Further on Non-Cartesian Systems

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Dedicated to Marie-Louise Nykamp

Abstract

A class of *non-Cartesian* physical systems, [7], are those whose composite state spaces are given by significantly extended tensor products. A more detailed presentation of the way such extended tensor products are constructed is offered, based on a step by step comparison with the construction of usual tensor products. This presentation clarifies the extent to which the extended tensor products are indeed more general than the usual ones.

"History is written with the feet ..."

Ex-Chairman Mao

"Science in not done scientifically, since

it is mostly done by non-scientists ... "

Anonymous

1. Preliminaries

As seen in [7], there is a natural division of physical systems in Cartesian and non-Cartesian according to the way their state spaces compose.
Namely, given two Cartesian systems X and Y with the respective state spaces E and F, then the composite system "$X \, and \, Y$" has the state space given by the Cartesian product $E \times F$. Classical physical systems are in this sense Cartesian.
On the other hand, quantum systems, for instance, have considerably larger state spaces for their composites. Namely, if X and Y are two such systems and their state spaces are the complex Hilbert spaces E and F, respectively, then the state space of the composite quantum system "$X \, and \, Y$" is the tensor product $E \bigotimes F$. And indeed, this is a considerably larger space than the Cartesian product $E \times F$, since we have the injective mapping, which for convenience we shall consider to be an embedding

$$(1.1) \qquad E \times F \ni (x, y) \longmapsto x \otimes y \in E \bigotimes F$$

and the difference between the two sets, in this case both complex Hilbert spaces is clearly illustrated already in the finite dimensional case when, if m, n are the dimensions of E and F, respectively, then $m+n$ is the dimension of $E \times F$, while $E \bigotimes F$ will have the dimension mn. Thus in general, the set of *entangled* elements

$$(1.2) \qquad (E \bigotimes F) \setminus (E \times F)$$

is considerably larger than the set $E \times F$ of non-entangled elements.

An essential difference, therefore, between Cartesian and non-Cartesian physical systems is that in the state spaces of the composites of two of the latter there are states which can be seen as entangled in a generalized sense, namely, those whose state *cannot* be expressed simply in terms of a pair of states, with each state in the pair taken from one of the two component systems. And as is well known in the case of quantum systems, entangled composite states are most important, for instance, in quantum computation, and in general, quantum information technology.

For convenience, we shall keep using the simpler traditional term *entangled* also for the states mentioned above which are, in fact, entangled in a generalized sense.

So far, it appears that the only known non-Cartesian physical systems are the quantum ones.
In this regard, in [7], the problem was formulated to find physical systems other than the quantum ones and which are non-Cartesian.
Needless to say, there may be various applicative advantages in such systems. Among others, they may be used to build computers which - due to the presence of entangled states - could have advantages over usual electronic digital computers.

2. Usual Tensor Products

Let E and F be abelian groups. Then their tensor product $E \bigotimes F$ is constructed in the following *five steps*, [2-6].

Step 1 :

Let G be the *free monoid* generated by the elements of the usual Cartesian product $E \times F$. In other words, the elements of G are all the finite sequences

$$(2.1) \qquad (a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_n)$$

where $n \geq 1$ and $a_1, a_2, a_3, \ldots, a_n \in E, b_1, b_2, b_3, \ldots, b_n \in F$. We also include the *empty sequence*, which thus corresponds to $n = 0$.

Step 2 :

We recall that the *monoidal composition* of these sequences is done simply by their concatenation. Furthermore, in order to simplify the notation, the comas in (2.1) will be omitted. It will be convenient to denote the resulting *monoid* by

$$(2.2) \qquad (G, \diamond)$$

Clearly, $(G, \diamond)$ is a *noncommutative* monoid, whenever at least one of the groups E or F has more than one single element. Further, we have the *injective* mapping

$$(2.3) \qquad E \times F \ni (a, b) \longmapsto (a, b) \in G$$

which in fact is the *embedding*

$$(2.4) \qquad E \times F \subseteq G$$

and this embedding is *strict*, whenever $m + n < mn$, where E has at least m elements, while F has at least n elements.

Step 3 :

We define an *equivalence relation* $\approx$ on G as follows. Given two elements

$$g = (a_1, b_1)(a_2, b_2)(a_3, b_3) \ldots (a_n, b_n),$$

$$h = (c_1, d_1)(c_2, d_2)(c_3, d_3) \ldots (c_m, d_m) \in G$$

they are equivalent, if and only if any of the following conditions holds :

$$(2.5) \qquad g = h$$

or one of the elements g or h can be obtained from the other by a finite number of applications of any of the following operations :

(2.6) a permutation of pairs (a, b) in g

(2.7) a permutation of pairs (c, d) in h

(2.8) replacement of a pair $((a' + a''), b)$ in g with the pair of pairs $(a', b)(a'', b)$, or vice-versa

(2.9) replacement of a pair $(a, (b' + b''))$ in g with the pair of pairs $(a, b')(a, b'')$, or vice-versa

(2.10) replacement of a pair $((c' + c''), d)$ in h with the pair of pairs $(c', d)(c'', d)$, or vice-versa

(2.11) replacement of a pair $(c, (d' + d''))$ in h with the pair of pairs $(c, d')(c, d'')$, or vice-versa

where $+$ is the group operation in the respective abelian groups E and F.

It follows easily that $\approx$ is an equivalence relation which is *compatible* with the monoid $(G, \diamond)$.

