On Reduction

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Abstract

In this paper I discuss "reduction", a.k.a. "reciprocal addition"; addition conjugated by reciprocal. I discuss reduction's definition, its laws, its graphs, its geometry, its algebra, its calculus, and its practical applications. This paper contains a problem set with answer key.

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1. Definitions and Laws

Define $$a \oplus b = \frac{1}{\frac{1}{a} + \frac{1}{b}}$$

Define $$a \ominus b = a \oplus (-b)$$

$\oplus$ is called "reduction"; it is the reciprocal of the sum of the reciprocals. $\ominus$ is called "protraction"; it is the reciprocal of the difference of the reciprocals. $\oplus$ is also denoted 1/+, and called "reciprocal addition", "harmonic addition" and "parallel addition". $\ominus$ is also denoted 1/-, and called "reciprocal subtraction", "harmonic subtraction" or "parallel subtraction".

$a\oplus b$ is pronounced "a with b", and $a\ominus b$ is "a without b".

The De Morgan Identities:

$$\frac{1}{a \oplus b} = \frac{1}{a} + \frac{1}{b}$$

$$\frac{1}{a \ominus b} = \frac{1}{a} - \frac{1}{b}$$

$$\frac{1}{a + b} = \frac{1}{a} \oplus \frac{1}{b}$$

$$\frac{1}{a - b} = \frac{1}{a} \ominus \frac{1}{b}$$

These follow directly from the definitions by inversion or by substituting reciprocals. I name them after the DeMorgan logic identities which distribute NOT over AND and OR, with alternation.

They imply the *Reciprocal Equations Theorem*:

$a \oplus b = c$ if and only if $1/a + 1/b = 1/c$
$a \ominus b = c$ if and only if $1/a - 1/b = 1/c$
$a + b = c$ if and only if $1/a \oplus 1/b = 1/c$
$a - b = c$ if and only if $1/a \ominus 1/b = 1/c$

Product Over Sum / Product Over Reverse Difference:

$$a \oplus b = \frac{ab}{b + a}$$

$$a \ominus b = \frac{ab}{b - a}$$

These follow directly from the definitions. They imply:

Complementarity:

$$(a+b)(a \oplus b) = ab$$

$$(a-b)(a \ominus b) = -ab$$

$$a + b = \frac{ab}{b \oplus a}$$

$$a - b = \frac{ab}{b \ominus a}$$

Distribution:

$$c(a \oplus b) = ca \oplus cb$$

Proof:

$$c(a \oplus b) = \frac{cab}{b + a} = \frac{ccab}{c(b + a)} = \frac{cacb}{cb + ca} = ca \oplus cb$$

Complementarity and Distribution imply *Triads:*

$$a(a+b) \oplus b(a+b) = ab,$$

$$a(b-a) \ominus b(b-a) = ab,$$

so if $a+b = c$ then $ac \oplus bc = ab$

and if $b-a = d$ then $ad \ominus bd = ab$

Therefore, these sums and differences:

$7+3 = 10$; $9+6 = 15$; $7-3 = 4$; $9-6 = 3$

yield these reductions and protractions:

$70 \oplus 30 = 21$; $135 \oplus 90 = 54$; $12 \ominus 28 = 21$; $18 \ominus 27 = 54$

which yield these sums and differences of reciprocals:

$1/70 + 1/30 = 1/21$; $1/135 + 1/90 = 1/54$;

$1/12 - 1/28 = 1/21$; $1/18 - 1/27 = 1/54$

Field Laws:

$$a \oplus 0 = 0$$

$$a \oplus \infty = a$$

$$a \ominus a = \infty$$

$$a \oplus b = b \oplus a$$

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c$$

$$a\ (b \oplus c) = ab \oplus ac$$

$$a \ominus a \oplus b = b$$

$$(a \oplus b)(c \oplus d) = ac \oplus ad \oplus bc \oplus bd$$

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{ad \oplus bc}{bd}$$

Note that $\oplus$ resembles + , except that its identity element is infinity, not zero. $\oplus$ and + are mirror images of each other, but not identical. For instance, $x \oplus x = x/2$, whereas $x+x = 2x$.

Consider these familiar-looking laws:

Common Denominators:

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

$$\frac{a}{c} \oplus \frac{b}{c} = \frac{a \oplus b}{c}$$

General Denominators (or, Rabbit Rule):

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{ad \oplus bc}{bd}$$

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/__\
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They have these less-familiar looking counterparts:

Common Numerators:

$$\frac{c}{a} + \frac{c}{b} = \frac{c}{a \oplus b}$$

$$\frac{c}{a} \oplus \frac{c}{b} = \frac{c}{a + b}$$

General Numerators (or, Stool Rule):

$$\frac{a}{b} + \frac{c}{d} = \frac{ac}{ad \oplus bc}$$

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{ac}{ad + bc}$$

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The last four rules are the same as the rules for addition, with the roles of numerator and denominator reversed, and addition alternating with reduction. Fractions stand on their heads when they pass through the reciprocal looking-glass.

2. Geometry of Reduction and Protraction

Consider this diagram, the “fish-tail”:

```
      *     *
      |\   /|
      | \ / |
    x |  *  | y
      | /|\ |
      |/ |z\|
      *--*--*
        a  b
```

The line segments x and y need not be equal; they and the z segment are only assumed to be parallel; so the following is a theorem in affine geometry.

