The general relativity precession of Mercury.

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Abstract

The solution to the unexplained anomalous precession of the perihelion of Mercury, was the first success of GR (Einstein 1915), event which is near to reach its first centenary. We propose in this paper, to update the classic test of relativity, studying the gradual progression of one-orbit precession, not only in its perihelion, but also along a complete trajectory around the Sun. Just to underline GR results, we have confronted it with other virtual and mathematical potentials which, leading to an identical secular advance of the perihelion, offer different equations of motion with only theoretical meaning. Spacecraft Messenger will begin to orbit Mercury next March 18, and during twelve months, both will make 4.2 revolutions around the Sun. That event should afterwards allow us, to measure and draw accurately, the geometry of the whole geodesic orbit as an open free-fall path, isolated from other planets gravitational interference. This update must verify the GR issues with modern standards, throughout an accessible test to perform, with clear results, unlike a complex test, expensive and with uncertain conclusions.

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1.- G.R. equations of the orbit of Mercury.

Short time ago, professor C. M. Will remarked that "... ironically, while Mercury's perihelion advance is one of the three "crucial" tests of general relativity, is almost impossible to find a modern paper that quotes the latest observational results.*"* [1].

In this paper, we propose to analyse the instantaneous relativistic precession of the orbit of Mercury, setting its gradual progression along a single orbit and not just focused on the final and secular shift of the perihelion. Astronomical observations of Mercury have been difficult in the past, so the precise detection of nodes and other points has not been easy and much less if we face small actions like the advance of its orbit. It was very remarkable for LeVerrier in 1859 to detect it, regardless the gravitational disturbances of other planets, particularly Venus, Jupiter and Earth with also the precession of the equinoxes.

Those 42.95" arc / century, are the result of a secular addition of only 5.019×10^{-7} rad. / revolution, roughly equivalent to a location shift of 23.1×10^{3} m at the end of each orbit. The question now, is how such action is achieved throughout the 88-day orbital period and what are the theoretical assumptions about the sequential and gradual progression of precession along one orbit.

Is it possible to measure it by modern radiometric observation techniques and compute positions, cut-off from other gravitational perturbations?

If Mercury was the only planet in the solar system, its path would follow a single geodesic track, inside a curved space-time with gravitational geometry, in which the so call "anomalous precession" would not be such, but the natural and expected evolution, due to a free-fall and open straight-line trajectory.

Many GR textbooks and articles define and characterize the trajectory, starting from the Schwarzschild solution, which develops a geometry and a metric on a space-time with spherical symmetry. He was the first in 1916 to solve the field equations that Einstein had proposed a year earlier. Point out that Einstein, developed few documents with his explanation of the perihelion advance, [2] although it was one of his most remarkable results and remains so, even today. The basic elements to determine the relativistic equations of motion of any particle with mass, and therefore also Mercury around the Sun, is set for the effective potential and the angular momentum.

$$Vef = -\frac{\mu c^2}{r} + \frac{h^2}{2r^2} - \frac{\mu h^2}{r^3}$$ and GR equation of motion :

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2 \qquad (0)$$

where $\mu = GM/c^2$, $u = 1/r$, *h= angular momentum, M = Solar mass and* $\phi =$ *true anomaly.*

The relativistic term that modifies the newtonian equation, is the last one that depends on the square of the speed of light *(c)*. On this basis, we can write the equation of the trajectory of Mercury's orbit

$$r = \frac{p}{1 + e\ \cos(\phi) + \alpha(\phi)} \qquad (1)$$

where $\alpha(\phi)$ is a small perturbation function that makes GR different from the classic kepler-ellipse. From equation (0), we obtain a first solution for $\alpha(\phi)$

$$\alpha_1(\phi) = \frac{3GM}{c^2 p}\left[\ 1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\phi\right) + e\phi \sin\phi\right] \qquad (2)$$

provided and checked by M. Hobson [3], S. Carroll [4], and many of the texts consulted. Thus, we obtain the equation of the orbit of Mercury.

The trajectory is equivalent to an orbit following a classic keplerian ellipse, that also turns around its focus by a slight precession measured by the advance of the perihelion after one orbit or secular accumulation after 415 orbits fulfilled during a century.