Step 4 :

Finally, one defines the tensor product as the *quotient space*

(2.12) $$E \bigotimes F \;=\; G / \approx$$

and in view of (2.3), (2.4), obtains the *injective* mapping

(2.13) $$E \times F \ni (a, b) \;\longmapsto\; a \otimes b \in E \bigotimes F$$

where $a \otimes b$ denotes the *coset*, or in other words, the *equivalence class* of $(a, b) \in G$, see (2.4), with respect to the equivalence relation $\approx$ on G.

Step 5 :

Since the equivalence $\approx$ is compatible with the monoid structure of $(G, \diamond)$, and in view of (2.8), (2-9), it follows that the tensor product $E \bigotimes F$ obtains an *abelian group* structure.

3. Comparison with Extended Tensor Products

The essential fact to note is that in the above steps 1 and 2, there is absolutely no need for any structure on the sets E and F, and thus they can be arbitrary nonvoid sets.

Furthermore, in step 3 above, the only place the structure on the sets E and F appears is in (2.8), (2.9). And the way this structure is involved allows for *wide ranging* generalizations, far beyond any algebra, [2-6]. These two facts are at the basis of the possibility to extend the construction of tensor products to a surprising extent beyond that familiar in linear and multi-linear algebra, and in particular, beyond Hilbert spaces, as customary in quantum mechanics.

Here we illustrate the above in the case of the tensor product of two arbitrary sets, while in the next section, that will further be extended to tensor products of infinitely many arbitrary sets.

In order to make the natural way of such an extension easier to grasp, let us note that a usual algebraic operation on a set E, such as for instance, addition or multiplication, is given by a mapping $\alpha : E \times E \longmapsto E$. And such a mapping is a particular case of a ternary relation $\widetilde{\alpha} \subseteq E \times E \times E$, defined by

$$(a, b, c) \in \widetilde{\alpha} \iff \alpha(a, b) = c, \quad a, b, c \in E$$

And clearly, in (2.8)-(2.11) one could also ask that

(3.1) a pair (a, b) in g or h be replaced with the pair of pairs $(a\,', b)(a\,'', b)$, or vice-versa, where $(a\,', a\,'', a) \in \widetilde{\alpha}$

And then, it is easy to go to the further level of extension of the tensor products $E \bigotimes G$. Namely, let E and F be two arbitrary nonvoid sets. Then we proceed with steps 1 and 2 as above. As for step 3, we only modify (2.8) and (2.9) as follows.

Given any m, n-ary relation $\alpha \subseteq E^m \times E^n$, together with any p, q-ary relation $\beta \subseteq F^p \times F^q$. As an extension of (3.1), instead of (2.8)-(2.11) we require

(3.2) replace in g or h any sequence of pairs $(a'_1, b) \ldots (a'_m, b)$ with the sequence of pairs $(a''_1, b) \ldots (a''_n, b)$, where $(a'_1, \ldots, a'_m, a''_1, \ldots, a''_n) \in \alpha$ and $b \in F$

(3.3) replace in g or h any sequence of pairs $(a, b'_1) \ldots (a, b'_p)$ with the sequence of pairs $(a, b''_1) \ldots (a, b''_q)$, where $a \in E$ and $(b'_1, \ldots, b'_p, b''_1, \ldots, b''_q) \in \beta$

However, the extension of the tensor products $E \bigotimes G$ can take place even further. Indeed, instead of a single multi-arity relation α on E, we can take an arbitrary finite or infinite family of them, say, α_i, with $i \in I$. And similarly, we can take an arbitrary finite or infinite family β_j, with $j \in J$, of multi-arity relations on F. And then, instead of (3.2) we can ask a similar operation for each α_i, with $i \in I$. And we can do likewise instead of (3.3) for each operation β_j, with $j \in J$. Furthermore, the arities of α_i and β_j can depend on $i \in I$ and $j \in J$, respectively.

Remarkably, the resulting equivalence relation $\approx$ on G will again be compatible with the monoid $(G, \diamond)$, thus the quotient

(3.4) $$E \bigotimes F \;=\; G/\approx$$

will, in view of (2.6), be a *commutative monoid* as well.

We note that, even if the sets E and F have no any kind of structure assumed on them, the tensor product (3.4) results from the multi-arity relations α_i, with $i \in I$ on E, respectively, β_j, with $j \in J$, on F.

4. Infinite Tensor Products

The above can obviously be further extended to tensor products of finite or infinite families of arbitrary nonvoid sets E_λ, with $\lambda \in \Lambda$, namely

$$(4.1) \qquad \bigotimes_{\lambda\in\Lambda} E_\lambda$$

with the preservation of the *injective* mapping

$$(4.2) \qquad \prod_{\lambda\in\Lambda} E_\lambda \ni (a_\lambda)_{\lambda\in\Lambda} \longmapsto \otimes_{\lambda\in\Lambda} a_\lambda \in \bigotimes_{\lambda\in\Lambda} E_\lambda$$

and thus the motivation for considering the *embedding*

$$(4.3) \qquad \prod_{\lambda\in\Lambda} E_\lambda \subseteq \bigotimes_{\lambda\in\Lambda} E_\lambda$$

with the resulting set

$$(4.4) \qquad \bigotimes_{\lambda\in\Lambda} E_\lambda \setminus \prod_{\lambda\in\Lambda} E_\lambda$$

of *entangled* elements in the composite system whose components are E_λ, with $\lambda \in \Lambda$.

Of course, the way these components are composed is given by the multi-arity relations assumed on each of them, as seen in the particular cases in section 3 above.

5. Non-Cartesian Systems

Needless to say, to the extent that non-cartesian physical systems other than the quantum ones may indeed exist, their composite state spaces may be obtained by constructions other than even the most general tensor products above.

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