Theorem: $x \oplus y = z$

Proof:

By similar triangles:

$$x/(a+b) = z/b$$

$$y/(a+b) = z/a$$

Reducing equals with equals, we get:

$$x/(a+b) \oplus y/(a+b) = z/b \oplus z/a$$

Therefore:

$$(x \oplus y)/(a+b) = z/(b + a)$$

The left side follows from the common denominators law, the right side follows from the common numerators law. Cancelling (a+b), we get:

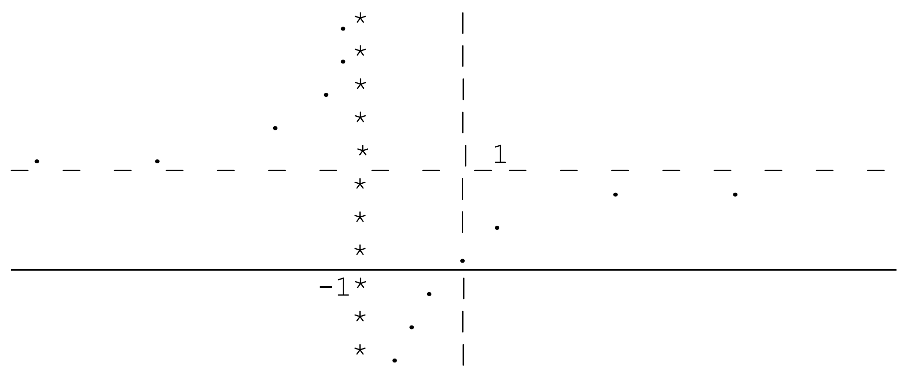
$$x \oplus y = z$$

QED.

From this it follows that $y = z \ominus x$ and $x = z \ominus y$.

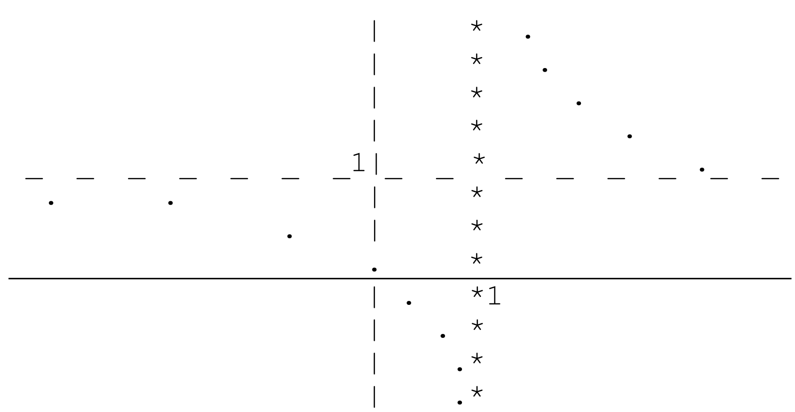
3. Graphs of Reduction and Protraction

Here's a graph of $y = x \oplus 1$:



In general the graph of $y=x\oplus a$ will have a vertical asymptote at $x=(-a)$, a horizontal asymptote at $y=a$, and it will go through (0,0).

Here's a graph of $y = 1 \ominus x$:



In general the graph of $y=a\ominus x$ will have a vertical asymptote at $x=a$, a horizontal asymptote at $y=a$, and it will go through (0,0).

4. Applications of Reduction

The "Many Hands" Theorem:
Assume that $r_1t_1 = r_2t_2 = d$. Then:

$$d/(r_1+r_2) = (t_1 \oplus t_2)$$
$$d/(t_1+t_2) = (r_1 \oplus r_2)$$
$$d/(r_1-r_2) = (t_1 \ominus t_2)$$
$$d/(t_1-t_2) = (r_1 \ominus r_2)$$

In a rate-time problem, when rates add, times reduce; and when rates subtract, times protract. "Many hands make light work."

If Alice can trim the hedges in A hours of work, and Bob can trim the hedges in B hours, then if they work together, and if their work rates add, then they will finish the job in $A \oplus B$ hours.

Two ships cross the same ocean, starting at the same time and heading for each other's home port. Let F be the time that the first ship took to go from port A to port B, let S be the time that the second ship took to go from port B to port A, and let M be the time it took for them to meet in mid-ocean. Approach-rates add, so times reduce: M equals $F \oplus S$.

Two racecars, when driving towards each other from a distance of a mile, pass each other in P seconds; when the faster car pursues the slower car from one mile behind, it takes C seconds to catch up. How quickly does each car run a mile?

Answer: Respectively, $2(P \oplus C)$ and $2(P \ominus C)$ seconds.

If you drive a mile at velocity v_1, and then another mile at velocity v_2, then on average you will go at velocity $2(v_1 \oplus v_2)$; the harmonic mean.

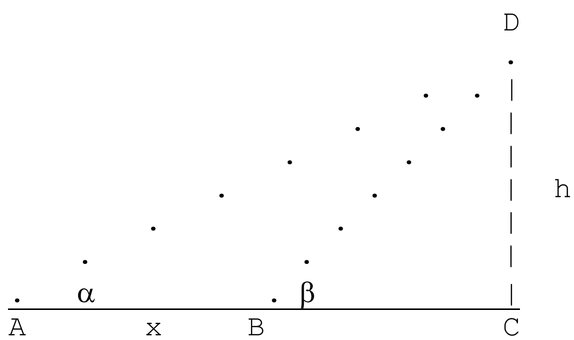
Kirkhoff's Law for parallel resistors goes:

$$R_{12P} = R_1 \oplus R_2 .$$

Resistances reduce in parallel.

Two simple lenses with focal lengths f_1 and f_2, when placed together, form a compound lens with focal length $f_1 \oplus f_2$.

A lens made of material with index of refraction n, and with lens radii r_1 and r_2, has focal length $(r_1 \ominus r_2)/(n-1)$.



In this diagram, $h = x\ (\tan(\alpha) \ominus \tan(\beta))$

5. Linear and Quadratic Reciprocal Algebra

The field laws imply *Transposition*:

If $x \oplus a = b$ then $x = b \ominus a$
If $x \ominus a = b$ then $x = b \oplus a$

Therefore we can solve reciprocal linear equations:

If $ax \oplus b = c$ then $x = (c \ominus b)/a$

We can also solve systems of reciprocal linear equations:

$$\text{If} \quad \begin{aligned} ax \oplus by &= E \\ cx \oplus dy &= F \end{aligned}$$

$$\text{then} \quad x = \frac{Ed \ominus bF}{ad \ominus bc} \quad , \quad y = \frac{aF \ominus Ec}{ad \ominus bc}$$

This is the *Reciprocal Cramer's Rule*, written with protractions rather than subtractions. It suggests a theory of 'reciprocal linear algebra', featuring reductive vectors, reductive matrices, etc.