This secular advance, can be characterized with the usual PPN formalism parameters, with the following result for one orbit

$$\delta\phi = \frac{6\pi GM}{c^2 p}\left[\frac{(2 - \beta + 2\gamma)}{3}\right] + \frac{6\pi}{2}R_{\oplus}^2\frac{\left(1 - 3/2\ \sin^2 i\right)}{p^2}J_2$$

(regardless the much smaller Lense-Thirring effect) , where J_2 is the quadrupole moment, i is the inclination referred to solar equator. Expressed in seconds-arc per century,

$$\Delta = 42.95''\left[\frac{2 - \beta + 2\gamma}{3} + 0.296\times 10^4 \,.\, J_2\right]$$

The results of this formulation are consistent with the values measured by radar detection,

[5] so we can state the adequate agreement with General Relativity [1]

The function $\alpha_1(\phi)$, shows how is the advance at the end of each orbit, how is precession being produced gradually, and how are the oscillations around its average value, linked with the forward sequence of the true-anomaly angle (ϕ). The key term is the last one $e\phi \sin(\phi)$, which produces a cumulative effect, constant and linear, true reason of the perihelion's advance. Also, the factor $\sin(\phi)$ allows the perturbing action, be incorporated with angular dimension –a slight angular precession $\Delta(\phi)$– .The remaining terms, provide only limited and periodic variations of small entity. The classic relativity textbook "Gravitation" by W. Misner et al [6], concludes in a linear progression

$$r = \frac{p}{1 + e\cos\left[(1 - \delta\phi_0/2\pi)\phi\right]} \qquad \text{with } \delta\phi_0 = \text{constant}$$

It follows that, gradual progress of precession is steady, with a fixed ratio, so that the advance along one orbit, has a linear and constant accumulation without oscillations, till its final value. Point out that the ratio of precession (K), is fixed referred to the angle (ϕ), which means that the angular velocity of Mercury (ω), will be identical and "drags" of the "angular velocity" of precession.

The constant rate of precession angle referred to ϕ is: $K = \frac{3GM}{c^2 p} = 7.99\ 10^{-8}$

With this linear advance, the perturbing function takes the form :

$$\alpha_2(\phi) = \frac{3GM}{c^2 p}\left(e\,\phi\,\sin\phi\right)$$

Other authors, develop a similar trajectory equation, with different terms also of small magnitude; among those, we emphasize the method of professor M. Berry [7] that concludes in the following expression

$$\alpha_3(\phi) = \frac{3GM}{c^2 p}\left[(1 + \frac{2}{3}e^2) + \frac{(1+3e^2)}{3e}\cos(\phi) - \frac{1}{3}e^2\cos^2(\phi) + e\phi\sin\phi\right]$$

Equations $\alpha_1(\phi)$ and $\alpha_3(\phi)$, have not-periodic terms that are " ... insignificant contribution and their only effect is to change slightly the interpretation of r_{min} and e." [7]. In this last case means that, r_{min}, the axes of the ellipse and the distances to the aphelion, decreases about $3.8 \cdot 10^3$ m regarding their theoretical value.

As shown in *Figure-1*, $\alpha_1(\phi)$ produces in the ascending and rising-up branch from the Sun, a further advance of precession regarding to the linear, constant and theoretical one (GR). This is because the periodic terms of $\alpha_1(\phi)$, is always positive and therefore, the radius r of the ellipse, will be always smaller than expected. In the ascending branch, the smaller radius is achieved in a slightly "previous" position and then, as a result of it, the precession must "move forward" to maintain Mercury in a dynamic equilibrium. During the falling and approaching branch to the Sun, the smaller radius is located ahead, so the keplerian ellipse is dragged a little "backward". The largest lead/lag angle with linear GR-precession, is when $\phi = \pi/2$ and $\phi = 3\pi/2$, with a value of $\Omega = \pm 0.016 \cdot K\,rad$.

In the case of $\alpha_3(\phi)$, the periodic oscillation is bigger, producing a breaking point in the lead/lag precession, relative to its linear and constant average value (GR), located in the aphelion and also in the perihelion.

