the same, q=k.

Theorem 1.4. Given a_j and defining as "d(p)" the length in digits of *p*, if d(p) is (strictly) less than d(r) - the digit length of *r* -, the natural number *q* linked to each $\tilde{r}$ such that $10^{d(p)-1} \leq \tilde{r} \leq 10^{d(p)}-1$ is equal to *k* [1].

Corollary 1.5. Let d(r) and d(p) be fixed. $\forall pr < 9 * 10^{d(r)-1}$, *q* and *k* (if these exist) are multiples of pr ($q:=pr*t$ and $k:=pr*s$, where $t,s \in \aleph \setminus \{0\}$) [2].

Using the results above, we can try to identify several patterns for given values of d(p) and d(r).
For example, choosing pr=7 and d(r)=d(p)=2, we have



In this case, d(r)=d(p) $\Rightarrow$ k=q=7*m, particularly m=3.
The grid shows that each column and each row contains three black squares, but this is not a general law. It is possible to deduce that the total number of the terms of the sequence, divisible by a given "pr" related to a pattern that recurs more than once in the chosen d(r)=constant interval, is at least $\frac{1}{pr}$. Under the previous restriction on pr, the non-strict inequality above includes all the possible combinations of d(r) and d(p). So we could exploit this relation to synthesize a probabilistic formula that tries to estimate the ratio of "candidate prime terms" (the elements that are not divisible by the prime factors taken into account in the pattern analysis) in comparison with the numerousness of the elements belonging to the d(r)=constant subset of the circular sequence:

$$0.1\bar{3} - 0.1\bar{3} * \frac{1}{7} - 0.1\bar{3} * \frac{6}{7} * \frac{1}{11} - 0.1\bar{3} * \frac{6}{7} * \frac{10}{11} * \frac{1}{13} - \cdots = \mathbf{0.1\bar{3}} * \left[\frac{6}{7} - \sum_{i=5}^{h} \frac{1}{pr_i} * \prod_{j=4}^{i-1} \left(1 - \frac{1}{pr_j} \right) \right] \quad (1)$$

[1] **Clearly, d(r)∈[1, +∞) and d(p)∈[1, d(r)].**

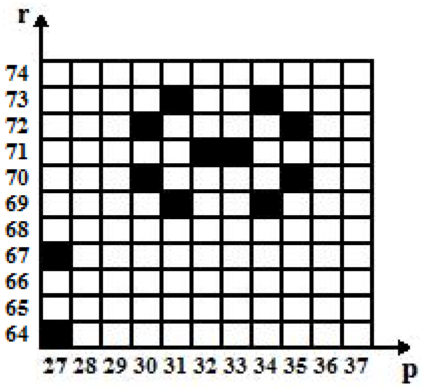
[2] **Since (by definition) d(r)≥d(p), it is possible that Corollary 1.5 is valid only for q (while there is not such a k related to the given pr). This occurrence implies d(r)>d(p), because we are able to identify k (empirically) only if it is less than $9 * 10^{d(p)-1}$.**

The index h refers to the h-th element of the prime number sequence **A000040** pr_j:=2,3,5,7,11,13,17…, for j=1,2,3,4,5,6,7,…, so the first term of the summation above is pr_5:=11, whereas the products start from pr_4:=7.

Specifically referring to the patterns' repetitiveness, the concrete constraint is that pr_h is forced to be less than $9 * 10^{d(r)-1}$ [3].

Moreover, excluding from the previous formula the pr_j for which the corresponding pattern does not repeat at least once, for the given value of d(r), we get a probabilistic overestimation of the candidate primes ratio [4] (in such a case, we have to consider only the multiplication factors linked to the primes that could be used in the pattern analysis, while the remaining subset of **A000040** must be deleted from the estimation formula (1)).

To show an example of a pattern that does not respect the previous prediction, we can take pr=11 and d(r)=d(p)=2:



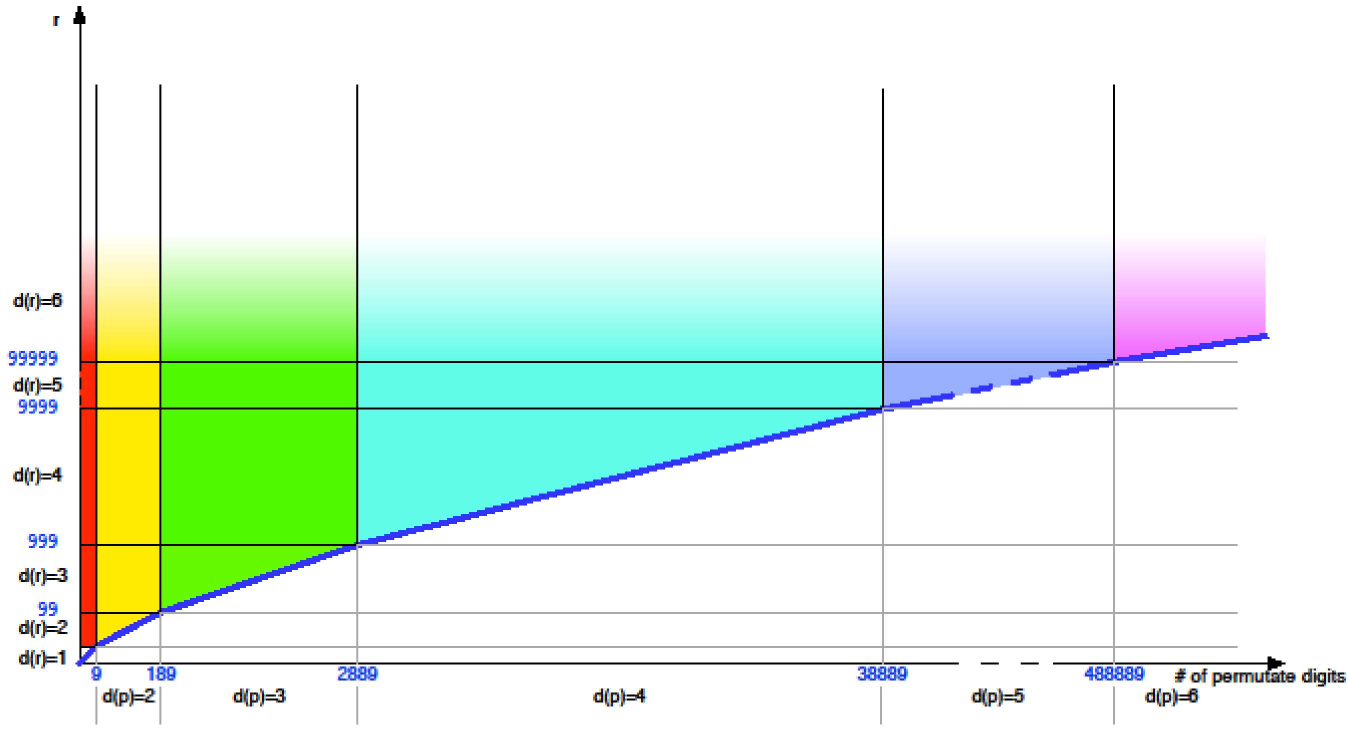
We could also think of the implicit result [7] related to pr=37 and r=3. In that case m=1, but there are 37*2 terms divisible by 37 for each 37x37 grid of elements. In fact, there are two different values of 100≤r<137 such that, for each possible value of p≤r, the factorization of a_j:=(p+1)_…_(r-1)_r_1_2_3_…_(p-1)_p includes the prime factor 37.

Section 4 contains a random collection of other patterns of the same kind, for pr∈{7,11,13,37} and d(r)≤4.

