# On Systems, Subsystems, Composite Systems and Entanglement

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*Dedicated to Marie-Louise Nykamp*

**Abstract**

It is shown that under suitable compositions of systems, arbitrary large amounts of entangled type states can easily be obtained.

## 1. Minimal Composite Systems

**Question 1 :**

Let two systems $S$ and $T$ have the respective state spaces $X$ and $Y$. Which are all the *minimal* systems $Q$ which have $S$ and $T$ as subsystems ?

$\square$

Obviously, the crucial issue in the question above is the concept of *minimality*. However, the definition of the concept of minimality requires the a priori specification of the concepts of *system, subsystem* and *composition* of systems or subsystems.

For the time being, let us assume that by a system $S$ we mean an entity which can be in different states, and correspondingly, has associated to it the set $X$ of such all such possible states.
Further, we assume that it general every subset $Y \subseteq X$ may possibly correspond to a subsystem $T$ of $S$.
Let us give two examples in this regard.

First, let $S, T$ be Classical Mechanical systems and $Q_c$ be their usual composition in Classical Mechanics, in which case the state space of $Q_c$ will be $Z_c = X \times Y$. This $Q_c$ is obviously minimal in the sense that every Classical Mechanical subsystem $Q'$ of $Q_c$ which has $S, T$ as subsystems must be $Q_c$ itself.

Second, let $S, T$ be Quantum Mechanical systems and $Q_q$ be their usual composition in Quantum Mechanics, in which case $X, Y$ are complex Hilbert spaces, and the state space of $Q_q$ will be $Z_q = X \bigotimes Y$. This $Q_q$ is again obviously minimal in the sense that every Quantum Mechanical subsystem $Q'$ of $Q_q$ which has $S, T$ as subsystems must be $Q_q$ itself.

Now in the second case, $Q_c$ can be seen as a *strict* subsystem of $Q_q$ since, for nontrivial quantum systems $S, T$, we have the strict inclusion

$$(1.1) \qquad X \times Y \ni (x, y) \longmapsto x \otimes y \in X \bigotimes Y$$

Let us denote by $\bigcirc_\kappa$ the composition of state spaces of *type* $\kappa$ of two systems, while the composition of respective states we denote by $\circ_\kappa$.
Thus in the first example above we have for $x \in X$, $y \in Y$

$$(1.2) \qquad Q_c = S \bigcirc_c T, \qquad x \circ_c y \in Z_c = X \bigcirc_c Y = X \times Y$$

while in the second above example we have

$$(1.3) \qquad Q_q = S \bigcirc_q T, \qquad x \circ_q y \in Z_q = X \bigcirc_q Y = X \bigotimes Y$$

**2. How to Obtain More Entanglement**

In (1.1), as is known, the states in

(2.1) $\quad (X \bigotimes Y) \setminus (X \times Y)$

are entangled. Also as is known, there is a major interest in entangled states both in Quantum Mechanics as a theory, as well as in Quantum Information. Therefore, the following

**Question 2 :**

Are there types $\kappa$ of compositions of systems, such that the injection

(2.2) $\quad X \times Y \ni (x, y) \longmapsto x \circ_\kappa y \in X \bigcirc_\kappa Y$

gives in

(2.3) $\quad (X \bigcirc_\kappa Y) \setminus (X \times Y)$

more entangled type states than in (2.1) ?

$\square$

We note that in (2.3) it is not a necessary requirement to have

(2.4) $\quad (X \bigotimes Y) \subset (X \bigcirc_\kappa Y)$

An obvious significant interest in larger and/or different amounts of entangled type states (2.3) comes from Quantum Information. And such an interest is further increased to the extent that, in the future, one may possibly identify corresponding effective physical realizations which do not suffer from major disadvantages, such as for instance, quantum decoherence.

We recall that in [1-4], a large variety of extensions of the usual tensor product were introduced and studied. And they provide a class of answers to the above question by the corresponding variety of enlarged sets of entangled type elements in (2.3).

Here, as a further considerable generalization of the approach in [1-4], we address the above question of finding types $\kappa$ of compositions of systems for which one has an injection

$$(2.5) \qquad X \bigotimes Y \ni x \otimes y \longmapsto x \circ_\kappa y \in X \bigcirc_\kappa Y$$

For that purpose, it is useful to recall the standard way the quantum composition given by the usual, or for that matter generalized, [1-4], tensor product $X \bigotimes Y$ is constructed.
Namely, for two systems $S$ and $T$ with the respective state spaces $X$ and $Y$, we define

$$(2.6) \qquad \mathcal{FM}(X, Y)$$

given by the *free monoid* generated by $X \times Y$. Thus $\mathcal{FM}(X, Y)$ is the set of all elements which are words

$$(2.7) \qquad (x_1, y_1)(x_2, y_2) \ldots (x_n, y_n)$$

where $n \geq 0$, while $x_1, x_2, \ldots, x_n \in X$, $y_1, y_2, \ldots, y_n \in Y$, with the case $n = 0$ corresponding to the empty word. Further, the monoid operation on $\mathcal{FM}(X, Y)$ is simply the concatenation of such words, thus the empty word is the *neutral* element.
Obviously, if $X$ or $Y$ have at least two elements, then $\mathcal{FM}(X, Y)$ is *noncommutative*.