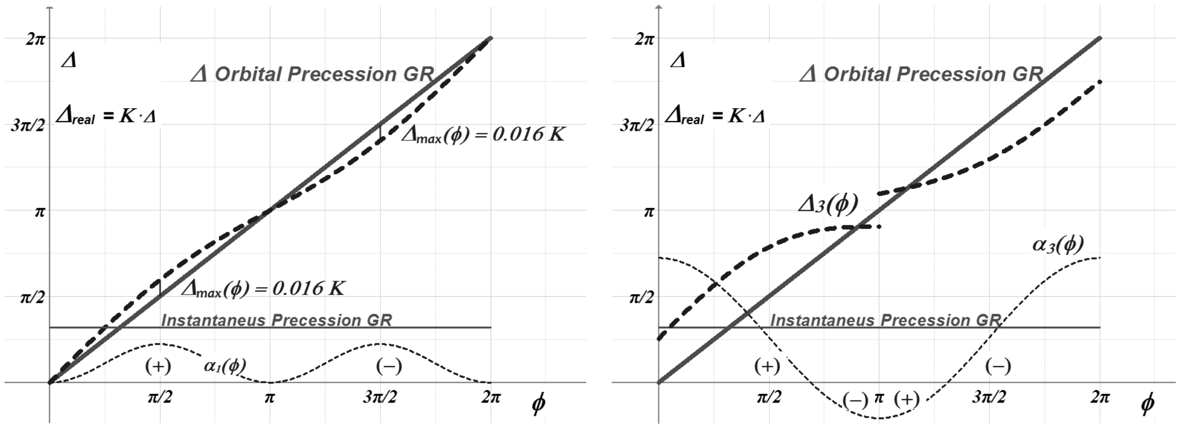

Figure-1 : Instantaneous and orbital precession for $\alpha_1(\phi)$ and $\alpha_3(\phi)$ perturbation functions

2.- Potential "S" equations of the orbit of Mercury

Potential "S" is defined as a slight perturbation to the newtonian gravitational potential, linked with the radial velocity of the object. Potential "S" is an update of the Tisserand mathematical potential developed in 1872. It would also need a gravitational basement to explain its real application; it is only a theoretical and virtual potential with unclear physical meaning :

$$\boldsymbol{S}(\phi) = -\frac{3GM}{r}\left(\frac{Vr}{c}\right)^2$$

where Vr is the radial velocity of the object. $S(\phi)$ is a perturbation added to the newtonian potential, and produces exactly the same one-orbit perihelion advance as GR, when $Vr << c$.

We can check the perihelion advance produced by $S(\phi)$, using any of the existing methods for calculating the effect of a perturbing central acceleration on an elliptic keplerian orbit: Gauss, G. Adkins [8], O.Chaschina [9] and B. Davies [10]. We will use this last one.

$$\Delta(\phi) = \frac{1}{e}\int_0^{2\pi} g(r)\cos(\phi)\, d\phi \qquad (3)$$

where $g(r)$ is the acceleration coefficient of the disturbing action versus the newtonian.
The potential S(ϕ) produces a radial perturbing acceleration Ar, which has the following expression [11] y [12]. ($Vr, \ddot{r}$ are the usual keplerian ellipse parameters.)

$$Ar(\phi) = -\frac{GM}{r^2}\left[-3\left(\frac{Vr}{c}\right)^2 + 6\frac{r\ddot{r}}{c^2}\right] = -\frac{GM}{r^2}\; g(r) \;\; ; \;\; Vr = \dot{r} = \frac{e\,h\sin\phi}{p} \;\; ; \;\; \ddot{r} = \frac{e\,h^2\cos\phi}{p\,r^2}$$

$$g(r) = -\frac{3GM}{c^2 p}\left[e^2\sin^2\phi - 2e\cos\phi\;\;(1 + e\cos\phi)\right];$$

inserting this in (3)

$$\Delta(\phi) = -\frac{3GM}{c^2 p}\,\frac{1}{e}\int_0^{2\pi}\left[e^2\sin^2\phi - 2e\cos\phi\,(1 + e\cos\phi)\right]\cos(\phi)\; d\phi$$

integrating we obtain

$$\Delta = \frac{6\pi\, GM}{c^2 p}$$, exactly the same precession as GR.