An obvious consequence is that, once we have detected the value of q:=q(pr, d(r), d(p)) for a given (pr*, d(r≤r*)), we automatically know what is the value of k:=k(pr*, d(r), d(p)) for the whole set of terms identified by d(r)≤d(r)* (reminder: d(p)≤d(r), ∀pr). The final result is that, under the previous conditions we have stated, k is implicit in q. Another way to formalize the same result is to postulate that, for a given pair (pr, d(p)), the value of k is constant at the variation of the parameter r. This is graphically shown in the figure below:

[3] **In Section 2, this result will be combined with another one to further reduce the candidate prime numbers set.**

[4] **Due to the inequality stated a few lines before.**



In this graph, sectors of the same color represents sets of elements characterized by an identical value of k.

There is another outcome (quite easy to demonstrate). Let pr=pr* (i.e., any prime number). Taking two generic terms of the consecutive sequence, say a':=(p=n', r=n')∈M(n') and a":=(p=n", r=n")∈M(n") – where n'>n" -, which are both divisible by pr*, the following element of the circular sequence is also divisible by pr*:

a''':=(n"+1)_(n"+2)_…_(n'-1)_n'_1_2_3_…_(n"-1)_n"∈M(n'). In fact, the number (n"+1)_…_(n'-1)_n' is divisible by pr* as well.

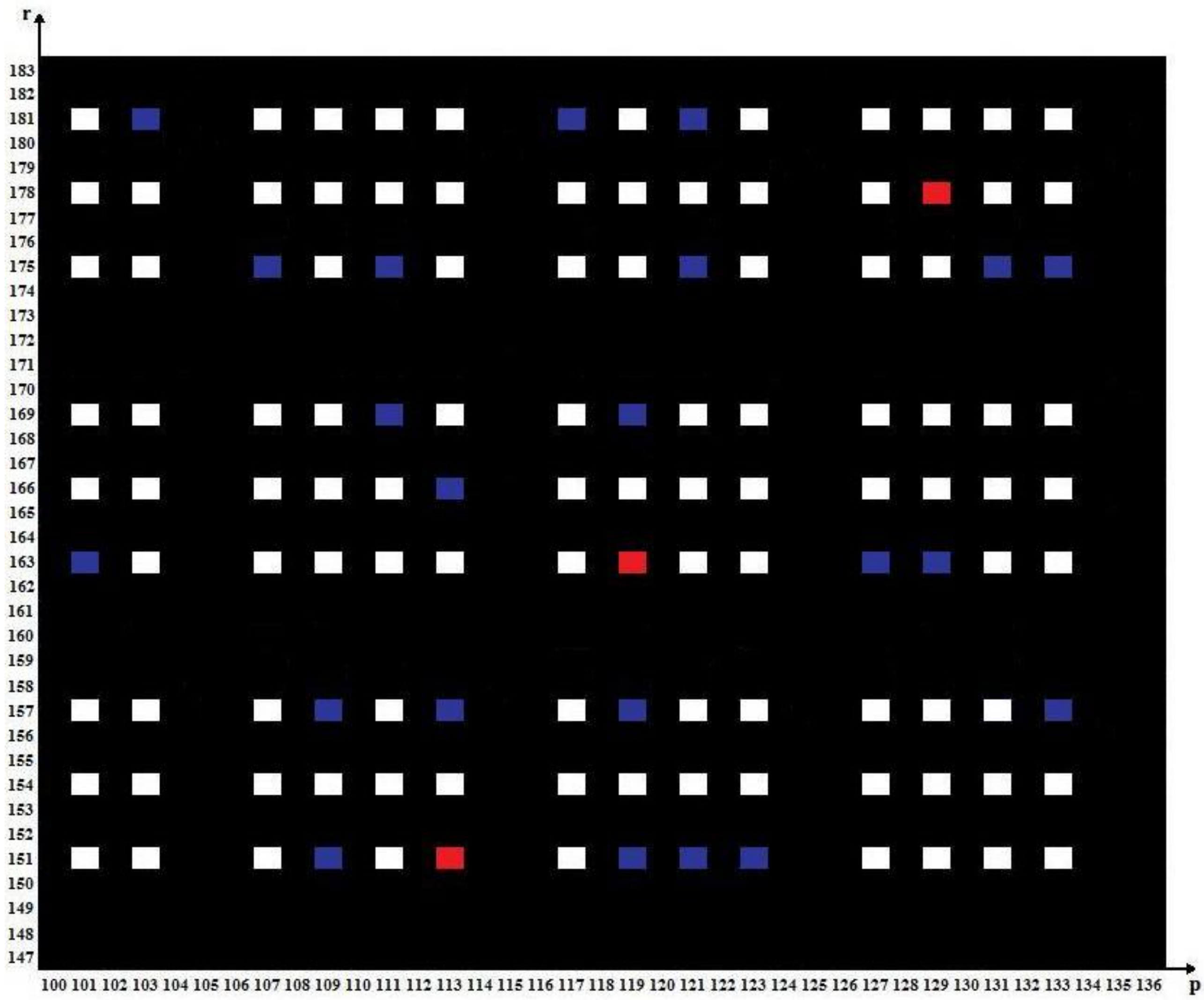
Clearly we can be sure that pr* divides the terms

(n"+1)_(n"+2)_…_(n'-1)_n'_…_(n'+q-1)_(n'+q)_1_2_3_…_(n"-1)_n"∈M(n'+q),

(n"-k+1)_…_(n'-1)_n'_1_2_3_…_(n"-k-1)_(n-k)"∈M(n') and

(n"-k+1)_…_n'_(n'+1)_…_(n'+q)_1_2_3_…_(n"-k-1)_(n"-k)∈M(n'+q) iff d(p) and d(r) will remain unchanged. In this way, knowing the distribution of a small prime factor inside a portion of the consecutive sequence, we are able to exclude another subset of non prime elements from the circular one. These data allow us to rule out some terms of the circular sequence characterized by d(p)=d(r), because the known period of the consecutive sequence is *q* and, in those sectors, q=k (Theorem 1.3).

To illustrate a practical application, we could adopt the same prime factors analyzed by Ripà [7] and study one particular square grid with 37x37 terms characterized by d(r)=d(p)=3. The pattern is as follows:



The white squares plus the red ones represent the candidate prime numbers (the red squares are effectively prime numbers), the black squares describe terms that are divisible by (at least) one prime factor belonging to {2,3,5,37}, whereas the blue items represent the elements of the circular sequence divisible by 7, 11 or 13. In this case, the candidate primes are 101 (with 3 prime numbers **A181073**) in a total of 1369 terms.

In a given (d(r),d(p)) sector, overlapping the pattern of the terms divisible by pr_{b+1} (for b≤h-1) on the pattern of the elements divisible by $pr_{(j\leq b)}$, we are able to reduce the total of the candidate prime numbers to verify via a primality test [2], with the advantage of studying only a few of the "shortest" terms of the sequence in our (d(r),d(p)) quadrant, for some $7\leq pr_j < 9 * 10^{d(r)-1}$.

§2. The finite sequence of the circular digit permutations and others

In this section we illustrate another important rule linked to the patterns of the circular sequence divisibility and then we present a few new sequences constructed starting from the result mentioned above.

