PDEs and Symmetry : an Open Problem

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Dedicated to Marie-Louise Nykamp

Abstract

A simple and basic problem is formulated about symmetric partial differential operators. The symmetries considered here are other than Lie symmetries.

1. A Starting Remark

Let

(1.1) $\mathcal{S}(\mathbb{R}^n)$

be the set of all $\mathcal{C}^\infty$-smooth functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ which are symmetric.

We consider $\mathcal{C}^\infty$-smooth partial differential operators of the form

(1.2) $P(x, D)U(x) = F(x, U(x), \ldots, D^p U(x), \ldots), \quad x \in \mathbb{R}^n$

acting on $\mathcal{C}^\infty$-smooth functions $U : \mathbb{R}^n \longrightarrow \mathbb{R}$, where F is a real valued $\mathcal{C}^\infty$-smooth function defined for all real values of all its arguments, while $p \in \mathbb{N}^n$, $|p| \leq m$, for a certain given $m \geq 1$.

We call $P(x, D)$ *symmetric*, if and only if

$$(1.3) \qquad \mathcal{S}(\mathbb{R}^n) \ni U \longmapsto P(x, D)U \in \mathcal{S}(\mathbb{R}^n)$$

Obviously, the partial differentials $D_{x_1}, \ldots, D_{x_n} : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$ are *not* symmetric. On the other hand, simple examples of symmetric partial differential operators are given by

$$(1.4) \qquad D_{x_1}U(x) + \ldots + D_{x_n}U(x) - F(U(x)), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

or by the Poisson operators

$$(1.5) \qquad D^2_{x_1}U(x) + \ldots + D^2_{x_n}U(x) - F(U(x)), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is any $\mathcal{C}^\infty$-smooth function.

One can note that the above symmetries are not of Lie type.

2. A Simple Example

For convenience, let us start by noting a few facts when $n = 2$.

First, as mentioned, the implication does *not* hold

$$(2.1) \qquad f \in \mathcal{S}(\mathbb{R}^2) \implies D_{x_1}f, D_{x_2}f \in \mathcal{S}(\mathbb{R}^2)$$

on the other hand, also as noted, we have

$$(2.2) \qquad f \in \mathcal{S}(\mathbb{R}^2) \implies D_{x_1}f + D_{x_2}f \in \mathcal{S}(\mathbb{R}^2)$$

Let us consider the *converse* of (2.2). Namely, given $g \in \mathcal{S}(\mathbb{R}^2)$, then let us see whether

$$(2.3) \qquad f \in \mathcal{C}^\infty(\mathbb{R}^2), \quad D_{x_1} f + D_{x_2} f = g \quad \Longrightarrow \quad f \in \mathcal{S}(\mathbb{R}^2)$$

As a particular case of (2.3), we recall that, for a given $g \in \mathcal{C}^\infty(\mathbb{R}^2)$, the solution of the PDE

$$(2.4) \qquad D_{x_1} f + D_{x_2} f = g$$

with the initial condition

$$(2.5) \qquad f(x_1, 0) = h(x_1), \qquad x_1 \in \mathbb{R}$$

for a specified $h \in \mathcal{C}^\infty(\mathbb{R})$, is given by

$$(2.6) \quad f(x_1, x_2) = h(x_1 - x_2) + \int_0^{x_2} g(x_1 + (\xi - x_2), \xi) d\xi, \quad (x_1, x_2) \in \mathbb{R}^2$$

Thus

$$(2.7) \quad f(x_2, x_1) = h(x_2 - x_1) + \int_0^{x_1} g(x_2 + (\xi - x_1), \xi) d\xi, \quad (x_1, x_2) \in \mathbb{R}^2$$

and therefore, f is symmetric, if and only if

$$h(x_1 - x_2) + \int_0^{x_2} g(x_1 + (\xi - x_2), \xi) d\xi =$$

$$= h(x_2 - x_1) + \int_0^{x_1} g(x_2 + (\xi - x_1), \xi) d\xi, \qquad (x_1, x_2) \in \mathbb{R}^2$$

or equivalently, if and only if, for $(x_1, x_2) \in \mathbb{R}^2$, we have

$$(2.8) \qquad \int_0^{x_1} g(x_2 + (\xi - x_1), \xi) d\xi - \int_0^{x_2} g(x_1 + (\xi - x_2), \xi) d\xi =$$

$$= h(x_1 - x_2) - h(x_2 - x_1)$$

However, $h \in \mathcal{C}^\infty(\mathbb{R})$ in (2.5) can be arbitrary, and (2.6) will give a corresponding solution $f \in \mathcal{C}^\infty(\mathbb{R}^2)$ of (2.4), (2.5).