What should be noted here is that the construction of the tensor product of two Hilbert complex spaces, groups, or in general of two arbitrary sets, [1-4], starts with the same first step in (2.6), (2.7). However, immediately next, a considerable reduction of the state space $\mathcal{FM}(X, Y)$ is implemented by replacing it with various quotient spaces obtained through certain equivalence relations defined on $\mathcal{FM}(X, Y)$. For the time being let us set aside any such equivalence relations and the corresponding quotients, and instead consider $\mathcal{FM}(X, Y)$ itself.

And then here one can note that there may as well be other ways to start than in (2.6), (2.7). Indeed, for the purpose of finding further types of compositions, we note that (2.6), (2.7) can also be seen in the following manner. Let

$$(2.8) \qquad \mathcal{N}$$

be the set of all subsets of $\mathbb{N} = \{1, 2, 3, \ldots\}$ of the form $I = \{1, 2, \ldots, n\}$, where $n \geq 1$. Then, as a set, $\mathcal{FM}(X, Y)$ can be identified with the set of all the mappings

$$(2.9) \qquad s : I \longmapsto X \times Y, \qquad I \in \mathcal{N}$$

to which the empty mapping from $\phi$ to $X \times Y$ is added. Clearly, the set $\mathcal{FM}(X, Y)$ need not necessarily be endowed with any particular structure, although it naturally has the above free monoid structure which is defined as follows. Given $s : \{1, 2, \ldots, n\} \longrightarrow X \times Y$ and $t : \{1, 2, \ldots, m\} \longrightarrow X \times Y$, then $st : \{1, 2, \ldots, n + m\} \longrightarrow X \times Y$ is defined by

$$(2.10) \qquad st(i) = \left| \begin{array}{lll} s(i) & \text{if} & 1 \leq i \leq n \\ & & \\ t(i-n) & \text{if} & n+1 \leq i \leq n+m \end{array} \right.$$

And now, in view of the above, a considerable variety of compositions of state spaces, larger than $\mathcal{FM}(X, Y)$, are suggested by the above. For instance, Let $\mathcal{I}$ be a set of nonvoid index sets $I$, and denote by

$$(2.11) \qquad \mathcal{M}_{\mathcal{I}}(X, Y)$$

the set of all the mappings

$$(2.12) \qquad s : I \longrightarrow X \times Y, \qquad I \in \mathcal{I}$$

to which again the empty mapping from $\phi$ to $X \times Y$ is added.

If in particular, we consider $\mathcal{I}$, such that

$$(2.13) \qquad \mathcal{N} \subseteq \mathcal{I}$$

then obviously $\mathcal{FM}(X,Y) \subseteq \mathcal{M}_{\mathcal{I}}(X,Y)$.

Furthermore, unlike with $\mathcal{FM}(X,Y)$, the sets $\mathcal{M}_{\mathcal{I}}(X,Y)$ can contain *infinite* families of pairs $(x,y) \in X \times Y$, which correspond to mappings $s$ in (2.12) that have as domains infinite index sets $I$, where $I \in \mathcal{I}$.

However, even if the set $\mathcal{I}$ only contains finite index sets $I$, the corresponding set $\mathcal{M}_{\mathcal{I}}(X,Y)$ can be arbitrarily *larger* than $\mathcal{FM}(X,Y)$, in view of (2.13).

We note that, unlike $\mathcal{FM}(X,Y)$, the more general composition $\mathcal{M}_{\mathcal{I}}(X,Y)$ does in general no longer have a natural monoid structure, such as for instance in (2.10).

And now, a yet more involved type of composition can be obtained as follows. Let $\mathcal{I}$ be as above, and for $i \in I \in \mathcal{I}$, let $J_i$ be a finite graph with the vertices $(j_1, j_2, \ldots, j_{p_i}) \in \{0,1\}^{p_i}$. For convenience, let us denote by $\mathcal{J}$ the set of all such graphs $J_i$, with $i \in I \in \mathcal{I}$.

Now, we denote by

$$\mathcal{M}_{\mathcal{I},\mathcal{J}}(X,Y) \tag{2.14}$$

the set of all mappings

$$s : I \ni i \longmapsto s(i) \in (X,Y)^{J_i} \tag{2.15}$$

where for $J = (j_1, j_2, \ldots, j_p) \in \{0,1\}^p$, we have

$$(X,Y)^J = \prod_{1 \leq q \leq p} Z^q \tag{2.16}$$

with

$$Z^q = \left| \begin{array}{l} X \;\text{ if }\; j_q = 0 \\ \\ Y \;\text{ if }\; j_q = 1 \end{array} \right. \tag{2.17}$$

## 3. Arbitrary Large Amounts of Entanglement

As seen already with the composition of systems which give above state spaces $\mathcal{M}_{\mathcal{I}}(X, Y)$, such state spaces can be arbitrarily large, provided that the sets $\mathcal{I}$ of index sets $I$ are suitably chosen. And in case (2.13) holds for such sets $\mathcal{I}$ of index sets, then clearly, the amount

$$\mathcal{M}_{\mathcal{I}}(X, Y) \setminus (X \times Y) \tag{3.1}$$

of entangled type states in such systems can also be arbitrarily large.

Obviously, a similar situation happens with compositions of systems whose state spaces are given by (2.14).

# References


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