Definition 2.1. Let $f_1\_f_2\_f_3\_\ldots\_f_{(n-1)}\_f_n$ be the n-tuple of consecutive digits resulting from the concatenation of the first *n* terms of the sequence **A007376**, where $f_1:=1$ and $f_2:=f_1+1$. We define as "circular permutations of the digits" the *n* elements that compound the subset of the permutations of the form:

$\bar{a}_{l+1}:=f_2\_f_3\_f_4\_\ldots\_f_{(n-1)}\_f_n\_f_1\equiv 2\_3\_4\_\ldots\_f_{(n-1)}\_f_n\_1$,

$\bar{a}_{l+2}:=f_3\_f_4\_\ldots\_f_{(n-1)}\_f_n\_f_1\_f_2\equiv 3\_4\_\ldots\_f_{(n-1)}\_f_n\_1\_2$,

…

$\bar{a}_{l+i}:=f_{(i+1)}\_f_{(i+2)}\_\ldots\_f_n\_f_1\_f_2\_\ldots\_f_{i-1}\_f_i\equiv f_{(i+1)}\_f_{(i+2)}\_\ldots\_f_n\_1\_2\_\ldots\_f_{i-1}\_f_i$,

…

$\bar{a}_{l+n}:=f_1\_f_2\_f_3\_\ldots\_f_{(n-1)}\_f_n\equiv 1\_2\_3\_\ldots\_f_{(n-1)}\_f_n$.

Theorem 2.2. For some pairs (j,r), $\exists$ M(r) | pr_j. For these (j,r), all the circular permutations of the digits of M(r) are also divisible by pr_j.

Let d(r)=constant, by Theorem 1.3, M(r) | pr_j $\Rightarrow$ M(r+q) | pr_j, but there exist others $pr>9*10^{d(r)-1}$ which are in possession of the same property. Besides, from Theorem 2.2, it follows that all the possible circular permutations of the digits of M(r+q) are also divisible by pr_j.

On the next pages, we analyze the sequence by Definition 2.1, related to every circular permutation of the digits of the canonical consecutive sequence **A007908**. Hence, we limit the number of the terms of the sequence fixing d(r) at 3, so r∈[100, 999]. Unbundling a chunk of the extended circular digit permutations sequence, there are a lot more new elements than in the canonical circular sequence. Thus:

$\bar{a}_1:=12345\ldots9899100 \in M(100)$,

$\bar{a}_2:=23456\ldots98991001 \in M(100)$,

…

$\bar{a}_{11}:=0111213\ldots98991001234567891\equiv111213\ldots98991001234567891 \in M(100)$,

…

$\bar{a}_{1386450}:=91234\ldots99799899 \in M(999)$.

The total of the elements of such a sequence is 192+195+198+…+2886+2889, that is

$$\sum_{m=0}^{899}(192+m*3)=192*900+3*\frac{899*900}{2}=900*(192+1.5*899)=\mathbf{1386450} \quad (2)$$

This represents only the empirical evidence of the properties we have already seen in the first section: there are several pr_j (**A180346**) that divide the whole set of the digit permutations of $M(\tilde{r})$, for $\tilde{r}\in[100, 999]$. Let $4\leq j\leq169$ (which implies $7\leq pr\leq1009$), the $\tilde{r}$ values are the following (and under our assumption, they belong to the sequence **A181373** of the OEIS [8]):

$M(\tilde{r}) \mid 7 \Rightarrow \tilde{r}=100+14*v$ (v=0,1,2,…,64)

$M(\tilde{r}) \mid 11 \Rightarrow \tilde{r}=106+22*v$ (v=0,1,2,…,40)

$M(\tilde{r}) \mid 13 \Rightarrow \tilde{r}=120+26*v$ (v=0,1,2,…,33)

$M(\tilde{r}) \mid 17 \Rightarrow \tilde{r}=196+272*v$ (v=0,1,2)

$M(\tilde{r}) \mid 19 \Rightarrow \tilde{r}=102+114*v$ (v=0,1,2,3,4,5,6,7)

$M(\tilde{r}) \mid 23 \Rightarrow \tilde{r}=542$

$M(\tilde{r}) \mid 29 \Rightarrow \tilde{r}=400$

$M(\tilde{r}) \mid 31 \Rightarrow \tilde{r}=181+155*v$ (v=0,1,2,3,4,5)

$M(\tilde{r}) \mid 37 \Rightarrow \tilde{r}=123+\sum_s d_s$, (where d_s=0,12,25,12,25,12,25,… for s=0,1,2,3,…,47)

$M(\tilde{r}) \mid 41 \Rightarrow \tilde{r}=216+205*v$ (v=0,1,2,3)

$M(\tilde{r}) \mid 43 \Rightarrow \tilde{r}=372+301*v$ (v=0,1,2)

$M(\tilde{r}) \mid 53 \Rightarrow \tilde{r}=127+689*v$ (v=0,1)

$M(\tilde{r}) \mid 61 \Rightarrow \tilde{r}=616$

$M(\tilde{r}) \mid 67 \Rightarrow \tilde{r}=399$

$M(\tilde{r}) \mid 73 \Rightarrow \tilde{r}=196+584*v$ (v=0,1)

$M(\tilde{r}) \mid 83 \Rightarrow \tilde{r}=118$

$M(\tilde{r}) \mid 97 \Rightarrow \tilde{r}=516$

$M(\tilde{r}) \mid 101 \Rightarrow \tilde{r}=416+404*v$ (v=0,1)

$M(\tilde{r}) \mid 107 \Rightarrow \tilde{r}=884$

$M(\tilde{r}) \mid 127 \Rightarrow \tilde{r}=106$

$M(\tilde{r}) \mid 163 \Rightarrow \tilde{r}=576$

$M(\tilde{r}) \mid 211 \Rightarrow \tilde{r}=306$

$M(\tilde{r}) \mid 271 \Rightarrow \tilde{r}=936$

$M(\tilde{r}) \mid 277 \Rightarrow \tilde{r}=174$

$M(\tilde{r}) \mid 1009 \Rightarrow \tilde{r}=960$

*Using these relations, we are able to reduce the percentage of candidate prime terms of the chart at the end of Section 1, since M(181) | 31 and the numerousness of candidate primes among our 1369 elements crashes at **90** (with a ratio of 0.06574).*

Lemma 2.3. In the interval we have set, the "fixed factors" from Theorem 2.2 recur with regularity when $\tilde{r}$ grows: they are periodical and the period is a multiple of pr_j itself (as stated in Corollary 1.5).

The inclusion of pr=1009 in the list, which is above the upper limit of $\tilde{r}$, represents the proof of the existence of primes that divide all the possible circular digit permutations of a given M(r), for which the method presented in the first section cannot be applied. It is possible to combine this property with the method

previously shown. The synergy of these two techniques would lead us to a very small percentage of candidate prime terms to test in the direct way [5].
We should initially know for which pairs (pr, r) there is a full match. For this purpose, it is sufficient to apply Pascal's method [3-4] to derive the divisibility criteria of the prime factors we want to consider.
Under the restriction on *r* made at the beginning, we can find a lot of values of pr>$9 * 10^{d(r)-1}$=900 characterized by the applicability of Lemma 2.3. For example, we can take pr_{2157}:=18973 and r=903.