Clearly, no matter how $g \in \mathcal{C}^\infty(\mathbb{R}^2)$ is given, there are $h \in \mathcal{C}^\infty(\mathbb{R})$ for which (2.8) need not hold.

Indeed, for given $g \in \mathcal{C}^\infty(\mathbb{R}^2)$, the relation (2.8) implies on $h \in \mathcal{C}^\infty(\mathbb{R})$

the following condition

$$(2.9) \quad h(x)-h(-x) = \int_0^{x_1} g(x_2+(\xi-x_1),\xi)d\xi - \int_0^{x_2} g(x_1+(\xi-x_2),\xi)d\xi$$

where

$$(2.10) \quad x = x_1 - x_2, \quad (x_1, x_2) \in \mathbb{R}^2$$

therefore

$$(2.11) \quad h(x) - h(-x) = \int_0^{(x+x_2)} g(\xi - x, \xi)d\xi - \int_0^{x_2} g(x+\xi, \xi)d\xi$$

and then the issue is whether the right hand term in (2.11) does indeed not depend on x_2.

In this regard we note that the derivative with respect to x_2 of the right hand term in (2.11) is

$$g(x + x_2 - x, x + x_2) - g(x + x_2, x_2)$$

thus it vanishes whenever g is a symmetric function.

Consequently, for every symmetric function $g \in \mathcal{C}^\infty(\mathbb{R}^2)$, the relation (2.11) takes the form

$$(2.12) \quad h(x) - h(-x) = G(x), \quad x \in \mathbb{R}$$

where $G \in \mathcal{C}^\infty(\mathbb{R})$ is defined by g through the right hand term in (2.11).

Clearly, in (2.12), we can choose h arbitrary on $(-\infty, 0)$, provided that it is $\mathcal{C}^\infty$-smooth, and then we obtain on $(0, \infty, 0)$ the $\mathcal{C}^\infty$-smooth function

$$(2.13) \quad h(x) = h(-x) + G(x), \quad x \in (0, \infty)$$

As for $x = 0$, the relation (2.12) gives

(2.14) $\quad G(0) = 0$

and leaves $h(0)$ undetermined. However, (2.12) allows as well the arbitrary $\mathcal{C}^\infty$-smooth choice of h on $(-\infty, 0]$. And then, with (2.13), we obtain h being $\mathcal{C}^\infty$-smooth on $\mathbb{R}$, and satisfying (2.12).

In this way we obtain

Proposition 1

Given $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, then

(2.15) $\quad f \in \mathcal{S}(\mathbb{R}^2) \implies D_{x_1} f + D_{x_2} f \in \mathcal{S}(\mathbb{R}^2)$

while conversely, the relation

(2.16) $\quad D_{x_1} f + D_{x_2} f \in \mathcal{S}(\mathbb{R}^2)$

does *not* imply

(2.17) $\quad f \in \mathcal{S}(\mathbb{R}^2)$

Remark 1

In view of the above, the mappings

(2.18) $\quad \mathcal{S}(\mathbb{R}^2) \ni f \longmapsto D_{x_1} f + D_{x_2} f \in \mathcal{S}(\mathbb{R}^2)$

(2.19) $\quad \mathcal{C}^\infty(\mathbb{R}^2) \ni f \longmapsto D_{x_1} f + D_{x_2} f \in \mathcal{C}^\infty(\mathbb{R}^2)$

are surjective.

3. A Problem

The above motivates the formulation of a general problem.

Problem 1

Given a $\mathcal{C}^\infty$-smooth symmetric partial differential operator $P(x, D)$, with $x \in \mathbb{R}^n$, such that the mapping

$$(3.1) \qquad \mathcal{C}^\infty(\mathbb{R}^n) \ni f \longmapsto P(x, D)f \in \mathcal{C}^\infty(\mathbb{R}^n)$$

is surjective.

Is then the mapping

$$(3.2) \qquad \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto P(x, D)f \in \mathcal{S}(\mathbb{R}^n)$$

also *surjective* ?

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