The fine structure constant derived from the broken symmetry of two simple algebraic identities

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The fine structure constant is shown to arise naturally in the course of altering the symmetry of two algebraic identities. Specifically, the symmetry of the identity $M^2 = M^2$ is "broken" by making the substitution $M \to M - y$ on its left side, and the substitution $M^n \to M^n - x^p$ on its right side, where p equals the order of the identity; these substitutions convert the above identity into the equation $(M-y)^2 = M^2 - x^2$. These same substitutions are also applied to the only slightly more complicated identity $(M/N)^3 + M^2 = (M/N)^3 + M^2$ to produce this second equation $(M-y)^3/N^3 + (M-y)^2 = \left(M^3 - x^3\right)/N^3 + M^2 - x^3$. These two equations are then shown to share a mathematical property relating to dy/dx, where, on the second equation's right side this property helps define the special case $\left(M^3 - x^3\right)/N^3 + M^2 - x^3 = \left(10^3 - 0.1^3\right)/3^3 + 10^2 - 0.1^3 = 137.036$, which incorporates a value close to the experimental fine structure constant inverse.

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I. INTRODUCTION

The value 137.036, which is within seven parts per billion of the experimental fine structure constant (FSC) inverse [1], is shown to occur naturally during the study of two basic algebraic identities whose symmetry is altered. As the FSC is precisely known and of fundamental importance to physics, it is nontrivial to see such a close approximation of its experimental value appear in a purely mathematical investigation, as opposed to being fit by mathematics that would be of no interest if the FSC had not appeared first in the experiments of physicists.

II. THE SUBSTITUTION MAP

In what follows the symmetry of this *second* order identity

$$M^2 = M^2 \tag{2.1}$$

and this *third* order identity

$$\left(\frac{M}{N}\right)^3 + M^2 = \left(\frac{M}{N}\right)^3 + M^2 \tag{2.2}$$

will be "broken" by making the substitution

$$M \to M - y$$

on their left sides, and the substitution

$$M^n \to M^n - x^p$$

on their right sides, where p equals the order of each identity. Above, y and x are variables such that

$$0 < y \leq 0.1 \tag{2.3}$$

$$0 < x \leq 0.1 \quad , \tag{2.4}$$

where N and M are positive integer constants fulfilling

$$M = \frac{N^3}{3} + 1 \tag{2.5}$$

$$M \geq 10 \quad . \tag{2.6}$$

The reason for altering these identities using the above *substitution map* (an admittedly unusual thing to do) is to change them from related *identities* that are true for *all* values of N and M, into related *conditional equations* that are true for *some* values of N and M, while simultaneously making clear their shared origin. It will be shown that these equations also share a property involving their derivatives, where, for a special case, the terms of the third order equation will combine to produce the distinctive value 137.036. In this way it will be shown that ~137.036 is not merely important to the electromagnetic interaction, but also relevant to pure mathematics.

III. THE SECOND ORDER EQUATION

Begin with the second order identity (for which $p = 2$)

$$M^2 = M^2$$

and break its symmetry by making the substitution

$$M \to M - y$$

on its left side, and the substitution

$$M^n \to M^n - x^p$$

on its right side. This produces this second order equation

$$(M-y)^2 = M^2 - x^2 \quad . \tag{3.1}$$

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IV. THE FSC PROPERTY

Theorem 1 in Section VII shows that for Eq. (3.1) the following relation holds

$$\frac{dy}{dx} \approx \frac{x}{M} \quad ,$$

whereas Theorem 2 in Section VIII will show that for Eq. (5.1), below, this similar relation holds

$$\frac{dy}{dx} \approx \frac{x^2}{M} \quad .$$

Accordingly, for both of these equations, at

$$x = \frac{1}{M} \tag{4.1}$$

we get

$$\frac{dy}{dx} \approx \frac{1}{M^p} \quad , \tag{4.2}$$

where p equals the order of each equation. Given that Eq. (5.1), introduced in the next section, will be associated with the FSC, this will be termed *The FSC Property.*

V. THE THIRD ORDER EQUATION

To produce 137.036 from the third order identity

$$\left(\frac{M}{N}\right)^3 + M^2 = \left(\frac{M}{N}\right)^3 + M^2$$

again apply the substitution

$$M \to M - y$$

on the left, and the substitution

$$M^n \to M^n - x^p$$

on the right, where now $p = 3$. This results in this third order equation

$$\left(\frac{M-y}{N}\right)^3 + (M-y)^2 = \frac{M^3 - x^3}{N^3} + M^2 - x^3 \quad . \tag{5.1}$$

Importantly, it is Eq. (2.5)

$$M = \frac{N^3}{3} + 1$$

that constrains this equation to have the FSC property, as is shown by Theorem 2 in Section VIII. And inspection reveals that the smallest positive integers fulfilling Eq. (2.5) are

$$N = 3 \quad \text{and} \quad M = 10 \quad ,$$

which, when substituted into Eq. (5.1), produce

$$\left(\frac{10-y}{3}\right)^3 + (10-y)^2 = \frac{10^3 - x^3}{3^3} + 10^2 - x^3 \quad ,$$

so that at $x = 1/M$ we get

$$\frac{10^3 - 0.1^3}{3^3} + 10^2 - 0.1^3 =$$
$$999.999/27 + 99.999 = 137.036 \quad .$$

Hence, 137.036 appears in Eq. (5.1) for the smallest positive integers N and M consistent with Eq. (2.5). As noted at the outset, this value is very close to the 2010 CODATA value for the FSC inverse of 137.035 999 074, differing by just 7 parts per billion [1].