Specifically referring to the fixed pr analysis (derived from Lemma 2.3), and considering the finite interval we are studying, we are able to synthesize the previous conditions (for all the given $pr_{j\leq 169}$ – implying the constraint a_i(mod 10):≡{1,3,7,9} -) in the following relations:

$$\begin{cases} \tilde{r} = 100 + 3 * v \\ \tilde{r} \neq 100 + 14 * v \\ \tilde{r} \neq 106 + 22 * v \\ \tilde{r} \neq 120 + 26 * v \\ \tilde{r} \neq 123 + 37 * v \\ \tilde{r} \neq 135 + 37 * v \\ \tilde{r} \neq \{118,127,181,196,400,421,673,820\} \end{cases}$$

So, there are only 233 M($\tilde{r}$) in this finite sequence, that do not have (at least) a pr<1009 that divides all the elements of the set r=costant, and these are

103,109,112,115,121,124,130,133,139,145,151,154,157,163,166,169,175,178,187,190,193,199,202,205,208,
211,214,217,220,223,229,232,235,241,244,247,256,259,262,265,274,277,280,283,286,289,292,295,298,301,
307,313,316,319,322,325,331,334,337,340,343,346,349,355,358,361,364,367,373,376,379,385,388,391,397,
403,409,412,415,418,424,427,430,433,439,442,445,448,451,454,457,460,463,466,469,472,475,481,487,490,
496,499,508,511,514,517,523,526,529,532,535,538,541,544,547,550,553,556,559,565,571,574,577,580,583,
586,589,592,595,598,601,607,610,613,619,622,625,628,631,637,643,649,652,655,658,661,664,667,670,676,
679,682,685,694,697,703,706,709,712,721,724,733,736,739,742,745,748,751,754,757,760,763,769,775,778,
781,784,787,790,793,799,802,805,808,811,817,823,829,835,841,844,847,850,853,859,862,865,868,871,877,
880,883,886,889,892,895,901,904,907,910,913,916,919,922,925,928,931,934,943,946,955,958,961,967,970,
973,976,979,985,988,991,994,997.

Considering the canonical circular sequence bounded by d(r)=3, without using the pattern analysis on the smallest pr_j, we are able to exclude most of the elements of the sequence, limiting the candidate prime terms subset to under 11.1% of the total sum of the elements. In fact, we have: $\frac{56039-1176}{139950-2936} \approx 0.4004189$ and $\frac{137014}{494550} \approx 0.2770478$. This implies a percentage of candidate prime terms equal to 0.4004189*0.2770478*100=**11.0935**%.

In the case of the extended circular sequence to 1386450 terms, the percentage is even lower than in the previous one (well below 11%) [6].

At this point, we could modify our finite sequence a little, without altering its representative properties. We have to arrange, in ascending order, the 1386450 elements of (2). Cutting out the details, it is clear that, for a

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[5] **We are clearly referring to primality tests.**

[6] **We have taken into account the constraint for the candidate primes that implies the congruence in (mod 10) to {1,3,7,9}.**

given r, the first terms of the corresponding sequence subset start with two zeros. In fact, they are composed by 2 digits fewer than most of the other “room-mates” (in short, for a given value of the parameter r).

At the end of this operation, we know that the result does not vary, because the order of the subsets r_i remains unchanged, due to the maximum spread (between one pair of the elements of this set) of 2 digits, since each unitary increase of r adds 3 new digits to the “standard term” linked to the preceding value of r. It is also superfluous to specify that the numerousness of the candidate primes does not change appreciably, as well as the considerations we have already made about the “fixed prime factors”.

Remember that the criterion at the bottom of the previous relations is general: for example, taking $r \in [1000, 9999]$ it will only vary the linear rules that describe the periodicity of the fixed factors, but the concept will survive the new d(r). The only constraint to adhere to is that d(r) must be constant.

Now we could once again extend the sequence of the circular digits permutations mentioned before, defining another sequence with the same features as the previous one, but in which, after each set of permutations, only one new digit is added. Starting from the infinite extended consecutive sequence 1,12,123,…1234556789,1234567891,12345678910,123456789101,… we consider the set of all the circular permutations of each one of these terms, as we have already done starting from the canonical circular sequence. We could also repeat the same process for a few other Smarandache sequences (e.g., the reverse one **A000422** [1-10]). Referring to the expanded circular digit permutations sequence, we might transpose the same questions introduced by Ripà [7]: the results are pretty interesting. Without taking away the fun of the discovery, choosing as example the extended consecutive sequence that we have just described, we can only say that it contains some prime terms [11] (it does not even require deep analysis). In fact, the shortest primes are **A176942**

Sm’_10:=1234567891,

Sm’_14:=12345678910111,

Sm’_24:=123456789101112131415161,

Sm’_235:=12345…1101111121131141,

Sm’_2804:=12345…96696796896997097,

Sm’_4347:=12345…1359136013611362136313,

Etc. *(Note that Sm’_n is exactly n digits long).*

Referring to the circular sequence extended to include all the circular permutations of its digits, it is clear that the outcome stated in the first section still remains valid considering that, ∀n’>n”, if $1\_2\_3\_\ldots\_f_{(n'-1)}\_f_{n'}$ is divisible by pr_j and that $1\_2\_3\_\ldots\_f_{(n''-1)}\_f_{n''}$ is also divisible by the same prime factor, then $f_{(n''+1)}\_\ldots\_f_{(n'-1)}\_f_{n'}\_1\_2\_3\_\ldots\_ f_{(n''-1)}\_f_{n''}$ results in being divisible by pr_j (and obviously pr_j is a prime factor of $f_{(n''+1)}\_\ldots\_f_{(n'-1)}\_f_{n'}$ as well). This is a general property of each number, not only for those belonging to the extended consecutive sequence set: taking T’ (a random sequence composed of t’ figures that is divisible by pr_j) and T” (another t” digit long sequence that is divisible by the same pr_j), T’_T” and T”_T’ are both divisible by pr_j.

If we take a look at the sequence constructed from the circular permutations of the digits of the extended consecutive sequence (1,12,123,…,123456789,1234567891,12345678910,1234456789101,…), we can easily find a lot of sets of circular arrangements that are entirely divisible by the same pr_j. So, 1_2_3_…_1005_1006_1007_1 plus the remaining 2921 circular permutations of its digits are divisible by pr_4=7 and pr_5=11, whereas every circular permutation of the digits of 1_2_3_…_36_37_3 is divisible by 7, 11 and 13. Section 3 contains further comments on the topic.

§3. Concluding observations

We could study what happens if we try to expand the set, giving the usual interval d(r)=3, of the circular digit permutations of the 900 terms belonging to the canonical consecutive sequence to the circular digit permutations of the 2698 elements that characterize the extended consecutive one (constructed - as already seen - adding only one figure of the canonical sequence at the end of the previous element).

Thus, $\bar{\bar{a}}_1$:=1234…9899100, $\bar{\bar{a}}_2$:=234…98991001, $\bar{\bar{a}}_{11}$:=0111213…98991001234567891, $\bar{\bar{a}}_{4153381}$:=1234…997998999 and $\bar{\bar{a}}_{4156269}$:=91234…99799899: only approximately one third of the terms belonging to this sequence are part of the sequence formed by the circular permutations of the digits of the canonical consecutive one, even if Lemma 2.3 is satisfied as well. In this case, we will find new fixed factors not included in the "short version" of the sequence, e.g., pr_{17}:=59 (resulting from the circular digit permutations of 1234…1121131), mixed with other prime factors that we have already met, as pr_5=11, which is a fixed factor for 1234…10710810.

Nevertheless, one problem arises: we cannot arrange the terms in ascending order or we will lose the validity of Lemma 2.3. Moreover some $\bar{\bar{a}}_j$ are coincident and their position inside the sequence could not be univocally identified, e.g., $\bar{\bar{a}}_{101470}$:=1234…1981992 is equal to $\bar{\bar{a}}_{102450}$:=01234…1981992 and $\bar{\bar{a}}_{102941}$:=001234…1981992.