For instance, solve this system:

$$\begin{aligned} 2x \oplus 5y &= 6 \\ 14x \oplus 10y &= 21 \end{aligned}$$

$$x = \frac{6*10 \ominus 21*5}{2*10 \ominus 14*5} = \frac{60 \ominus 105}{20 \ominus 70} = \frac{140}{28} = 5$$

$$y = \frac{2*21 \ominus 14*6}{2*10 \ominus 14*5} = \frac{42 \ominus 84}{20 \ominus 70} = \frac{84}{28} = 3$$

This *reduction-protraction system*:

$$\begin{aligned} x \oplus y &= a \\ x \ominus y &= b \end{aligned}$$

has the solution: $x = 2(a \oplus b)$, $y = 2(a \ominus b)$

The first is called the "harmonic mean"; so let us call the second the "harmonic radius".

To solve second-degree reductive equations, we need factoring laws, plus the *Infinite Factors Rule*:

$$\text{If } U*V = \infty \quad \text{then} \quad U = \infty \quad \text{or} \quad V = \infty$$

Here is the *pq method:*

$$x^2 \oplus bx \oplus c = (x \oplus p)(x \oplus q)$$

if $pq = c$ and $p \oplus q = b$; and hence $p+q = c/b$.

For example, solve:

$$x^2 \ominus 90x \ominus 90 = \infty$$

We need $pq = -90$ and $p \oplus q = -90$; so $p+q = -90/-90 = 1$

-9 and 10 work, so;

$$(x \ominus 9)(x \oplus 10) = \infty$$

$$(x \ominus 9) = \infty \quad \text{or} \quad (x \oplus 10) = \infty$$

$$x = 9 \quad \text{or } x = -10$$

Here is the *ac method:*

$$ax^2 \oplus bx \oplus c = (ax \oplus p)(ax \oplus q)/a$$

if $pq = ac$ and $p \oplus q = b$; and hence $p+q = ac/b$.

For example, solve:

$$9x^2 \oplus 4x \ominus 4 = \infty$$

We need $pq = 9*-4 = -36$ and $p \oplus q = 4$; so $p+q = -36/4 = -9$

3 and -12 work, so;

$$(9x \ominus 12)(9x \oplus 3)/9 = \infty$$

$$(9x \ominus 12) = \infty \quad \text{or} \quad (9x \oplus 3) = \infty$$

$$X = 12/9 = 4/3 \quad \text{or} \quad x = -3/9 = -1/3$$

Here is *completing the square*:

$$x^2 \oplus bx \oplus 4b^2 = (x \oplus 2b)^2$$

Example: solve

$$3x^2 \oplus 15x \ominus 100 = \infty$$

$$x^2 \oplus 5x \ominus 100/3 = \infty$$

$$x^2 \oplus 5x = 100/3$$

$$x^2 \oplus 5x \oplus 100 = 100/3 \oplus 100$$

$$(x \oplus 10)^2 = 100/3 \oplus 100/1 = 100/(3+1) = 100/4 = 25$$

$$(x \oplus 10) = \pm 5$$

$$x = -10 \oplus/\ominus 5 \quad [\text{ “}\oplus/\ominus\text{“ means “with or without” }]$$

$$x = 10 \text{ or } -10/3$$

If we apply completing the square to

$$ax^2 \oplus bx \oplus c = \infty$$

then we get the *Reductive Quadratic Formula*:

$$x = \frac{-2b \oplus/\ominus \sqrt{(4b^2 \ominus ac)}}{a}$$

To translate this into addition, invert the equation:

$$1/(ax^2) + 1/(bx) + 1/c = 0$$

Multiply throughout by $abcx^2$;

$$abcx^2/(ax^2) + abcx^2/(bx) + abcx^2/c = 0$$

$$bc + acx + abx^2 = 0$$

$$abx^2 + acx + bc = 0$$

Then the Quadratic Formula yields:

$$x = \frac{-ac \pm \sqrt{(a^2c^2 - 4ab^2c)}}{2ab}$$

Addition and reduction combine to make a quadratic equation in this *summation-reduction system*:

$$x + y = S$$
$$x \oplus y = R$$

This system implies:

$$xy = (x + y)(x \oplus y) = SR \;;$$

$$xx + xy = xS \quad ; \quad xx \oplus xy = xR$$

$$x^2 + SR = xS \quad ; \quad x^2 \oplus SR = xR$$

$$x^2 - Sx + SR = 0 \quad ; \quad x^2 \ominus Rx \oplus SR = \infty$$

By the quadratic formulas, these have these solutions:

$$x = \frac{S + \sqrt{(S^2-4RS)}}{2} = 2R \ominus \sqrt{(4R^2 \ominus RS)}$$

$$y = \frac{S - \sqrt{(S^2-4RS)}}{2} = 2R \oplus \sqrt{(4R^2 \ominus RS)}$$

Here is an *alternating-addition equation*:

$$(A \oplus x) + x = 2B$$

It implies:

$$(A+x)(A \oplus x) + (A+x)x = (A+x)2B$$

$$Ax + Ax + x^2 = 2AB + 2Bx$$

$$x^2 + 2Ax - 2Bx = 2AB$$

$$x^2 + 2(A-B)x + (A-B)^2 = 2AB + (A-B)^2$$

$$(x + A-B)^2 = 2AB + A^2 - 2AB + B^2$$

$$(x + A-B)^2 = A^2 + B^2$$

$$x + A-B = \pm \sqrt{(A^2+B^2)}$$

$$x = (B-A) \pm \sqrt{(A^2+B^2)}$$

6. Reciprocal Complex Numbers

Reciprocal complex numbers are of the form $a \oplus ib$, where i equals the square root of negative one. This equals:

$$a \oplus ib = \frac{aib}{a + ib} = \frac{iab(a-ib)}{a^2 + b^2}$$

$$= \frac{ab}{a^2 + b^2} \quad (b + ia)$$

$$= \left(\frac{b}{a} \oplus \frac{a}{b} \right) \quad (b + ia)$$

Conversely,

$$a + ib = \frac{ab}{a^2 \oplus b^2} \quad (b \oplus ia)$$

$$= \left(\frac{b}{a} + \frac{a}{b} \right) \quad (b \oplus ia)$$

They follow the usual rules of complex arithmetic, in reductions:

$$(a \oplus ib) \oplus (c \oplus id) = (a \oplus c) \oplus i(b \oplus d)$$

$$(a \oplus ib) \ominus (c \oplus id) = (a \ominus c) \oplus i(b \ominus d)$$

$$(a \oplus ib) * (c \oplus id) = (ac \ominus bd) \oplus i(ad \oplus bc)$$

$$1 \;/\; (c \oplus id) = (c/(c^2 \oplus d^2)) \ominus i(d/(c^2 \oplus d^2))$$

The reciprocal form of the cis function is $\sec(\alpha) \oplus i \csc(\alpha)$. Call this "sic(α)"; these equations apply:

$$\mathrm{sic}(\alpha) = \sec(\alpha) \oplus i \csc(\alpha)$$

$$= (1/\cos(\alpha)) \oplus (1/(-i\sin(\alpha))$$

$$= 1/ (\cos(\alpha) - i\sin(\alpha))$$

$$= (\cos(\alpha) + i\sin(\alpha))/(\cos^2(\alpha) + \sin^2(\alpha))$$

$$= (\cos(\alpha) + i \sin(\alpha)) = \mathrm{cis}(\alpha)$$

So sic equals cis, and has the same identities:

$$\mathrm{sic}(\alpha+\beta) = \mathrm{sic}(\alpha) * \mathrm{sic}(\beta)$$

$$\mathrm{sic}(\alpha-\beta) = \mathrm{sic}(\alpha) / \mathrm{sic}(\beta)$$

$$\mathrm{sic}(N\alpha) = \mathrm{sic}(\alpha)^N$$

In general $a \oplus ib$ equals

$$\frac{ab}{a^2 + b^2} (b + ia)$$

$$= \frac{ab}{a^2 + b^2} \sqrt{(a^2 + b^2)} \; (\cos(\theta) + i \sin(\theta))$$

$$= \frac{ab}{\sqrt{(a^2+b^2)}} (\cos(\theta) + i \sin(\theta))$$

$$= \sqrt{(a^2 \oplus b^2)} * (\sec(\theta) \oplus i \csc(\theta))$$

- where θ is the angle that $a \oplus ib$ makes with the positive real axis.

From sic and reciprocal-complex algebra we can derive these and other reciprocal-trigonometric identities:

$$\sec^2(\alpha) \oplus \csc^2(\alpha) = 1$$

$$\sec(\alpha+\beta) = \sec(\alpha)*\sec(\beta) \ominus \csc(\alpha)*\csc(\beta)$$

$$\csc(\alpha+\beta) = \sec(\alpha)*\csc(\beta) \oplus \csc(\alpha)*\sec(\beta)$$

$$\sec(2\alpha) = \sec^2(\alpha) \ominus \csc^2(\alpha)$$

$$\csc(2\alpha) = ½ \sec(\alpha)*\csc(\alpha)$$

$$\sec(3\alpha) = \sec^3(\alpha) \ominus (1/3) \sec(\alpha)\csc^2(\alpha)$$

$$\csc(3\alpha) = (1/3) \sec^2(\alpha)*\csc(\alpha) \ominus \csc^3(\alpha)$$

7. When Additions Collide

Addition and reduction are both linear relative to themselves, but curved relative to the other. They're nonlinear when combined. For instance, any quadratic equation is an alternating-addition equation:

$$ax^2 + bx + c = 0 \quad \text{if and only if}$$

$$(x + b/a) \ominus c/b = x \quad \text{if and only if}$$

$$(x \oplus c/b) - b/a = x$$

Conversely,

$$(x + A) \ominus B = x \quad \text{if and only if}$$

$$(x \oplus B) - A = x \quad \text{if and only if}$$

$$x^2 + Ax + AB = 0$$

Alternating addition and reduction yields linear fractional functions:

$$(Ax + B) \oplus C = \frac{ACx + BC}{Ax + (B+C)}$$

$$(Ax \oplus B) + C = \frac{A(B+C)x + BC}{Ax + B}$$

$$\frac{ax + b}{cx + d} = \left(\left(\frac{a^2}{ad-bc}\right) x + \frac{ab}{ad-bc} \right) \oplus \frac{a}{c}$$

$$\frac{ax + b}{cx + d} = \left(\left(\frac{ad-bc}{d^2}\right) x \oplus \frac{ad-bc}{cd} \right) + \frac{b}{d}$$

Continued fractions arise from an infinite alternation of addition and reduction:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d \ldots}}} = a + (1/b \oplus (c + (1/d \oplus \ldots)))$$

We can simplify these series via the *Flipper Identities*:

$a + (b \oplus c) = a + b + (-b \oplus -b^2/c)$

$a \oplus (b + c) = a \oplus b \oplus (-b + -b^2/c)$

I call these "flipper identities" because they flip c upside down.

We can define multiplication, squaring and inverse by alternating addition and reduction.

$((u-v) \ominus (u+v)) = (u^2 - v^2) / (2v)$

$((u \ominus v) - (u \oplus v)) = (2u^2 \ominus 2v^2) / v$

Therefore:

$((a+b-c) \ominus (a+b+c)) = (a^2 + 2ab + b^2 - c^2) / (2c)$

$((a-b-c) \ominus (a-b+c)) = (a^2 - 2ab + b^2 - c^2) / (2c)$

$((a \oplus b \ominus c) - (a \oplus b \oplus c)) = (2a^2 \oplus ab \oplus 2b^2 \ominus 2c^2) / c$

$((a \ominus b \ominus c) - (a \ominus b \oplus c)) = (2a^2 \ominus ab \oplus 2b^2 \ominus 2c^2) / c$

Therefore:

$2ab/c = ((a+b-c) \ominus (a+b+c)) + ((a-b+c) \ominus (a-b-c))$

$ab/2c = ((a \oplus b \ominus c) - (a \oplus b \oplus c)) \oplus ((a \ominus b \oplus c) - (a \ominus b \ominus c))$

$ab = ((a+b-2) \ominus (a+b+2)) + ((a-b+2) \ominus (a-b-2))$

$ab = ((a \oplus b \ominus ½) - (a \oplus b \oplus ½)) \oplus ((a \ominus b \oplus ½) - (a \ominus b \ominus ½))$

$$a/c = ((a+½-c)\ominus(a+½+c)) + ((a-½+c)\ominus(a-½-c))$$

$$a/c = ((a\oplus 2\ominus c)-(a\oplus 2\oplus c)) \oplus ((a\ominus 2\oplus c)-(a\ominus 2\ominus c))$$

$$a^2 = 2((a-1)\ominus(a+1)) + 1 = ((a\ominus 2)-(a\oplus 2)) \oplus 4$$

$$a^2 = ½((a\ominus 1)-(a\oplus 1)) \oplus 1 = ((a-½)\ominus(a+½)) + ¼$$

There are other formulas. For instance:

$$\begin{aligned} -a^2/x &= ((x-a)\oplus a)- a \\ &= ((x+a)\ominus a)+ a \\ &= ((x\ominus a)+a)\ominus a \\ &= ((x\oplus a)-a)\oplus a \end{aligned}$$

$$\begin{aligned} 1/x &= (((-x)-1)\oplus 1)-1 \\ &= (((-x)+1)\ominus 1)+1 \\ &= (((-x)\ominus 1)+1)\ominus 1 \\ &= (((-x)\oplus 1)-1)\oplus 1 \end{aligned}$$

I speculate that any formula defining x*y, x/y or x^2 from the additions must have at least one minus, at least one constant, and be at least three levels deep. Also, a formula defining x*y/z can lack constants, but it must have a minus, and be at least four levels deep.

Consider this iteration:

$$(x_0,\ y_0) = (a,\ b)$$

$$(x_{N+1},\ y_{N+1}) = (\ (x_N+y_N)/2\ ,\ 2(x_N\oplus y_N)\ )$$

The pair takes arithmetic and harmonic means of itself. If a and b are both positive, then the sequence rapidly converges to (√(ab), √(ab)); if a and b are both negative, then the sequence converges to (- √(ab), - √(ab)); and if a and b have opposite signs, then the sequence is chaotic.

Arithmetic mean equals $(x_N+y_N)\oplus(x_N+y_N)$; harmonic mean equals $(x_N\oplus y_N)+(x_N\oplus y_N)$; so addition and reduction, without constants, but with recursion, suffice to generate square roots and chaos.

8. Projective Reduction

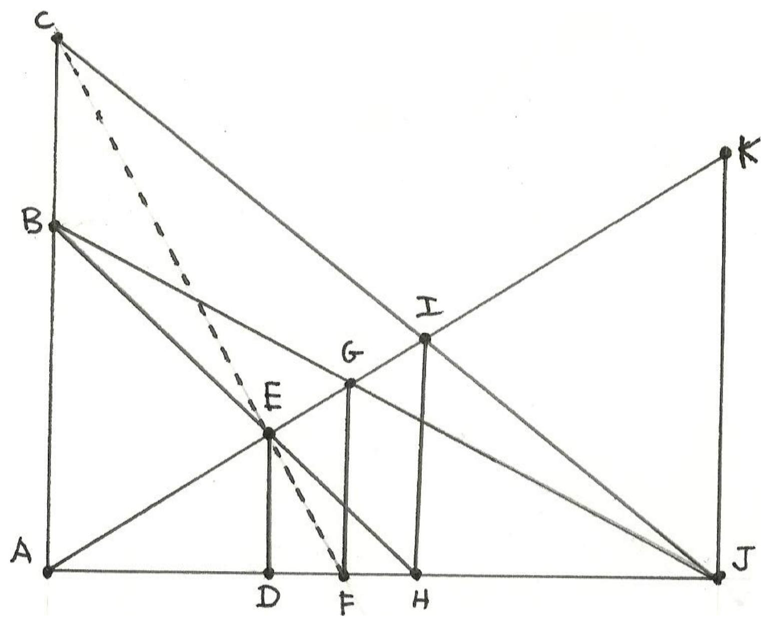
Recall the “fish-tail” diagram:

```
      *     *
      |\   /|
      | \ / |
    x |  *  | y        z  =  x⊕y
      | /|\ |
      |/ |z\|
      *--*--*
```

If you flip this diagram left for right, then the lengths x, y and z remain the same; so x⊕y = y⊕x. So commutativity is illustrated by reflection.

If you stretch it vertically then x, y and z become kx, ky and kz; so k(x⊕y) = kx ⊕ ky. So distribution is illustrated by stretching.

So commutativity and distribution for reduction relate to the symmetries of the fish-tail diagram. The associative law is more complicated. It is illustrated by this diagram:

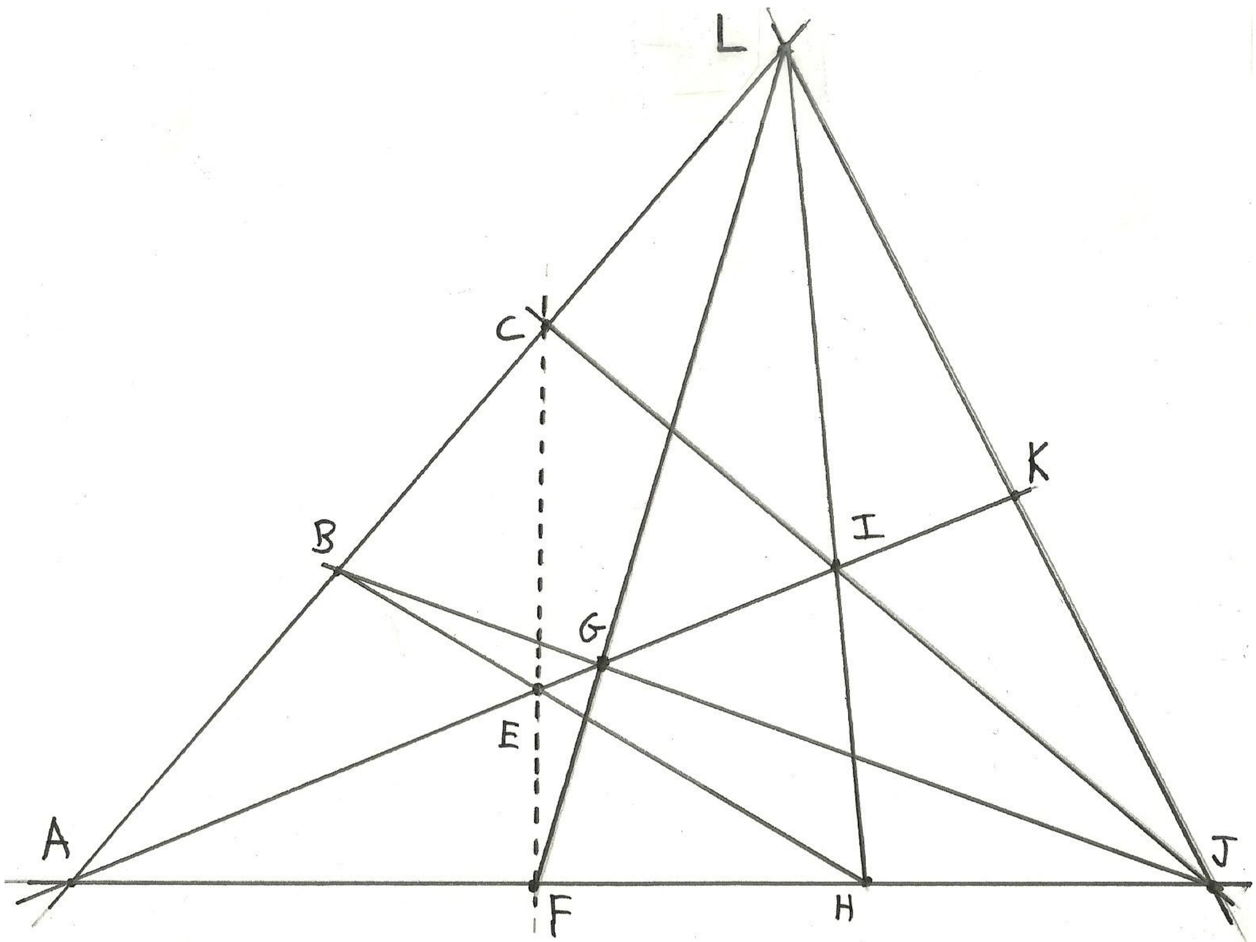


AC, DE, FG, HI and JK are parallel;
ABC, CIJ, BGJ, BEH, AEGIK and ADFHJ are collinear.

If AC = x, JK = y and AB = z, then
IH = x⊕y ; GF = y⊕z ; DE = (x⊕y)⊕z
But (x⊕y)⊕z = x⊕(y⊕z);
So the AC and FG fishtail meets at E;
Therefore C, E and F are collinear.

If AC, DE, FG, IH and JK are parallel;
and ABC, CIJ, BGJ, BEH, AEGIK and ADFHJ are collinear;
then CEF are collinear.

That theorem in affine geometry is equivalent to the associative law for reduction. I therefore call this the Affine Reductive Associative Theorem. Here is a projective version of the diagram, with D omitted, and including L as a point at infinity:



Given these collinearities:
ABCL, FGL, HIL, JKL, CIJ, BGJ, BEH, AEGIK, AFHJ
Then CEF are collinear.

This theorem is a consequence of *Pappus's Hexagon Theorem*: for BCL are collinear points and FHJ are collinear points; therefore the opposite sides of the hexagon BHLFCJ meet in three collinear points: EGI. This proof from Pappus does not assume that the three lines BCL, FHJ and EGI meet at a point, as they do here.

So the associativity of reduction is a consequence of a special case of Pappus's Hexagon Theorem!

9. Reciprocal Calculus

Define a function's "protractional fluxion", or "pluxion", this way:

$$Pf = pf/px = \lim_{y \to x} (f(x) \ominus f(y)) / (x \ominus y)$$

$$= \lim_{H \to \infty} (f(x \oplus H) \ominus f(x)) / H$$

I think Newton's term "fluxion" is a fine one, and it deserves more use; though here it is an abbreviation for "differential fluxion". Where the differential fluxion is the ratio of two infinitesimal differences, the protractional fluxion is the ratio of two infinite protractions. Since the algebraic laws of reduction mirror those of addition, we can derive the following pluxion laws in the usual ways:

$Px = 1$

$PK = \infty$ for any constant K

$P(Kf) = K\,P(f)$ for any constant K

$P(f \oplus g) = P(f) \oplus P(g)$

$P(f \ominus g) = P(f) \ominus P(g)$

$P(f*g) = P(f)*g \oplus f*P(g)$

$P(1/g) = - P(g)/g^2$

$P(f/g) = (P(f)*g \ominus f*P(g)) / g^2$

$P(x^2) = x/2$

$P(x^3) = x^2/3$

And in general; $P(x^N) = x^{N-1}/N$

So we can protractionate reductive polynomials. For instance:

$$P(7\, x^3 \oplus 5x^2 \ominus 3x \oplus 11) = (7/3)x^2 \oplus (5/2)x \ominus 3$$

Pluxions have the chain rule:

$$pz/px = pz/py * py/px$$

$$P(f(g(x))) = ((Pf)(g(x))) * (Pg)(x)$$

Recall the Complementarity rule;

$$(a-b)(a\ominus b) = -ab;$$

protraction times difference equals negative product.