VI. SUMMARY AND CONCLUSION

It has been shown that Eqs. (3.1) and (5.1) have a shared origin and likewise share the FSC property. Equation (5.1), however, differs from Eq. (3.1) in one key respect: only Eq. (5.1) is constrained by $M = N^3/3 + 1$. Moreover, it is only because it fulfills this additional requirement that Eq. (5.1) has the FSC property and thereby behaves like Eq. (3.1).

The smallest pairs of positive integers (N, M) meeting the above requirement are (3,10), (6,73), (9, 244), (12, 577), (15,1126), etc. Hence, the *smallest pair* for which the equations' behavior merges is (3,10). But for this pair Eq. (5.1) produces exactly 137.036, a number that is unique in deriving from minimal N and M. In this way this precise FSC inverse approximation emerges as a natural and unique byproduct of the analysis of two basic symmetric identities and their associated equations, making 137.036 relevant to pure mathematics independent of its role as a constant famous to physicists.

VII. THE FSC PROPERTY AND THE SECOND ORDER EQUATION

Theorem 1. *Assume Eq.* (3.1)

$$(M-y)^2 = M^2 - x^2 \quad ,$$

where y and x are variables such that $0 < y \leq 0.1$ and $0 < x \leq 0.1$, and M is an integer constant such that $M \geq 10$ (see Eqs. (2.3)*,* (2.4)*, and* (2.6)*). Then*

$$\frac{dy}{dx} \approx \frac{x}{M} \quad , \tag{7.1}$$

so that at $x = 1/M$ we get

$$\frac{dy}{dx} \approx \frac{1}{M^2} \quad ,$$

fulfilling the FSC property as defined in Section IV.

Proof. Equation (3.1)

$$(M-y)^2 = M^2 - x^2$$

expands and simplifies to

$$2My - y^2 = x^2 \quad .$$

Given $y \leq 0.1$ and $M \geq 10$ then $M/y \geq 100$. Hence y^2 is small compared to My. It follows that the approximation

$$2My \approx x^2$$

holds. Solving for y gives

$$y \approx \frac{x^2}{2M} \quad , \tag{7.2}$$

so that

$$\frac{dy}{dx} \approx \frac{x}{M} \quad .$$

Hence, at $x = 1/M$ we get

$$\frac{dy}{dx} \approx \frac{1}{M^2} \quad . \qquad \square$$

Remark 1. It is particularly instructive to compare Eq. (7.2) against Eq. (8.2) in the next section. Both imply $dy/dx = x^{p-1}/M$, where only their values for p differ (2 and 3, respectively).

VIII. THE FSC PROPERTY AND THE THIRD ORDER EQUATION

Theorem 2. *Assume Eq.* (5.1)

$$\left(\frac{M-y}{N}\right)^3 + (M-y)^2 = \frac{M^3 - x^3}{N^3} + M^2 - x^3 \quad ,$$

where y and x are variables fulfilling Eqs. (2.3) *and* (2.4)

$$0 < y \leq 0.1 \qquad 0 < x \leq 0.1 \quad ,$$

and N and M are positive integer constants fulfilling Eqs. (2.5) *and* (2.6)

$$M = N^3/3 + 1 \qquad M \geq 10 \quad .$$

Then

$$\frac{dy}{dx} \approx \frac{x^2}{M} \quad , \tag{8.1}$$

so that at $x = 1/M$ we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} \quad ,$$

fulfilling the FSC property as defined in Section IV.

Proof. Equation (5.1)

$$\left(\frac{M-y}{N}\right)^3 + (M-y)^2 = \frac{M^3 - x^3}{N^3} + M^2 - x^3$$

expands and simplifies to

$$-\frac{3M^2y}{N^3} + \frac{3My^2}{N^3} - \frac{y^3}{N^3} - 2My + y^2 \approx -\frac{x^3}{N^3} - x^3$$

or

$$\frac{3M}{N^3}\left(My - y^2\right) + \frac{y^3}{N^3} + 2My - y^2 \approx \frac{x^3}{N^3} + x^3 \quad .$$

Given $y \leq 0.1$ and $M \geq 10$ then $M/y \geq 100$. Hence y^3/N^3 and y^2 are small compared to My. It follows that the approximation

$$\frac{3M^2y}{N^3} + 2My \approx \frac{x^3}{N^3} + x^3$$

holds. Solving for y gives

$$\left(3M + 2N^3\right) My \approx \left(N^3 + 1\right) x^3$$

$$y \approx \frac{N^3+1}{3M + 2N^3}\frac{x^3}{M} \quad .$$

But Eq. (2.5) provides that

$$M = N^3/3 + 1 \quad ,$$

so that by substitution

$$\begin{aligned} y &\approx \frac{N^3+1}{3(\frac{N^3}{3}+1) + 2N^3}\frac{x^3}{M} \\ &\approx \frac{N^3+1}{N^3+3+2N^3}\frac{x^3}{M} \\ &\approx \frac{N^3+1}{3N^3+3}\frac{x^3}{M} \\ &\approx \frac{N^3+1}{3(N^3+1)}\frac{x^3}{M} \quad , \end{aligned}$$

which gives

$$y \approx \frac{x^3}{3M} \quad . \tag{8.2}$$

It follows that

$$\frac{dy}{dx} \approx \frac{x^2}{M} \quad .$$

Hence, at $x = 1/M$ we get

$$\frac{dy}{dx} \approx \frac{1}{M^3} \quad . \qquad \square$$

[1] P.J. Mohr, B.N. Taylor, and D.B. Newell (2011), "The 2010 CODATA Recommended Values of the Fundamental Physical Constants" (Web Version 6.0). This database was developed by J. Baker, M. Douma, and S. Kotochigova. Available: `http://physics.nist.gov/constants` [Friday, 22-Jul-2011 10:04:27 EDT]. National Institute of Standards and Technology, Gaithersburg, MD 20899.