The prime factors ≥7 that divide all the terms of the general (unlimited) sequence, constructed from the circular permutations of the digits of the extended consecutive one, exist starting from 16 digit long numbers (all the circular digits permutations of 1234567891011121 are divisible by 17) without an upper limit. This is testified by the example we have already introduced: each one of the 2922 circular permutations of the string 1234…1005100610071 is divisible by pr=11 and we can say the same for the circular permutations of the elements formed by 2922+44*v digits, for v=0,1,2,3,…,817. In this case, the period of pr=11 is identical to the one linked to the sequence derived by an integer value of r∈[1000, 9999], for d(p)=d(r)=4 and pr=11.

The consideration above, leads us to the following observation involving all the subsequences constructed, as previously described, from the extended consecutive sequence.

Starting from the sequence that includes every circular permutation of the extended consecutive one (1,12,123,…,123456789,1234567891,12345678910,1234456789101,…), given the value of d(r) plus the initial $\bar{\bar{a}}_j$, we obtain exactly "d(r)" interrupted subsequences (one of which – the canonical circular sequence - is complete). For these subsequences, most of the properties already explained in Section 1, plus Theorem 2.2, still remain valid.

In conclusion, we might even give free rein to the imagination and transpose some of the questions [6-9] (totally or partially) answered by Ripà [7] to the new sequences designed "ex-novo" starting from the considerations we have made in these pages: thus we can define a new sequence obtained taking into account only the odd figures of the circular one (considering all the variations we have seen so far) plus a lot of others on the same line.

§4. Several patterns belonging to the Smarandache circular sequence

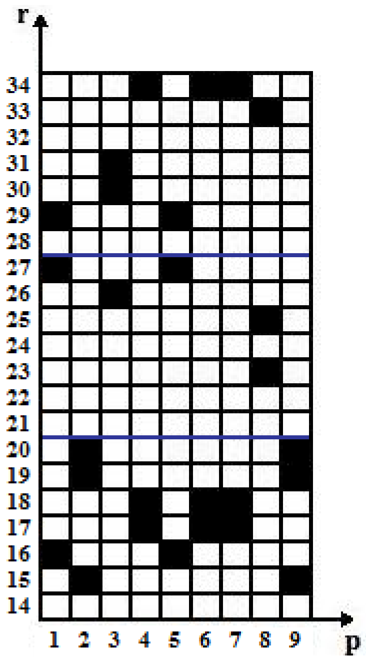
This is a small collection of the patterns of some pr_j that divides the corresponding terms of the canonical circular sequence (1,12,21,123,231,312,...).

In the following charts, each dark square represents an element of the sequence that is divisible by the given pr – remember that the generic term can be written as (p+1)_(p+2)_..._(r-1)_r_1_2_3_..._(p-1)_p -.

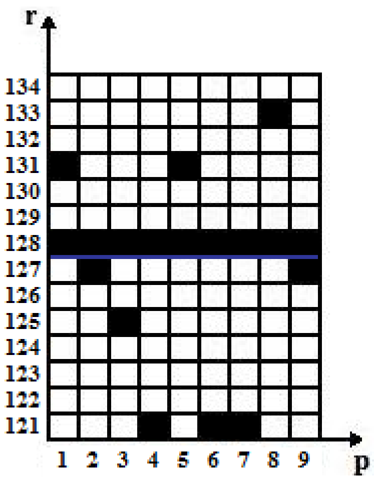
Often these patterns show some intrinsic regularity, that depends, for the most part, on the chosen pr_j (a clear example is represented by pr_6:=13).