Apply this to the product of pluxion and fluxion:

For y near to x, (pf/px)*(df/dx) approximately equals:

$$\{(f(x)\ominus f(y))/ (x\ominus y) \}*\{(f(x)-f(y))/ (x-y) \}$$

$$= \{(f(x)\ominus f(y))*(f(x)-f(y))\} / \{(x\ominus y)*(x-y)\}$$

$$= \{ - f(x) * f(y) \} / \{ - x * y \}$$

Which in the limit approaches

$$(f(x))^2 / x^2$$

Therefore in general:

$$(pf/px)*(df/dx) = (f(x))^2 / x^2$$

$$Pf * f' = f^2/x^2$$

Therefore the *Pluxion from Fluxion* law:

$$pf/px = (f(x))^2 / (x^2 * df/dx)$$

$$Pf = f^2 /(x^2 \cdot f')$$

In particular:

$$P(e^x) = e^x /x^2$$

$$P(e^{-1/x}) = e^{-1/x}$$

$$P(\ln(x)) = (\ln(x))^2 / x$$

$$P(-1/\ln(x)) = 1 / x$$

We can now derive P(f+g):

$$P(f+g) = (f+g)^2 / (x^2(f+g)')$$

$$= (f+g)^2 / (x^2(f'+g'))$$

$$= (f+g)^2(1/(x^2f') \oplus 1/(x^2g'))$$

$$= (f+g)^2 (Pf/f^2 \oplus Pg/g^2)$$

Similarly;

$$(f\oplus g)' = (f\oplus g)^2 (f'/f^2 + g'/g^2)$$

For instance;

$$(x\oplus x^2)' = (x\oplus x^2)^2 (x'/x^2 + x^{2\prime}/(x^2)^2)$$

$$= x^2(1\oplus x)^2 (1/x^2 + 2x/x^4)$$

$$= (x^2 * x^2/(x+1)^2)(1/x^2 + 2/x^3)$$

$$= (x^4/(x+1)^2)((x + 2) / x^3)$$

$$= x(x+2)/(x+1)^2$$

$$P(x+x^2) = (x+x^2)^2 (Px/x^2 \oplus Px^2/(x^2)^2)$$

$$= x^2(1+x)^2 (1/x^2 \oplus (x/2)/x^4)$$

$$= x^2(1+x)^2 (1/x^2 \oplus (1/2x^3)$$

$$= (1+x)^2 (1 \oplus (1/2x))$$

$$= (1+x)^2 (1 / (1+2x))$$

$$= (1+x)^2 / (1+2x)$$

For longer sums we have:

$$P(f+g+h) = (f+g+h)^2 (Pf/f^2 \oplus Pg/g^2 \oplus Ph/h^2)$$

$$(f\oplus g\oplus h)' = (f\oplus g\oplus h)^2 (f'/f^2 + g'/g^2 + h'/h^2)$$

And so on.

One of the uses of the fluxion is in Newton's Method for finding the roots of a function. To solve $f(x) = 0$, start with an estimate x_1, then run this iteration:

$$X_{N+1} = X_N - f(X_N)/f'(X_N)$$

But $\quad Pf = f^2 / (x^2 \cdot f')$

so $\quad f/f' = x^2 Pf / f$

and therefore

$$X_{N+1} = X_N - X_N^2 P(X_N)/f(X_N)$$

$$X_{N+1} = X_N (1 - X_N*Pf(X_N)/f(X_N))$$

This is the pluxion version of Newton's Method.

Another use of the fluxion is finding extrema. But since $Pf*f' = f^2/x^2$, $f'>0$ only if $Pf>0$; $f'=0$ only if $Pf=\infty$; and $f'<0$ only if $Pf<0$. So you may find critical points by solving $Pf(x)= \infty$; then examine Pf near the critical point to tell if it is an extremum or not. If Pf goes from – through ∞ to +, then the critical point is a minimum; if Pf goes from + through ∞ to – then the critical point is a maximum. This is the First Pluxion Test.

Inverse to the Pluxion is the *Indefinite Reciprocal Integral*, also called the *Anti-pluxion*:

$$R(f(x) px) = F(x) \oplus C \quad \text{if and only if} \quad P(F(x)) = f(x)$$

C is the constant of reciprocal integration.

There's also a Definite Reciprocal Integral:

$$R_a^b(f(x)\ px) =$$

$$\lim_{H\to\infty} (f(a)\oplus f(a\oplus H)\oplus f(a\oplus H/2)\oplus f(a\oplus H/3)\oplus \ldots \oplus f(b))* H)$$

These two are united by the *Fundamental Theorem of Reciprocal Calculus:*

If $F = R_a^x(f(y)\ py)$ then $pF/px = f$

If $pF/px = f$ then $R_a^b(f(x)\ px) = F(b) \ominus F(a)$

Here is a short table of reciprocal integrals:

$R(\ 1\ px\ ) = x$

$R(\ \infty\ px\ ) = C$, a constant.

$R(K*f\ px\ ) = K * R(f\ px)$, for any constant K.

$R((f \oplus g)px\ ) = R(f\ px) \oplus R(g\ px)$

$R((f \ominus g)px\ ) = R(f\ px) \ominus R(g\ px)$

$R(\ x^N\ px\ ) = (N+1)\ x^{N+1}$

$R(\ (1/x)\ px\ ) = -1 / \ln(x) = \log_x(1/e)$

$R(\ e^{-1/x}\ px\ ) = e^{-1/x}$

Here's the Chain Rule:

$R(\ f(g(x))\ Pg(x)\ px\ ) = (Rf)(g(x))$

Here's Reductive Integration by Parts:

$R(\ Pf * g * px\ ) = f*g \ominus R(\ f * Pg * px\ )$

$R\ (\ g * pf\ ) = f*g \ominus R(\ f * pg\ )$

10. Logistic

Zero is a problem for reduction, just as infinity is a problem for addition. By the Rabbit Rule;

$$\infty + \infty = 1/0 + 1/0 = (0*1+1*0)/(0*0) = 0/0,$$

the indefinite ratio; and by the Stool Rule;

$$0 \oplus 0 = 0/1 \oplus 0/1 = (0*0)/(0*1+1*0) = 0/0.$$

This makes sense if infinity and zero are unsigned, for then

$$1/0 + 1/0 = 1/0 - 1/0 = 0/0 \text{ , by common denominators;}$$
$$0/1 \oplus 0/1 = 0/1 \ominus 0/1 = 0/0 \text{ , by common numerators.}$$