Once we know that the final one-orbit precession is same as GR, we will study the gradual progression throughout one orbit. We will use another formula that defines the precession for small disturbing potentials [13], [14]. We define $\delta\phi$ as the instantaneous precession at each point of the orbit, that gradually builds up until it reaches a final value (Δ) at the end of one orbit.

$$\delta(\phi) = \frac{\partial}{\partial h}\left[\frac{1}{h}\; r^2\; S(\phi)\;\; d\phi\right] = -\frac{\partial}{\partial h}\left[3\frac{GM}{c^2}\;\frac{\sin^2\phi}{1+e\cos\phi}\;\frac{e^2\, h}{p}\;\; d\phi\right]$$

derivates referred to h are : [8]

$$\frac{\partial p}{\partial h} = 2\,\frac{p}{h} \quad ; \quad \frac{\partial e}{\partial h} = -\frac{1}{h}\,\frac{1-e^2}{e} \quad ;$$ and then, neglecting 2º order terms, we have

$$\delta(\phi) = \left[3\frac{GM}{c^2 p}\left(2-e^2\right)\frac{\sin^2\phi}{1+e\cos\phi}\;\; d\phi\right]$$

result whose integral, is also consistent with the same final precession.

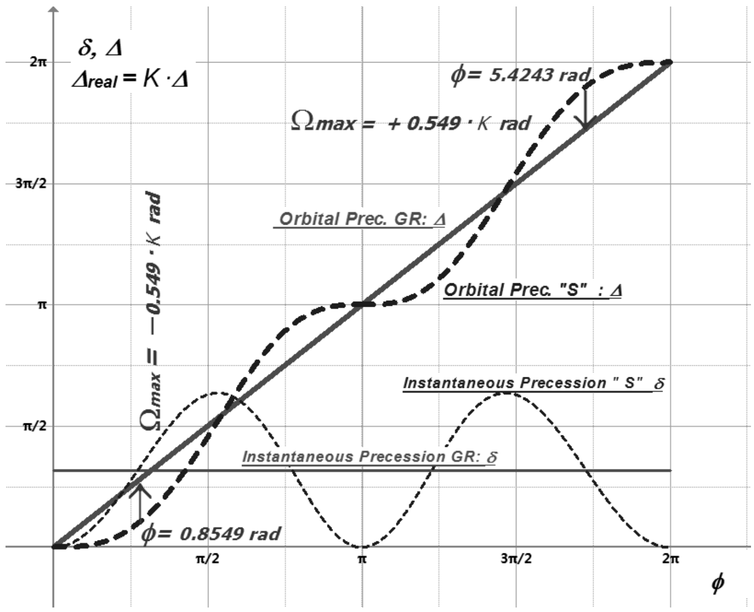

Figure-2 : Instantaneous and orbital precession for S (ϕ), perturbation potential

As seen in *Figure-2*, the instantaneous precession $\delta\phi$ is always positive, producing a forward advance in both branches of the orbit. This is because the perturbing potential produces a stable position which is always located in a "previous" position in the keplerian trajectory, and therefore the precession is always positive and with a symmetrical magnitude about the semi-major axis. However, progression is not constant nor linear, causing an angular lead/lag advance (Ω) related to the fixed and linear GR movement.

The peak instantaneous precession is at $\phi \gtrapprox \pi/2$ and $\phi \approx <3\pi/2$, very close to the peak values of *Vr*. The maximum angular lead/lag is when ϕ= 5,424 *rad* and ϕ= 0.855 *rad*, with $\Omega = \cdot$ K ± 0.549 *rad*.

The peak positional lead/lag of Mercury, would happen in **A** [ϕ= 2.462 *rad*.] and **B** [ϕ= 3.821 *rad*.] This is because in these points, the radius is significantly larger. In case **A**, Mercury would be in a forward position regarding a GR precession. This relative position would be $\boldsymbol{i} = 2.31 \cdot 10^3$ m and $\boldsymbol{j} = -0.36 \cdot 10^3$ m, values which would be equal but with opposite sign in **B**. Also point out that in about 23 days, Mercury would move from the peak forward position (**A**) to the most delayed (**B**), always referred to the relative location with a constant GR precession.