Pr_4:=7

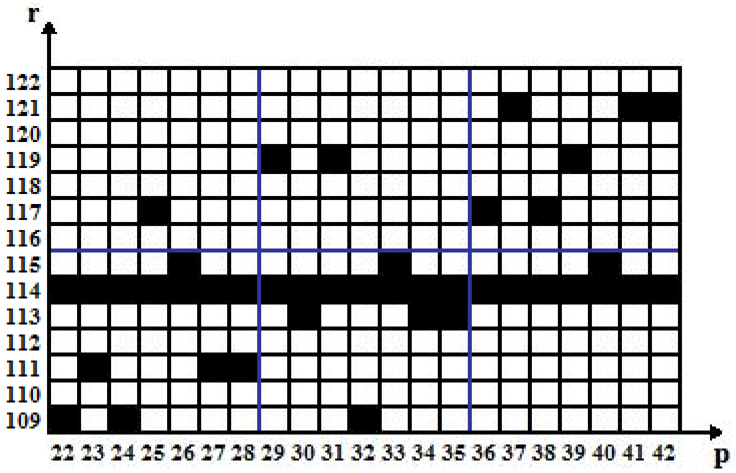
d(r)=2, d(p)=1



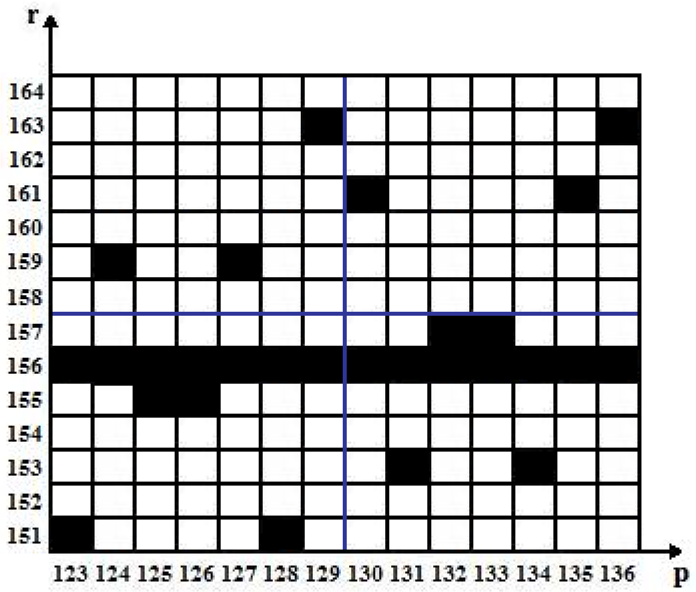
d(r)=3, d(p)=1



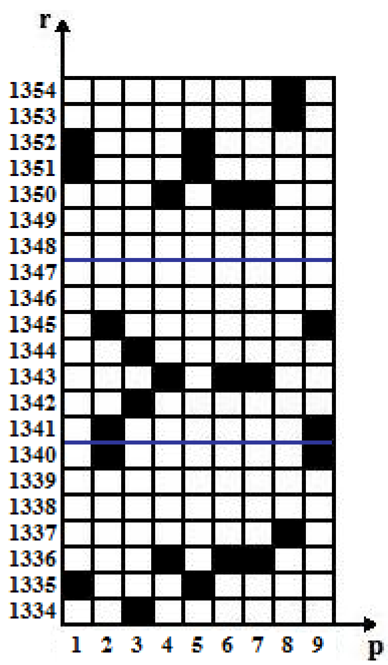
d(r)=3, d(p)=2



d(r)=3, d(p)=3

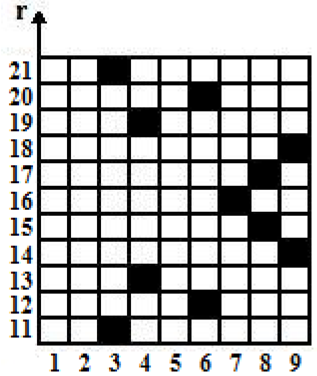


d(r)=4, d(p)=1

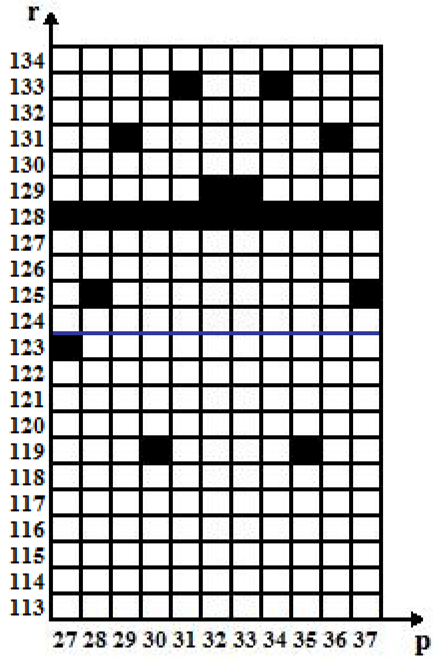


$Pr_5:=11$

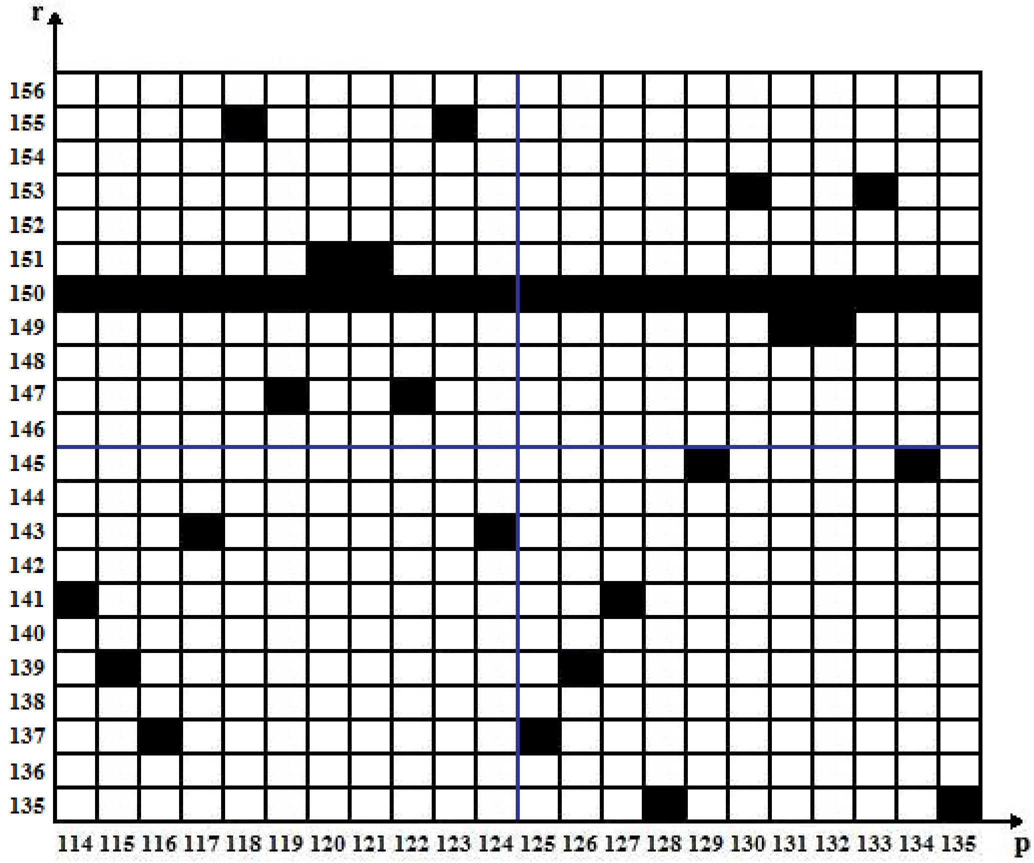
d(r)=2, d(p)=1



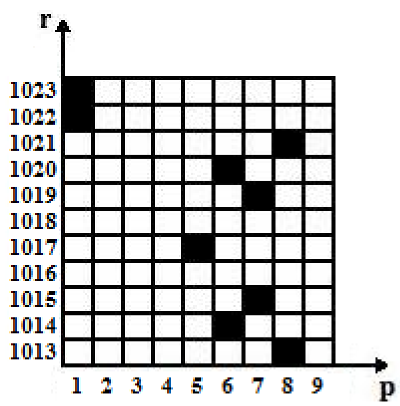
d(r)=3, d(p)=2



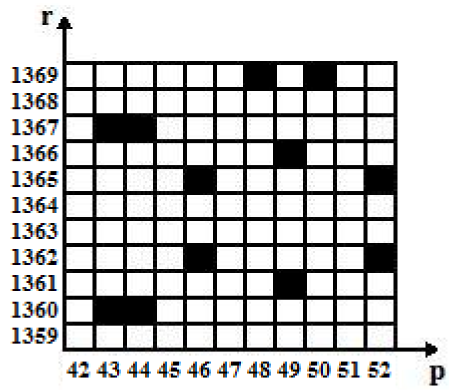
d(r)=3, d(p)=3



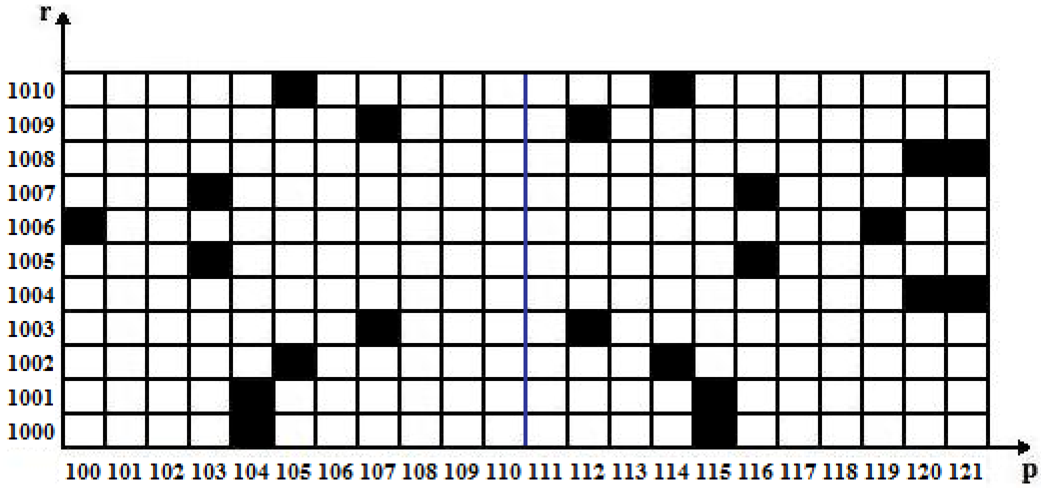
d(r)=4, d(p)=1



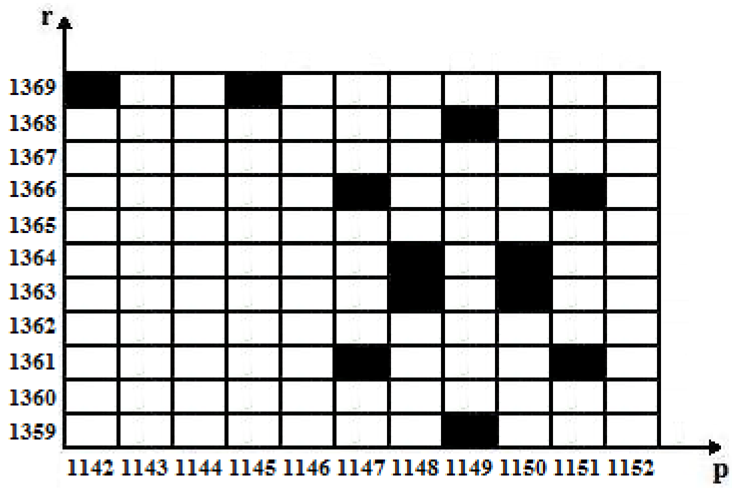
d(r)=4, d(p)=2



d(r)=4, d(p)=3

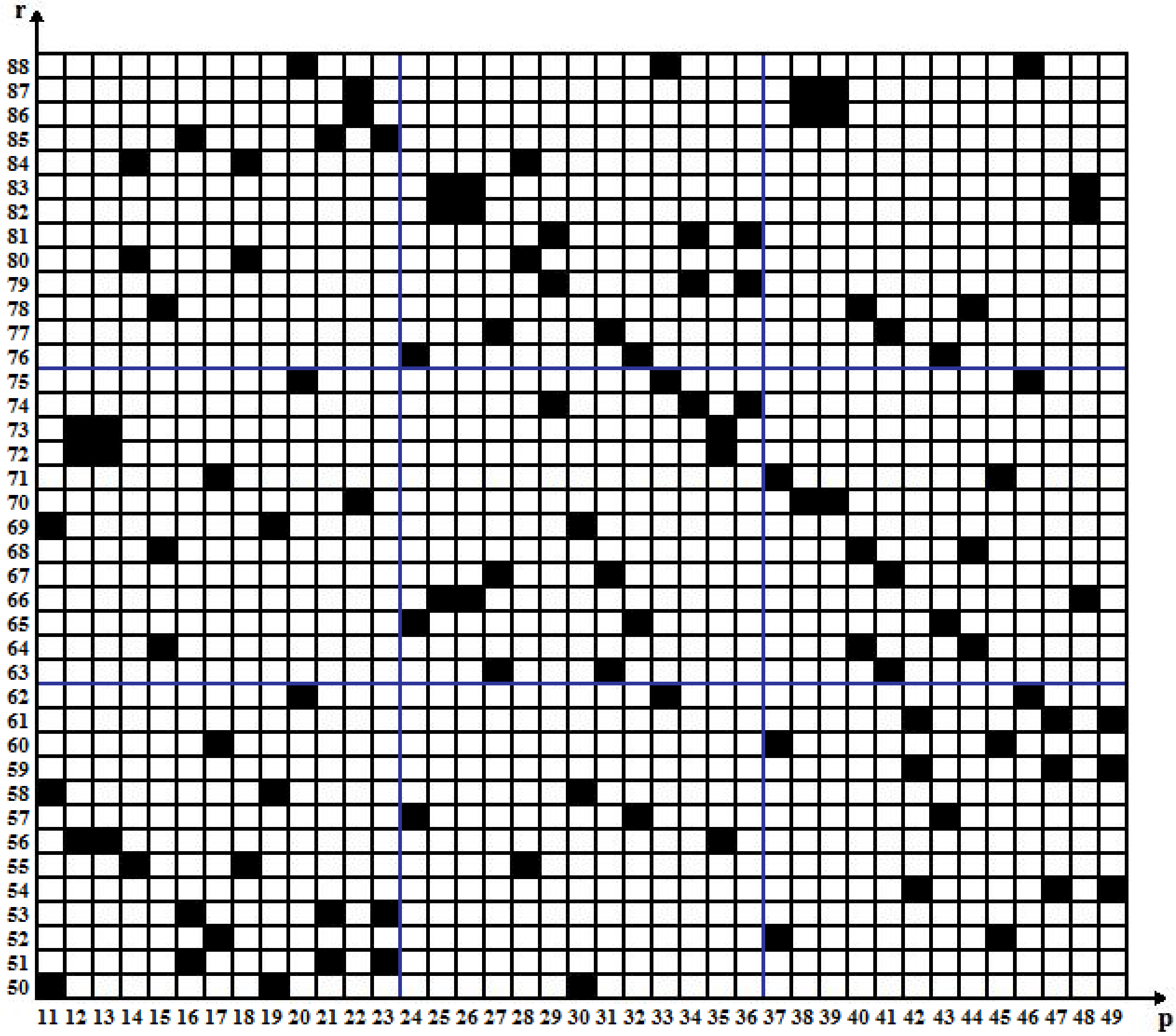


d(r)=4, d(p)=4

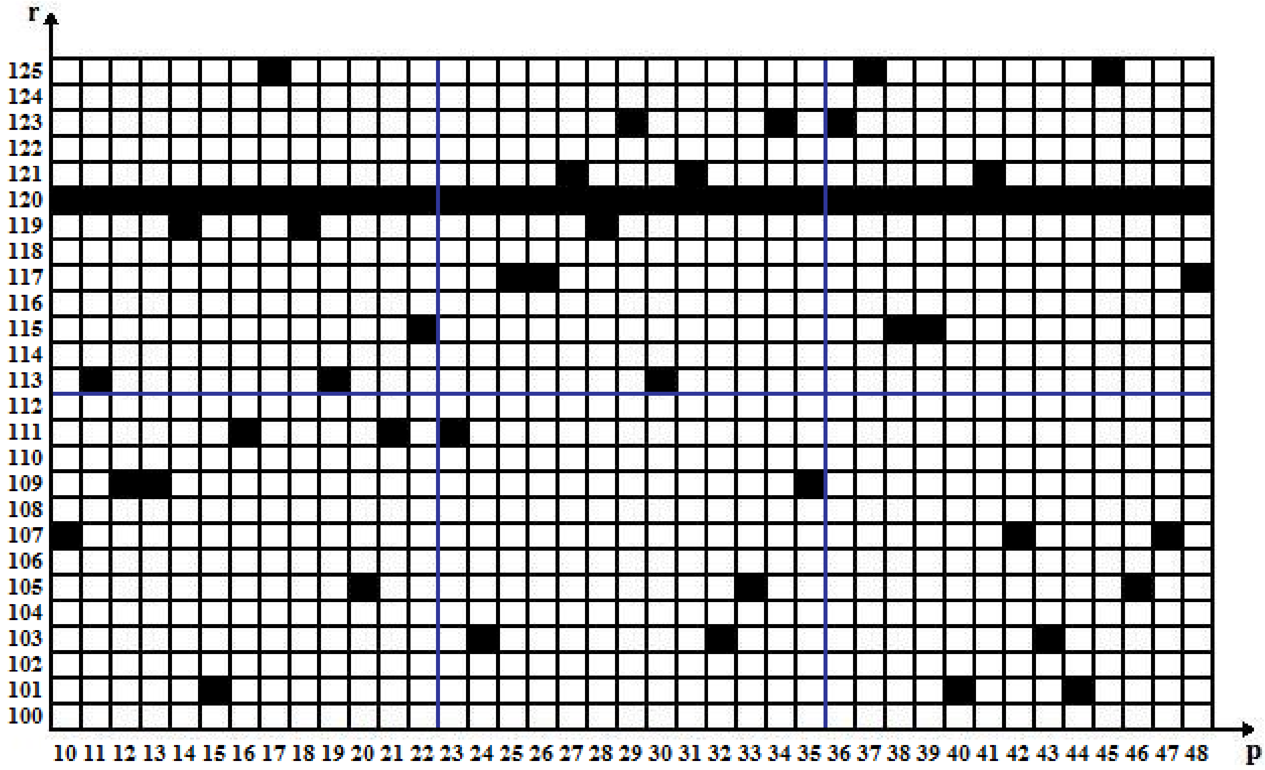