3. The orbit precession of Mercury for the Yukawa potential

The Yukawa potential is a not-newtonian action, characterized by a strength α and a range λ, that produces a slight disturbance to the classic gravitational field. It is also a theoretical and virtual potential with the following expression :

$$U_{\text{Yuk}} = -\frac{GM}{r}\left[1 + \alpha \exp(-r/\lambda)\right]$$

Being consistent with the real perihelion advance, the coefficients should be : $\alpha = 3.57 \cdot 10^{-10}$ and $\lambda = 2.89 \cdot 10^{10}$ [15]. Once we know the appropriate shift at the end of one orbit, we will analyse the gradual progression. The precession is determined by [15]

$$\delta\phi = \alpha \frac{2}{e}\int_0^{\pi} \frac{\mathrm{e}^{-r/\lambda}}{(1 + e\cos\phi)^2}\left\{ r/\lambda \left[2e + (1+e^2)\cos\phi\right] - (e + \cos\phi)\right\} \, d\phi$$

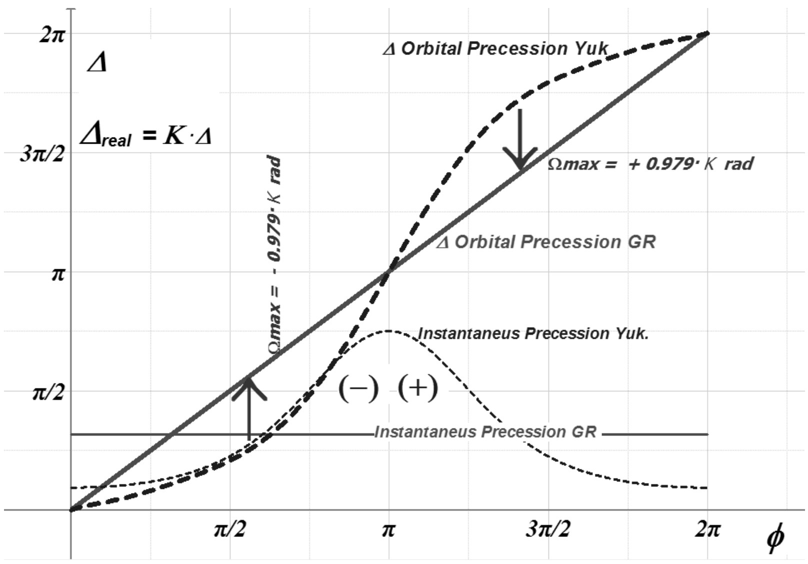

Figure-3 : Instantaneous and orbital precession for Yukawa potential

Yukawa precession rate *(Figure – 3)*, has a slight delay in the ascending and rising-up branch from the Sun and slight advance when Mercury approaches to the Sun, always referred to the linear precession GR. The maximum instantaneous precession comes in the aphelion. The maximum angular precession is when $\phi = 1.91$ *rad*. and $\phi = 4.38$ *rad*. being the difference referred to GR orbit, of $\Omega = 0.979 \pm \cdot$ K *rad*.

The peak positional lead/lag would happen when $\phi = 2{,}05$ *rad*. and $\phi = 4{,}23$ *rad* with a magnitude of $\boldsymbol{i} = \pm 4.66 \cdot 10^3$ m and $\boldsymbol{j} = \mp 0{,}51 \cdot 10^3$ m

4 .- Conclusions and proposals.

We propose in this paper, to update the classic test of relativity, studying the gradual progression of one-orbit precession, not only in its perihelion, but also along a complete trajectory around the Sun. Just to underline GR results, we have confronted it with other virtual and theoretical gravitational potentials which, leading to an identical secular advance of the perihelion, offer different equations of motion and different progression of the orbital instantaneous precession. However, these mathematical potential have only theoretical meaning.

Spacecraft Messenger will begin to orbit Mercury next March 18, and during twelve months, both will make 4.2 revolutions around the Sun. That event should afterwards allow us, to measure and draw accurately, the geometry of the whole geodesic orbit as an open free-fall path, isolated from other planets gravitational interference. This update should verify the GR issues with modern standards, throughout an accessible test to perform, with clear results, unlike a complex test, expensive and with uncertain conclusions.

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