Pr_6:=13

d(r)=2, d(p)=2



d(r)=3, d(p)=2


r
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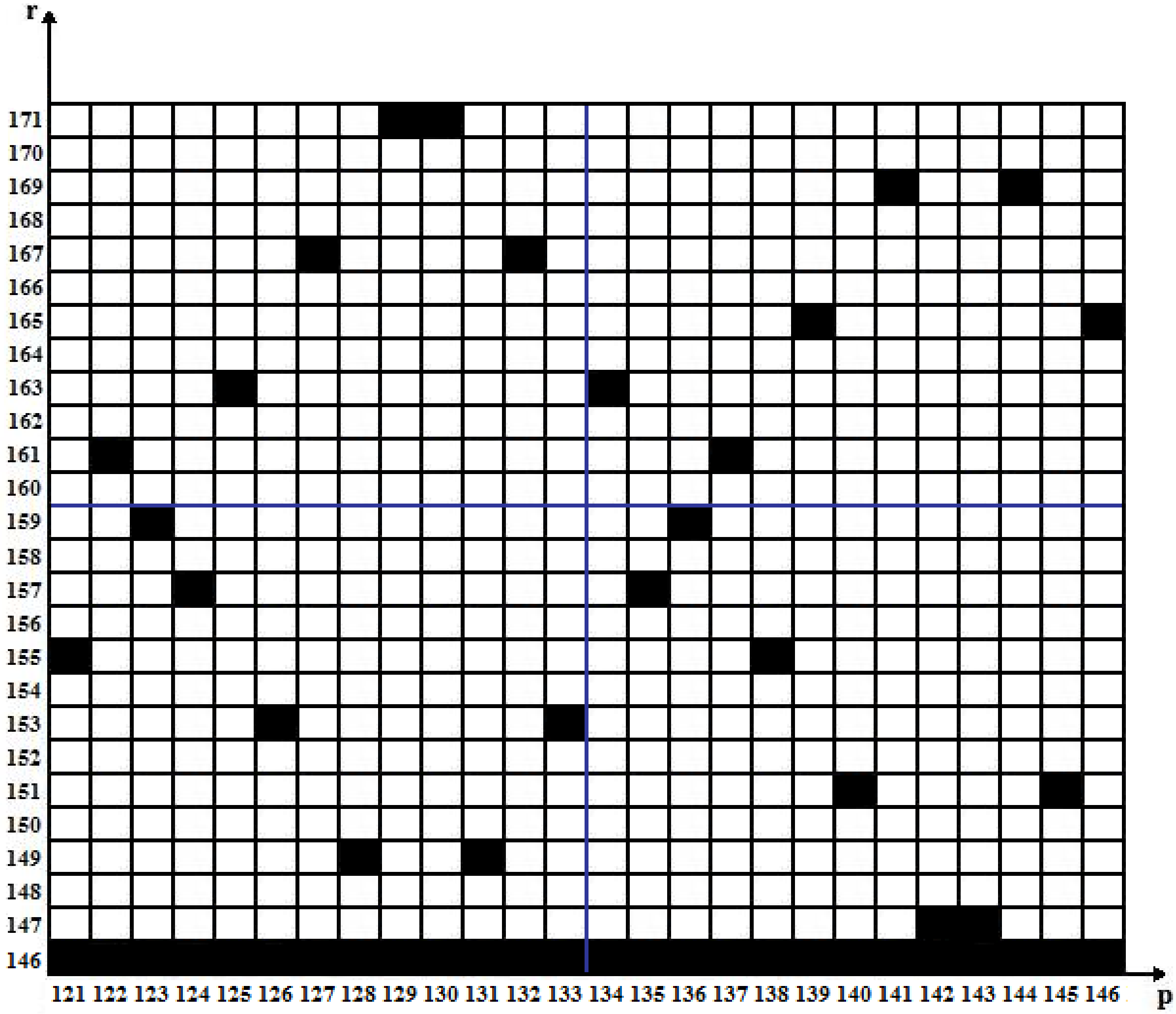
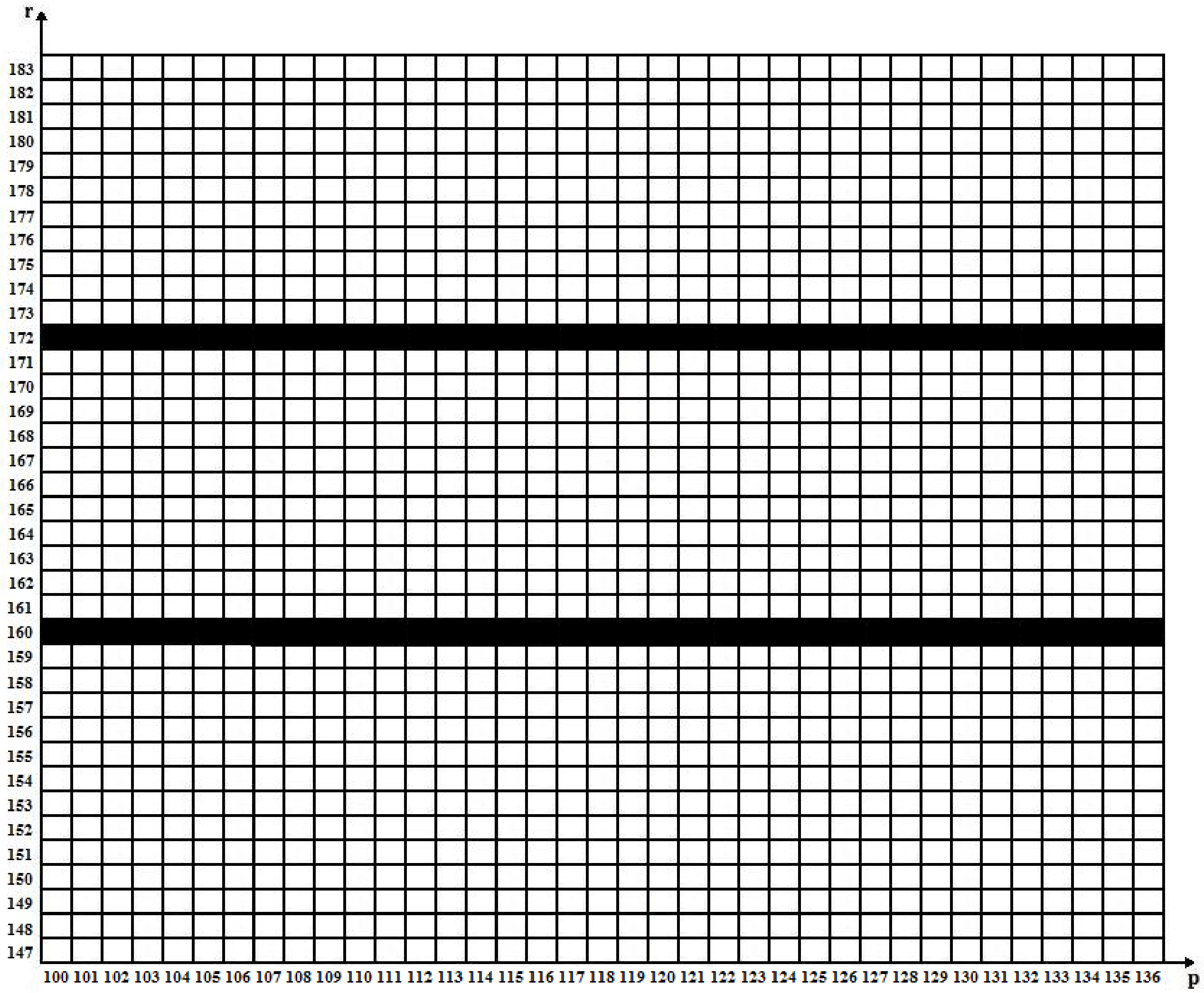
10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48
p

d(r)=3, d(p)=3



$Pr_{12}:=37$

d(r)=3, d(p)=3



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