If we give signs to infinity and zero, then we can reduce some of the indefiniteness:

$$+\infty + +\infty = +\infty$$
$$-\infty + -\infty = -\infty$$
$$+\infty + -\infty = 0/0$$
$$+0 \oplus +0 = +0$$
$$-0 \oplus -0 = -0$$
$$+0 \oplus -0 = 0/0$$

That implies these tables:

+	+0	+∞
+0	0	∞
+∞	∞	∞

⊕	+0	+∞
+0	0	0
+∞	0	∞

x	1/x
+0	∞
+∞	0

This is isomorphic to Boolean logic, under this matching:

0 --------------- true

∞ --------------- false

\+ --------------- and

⊕ --------------- or

1/x --------------- not

Let F be the generic positive finite quotient. (So F = n/m, with n and m both positive integers.) Then we get these tables:

+	0	F	∞
0	0	F	∞
F	F	F	∞
∞	∞	∞	∞

⊕	0	F	∞
0	0	0	0
F	0	F	F
∞	0	F	∞

x	1/x
0	∞
F	F
∞	0

This duplicates 3-valued Kleenean logic, under this matching:

0 ------------ true

F ------------ intermediate

∞ ------------ false

\+ ------------ and

⊕ ------------ or

1/x ------------ not

Addition, reduction and reciprocal are a fragment of arithmetic; when applied to {0, finite, ∞}, they are isomorphic to Kleenean logic; when applied to the extended positives, they resemble a fuzzy logic.

Under this matching, the DeMorgan laws:

not(x and y) = (not x) or (not y)

not(x or y) = (not x) and (not y)

have these arithmetical counterparts:

$$1/(x + y) = (1/x) \oplus (1/y)$$

$$1/(x \oplus y) = (1/x) + (1/y)$$

Therefore the name, "De Morgan distribution".

Define the “Mediant” operator $(x@y@z)$ this way:

$$x @ y @ z = (x*y*z)/((x+y+z)(x\oplus y\oplus z))$$

This implies:

$$x*y*z = (x+y+z)(x@y@z)(x\oplus y\oplus z)$$

$$x@y@z = (xy+yz+zx)/(x+y+z)$$

$$x@y@z = (xy\oplus yz\oplus zx)/(x\oplus y\oplus z)$$

$$w*(x@y@z) = (w*x)@(w*y)@(w*z)$$

$$1/(x@y@z) = (1/x)@(1/y)@(1/z)$$

$$x@0@y = x \oplus y$$

$$x@\infty@y = x + y$$

The last three laws show that the mediant corresponds to the majority operator in logic, M(x,y,z); since

Not(M(x,y,z)) = M(not x, not y, not z)

M(x,T,y) = x or y

M(x,F,y) = x and y

Now consider the “equivalence” operator:

x iff y = (x or not y) and (y or not x)

= (x and y) or (not x and not y)

It has this arithmetical counterpart:

$$(x \oplus 1/y) + (y \oplus 1/x)$$

$$= (x + y) \oplus (1/x + 1/y)$$

$$= (1\oplus xy)/(x\oplus y)$$

$$= (x+y)/(1+xy)$$

- otherwise known as relativistic velocity addition! Denote it by “$x\leftrightarrow y$”, and call it “equivalence”, or “Einstein addition”. Its laws are:

$$x \leftrightarrow 0 = x$$

$$x \leftrightarrow -x = 0$$

$$x \leftrightarrow y = y \leftrightarrow x$$

$$(x \leftrightarrow y) \leftrightarrow z = x \leftrightarrow (y \leftrightarrow z) = x \leftrightarrow y \leftrightarrow z$$

$$x \leftrightarrow y \leftrightarrow z = (x+y+z+xyz)/(xy+yz+zx+1)$$

$$x \leftrightarrow y \leftrightarrow z = (x \oplus y \oplus z \oplus xyz)/(xy \oplus yz \oplus zx \oplus 1)$$

$$x \leftrightarrow \infty = 1/x$$

$$x \leftrightarrow -1/x = \infty$$

$$\infty \leftrightarrow \infty = 0$$

$$1/(x \leftrightarrow y) = (1/x) \leftrightarrow y = x \leftrightarrow (1/y) = x \leftrightarrow y \leftrightarrow \infty$$

$$1/(x \leftrightarrow y) = (x \oplus y) + (1/x \oplus 1/y) = (x + 1/y) \oplus (1/x + y)$$

$$1/(x \leftrightarrow y) = (x \oplus y)/(1 \oplus xy) = (1+xy)/(x+y)$$

$$(1/x) \leftrightarrow (1/y) = x \leftrightarrow y$$

$$-(x \leftrightarrow y) = (-x) \leftrightarrow (-y)$$

$$-(x \leftrightarrow y \leftrightarrow z) = (-x) \leftrightarrow (-y) \leftrightarrow (-z)$$

$$1/(x \leftrightarrow y \leftrightarrow z) = (1/x) \leftrightarrow (1/y) \leftrightarrow (1/z)$$

$$x \leftrightarrow 1 = 1$$

$$x \leftrightarrow -1 = -1$$

$$1 \leftrightarrow -1 = 0/0$$

$$(A \leftrightarrow x) = B \quad \text{if and only if} \quad x = (-A \leftrightarrow B)$$

$$\tanh(x) \leftrightarrow \tanh(y) = \tanh\,(x + y)$$

$$x \leftrightarrow y = \tanh\,(\tanh^{-1} x + \tanh^{-1} y)$$

So Einstein addition is addition conjugated by hyperbolic tangent.

Problem Set: Elementary Reduction

#1. Find in lowest terms:

$6/5 \oplus 8/3 = ?$

$11/3 \oplus 11/4 = ?$

$3/11 \oplus 4/11 = ?$

$6/7 \ominus 9/5 = ?$

#2. $3\oplus 7=?$, $52\oplus 117=?$, $70\oplus 126=?$, $2\oplus 2=?$

#3. $5\ominus 15=?$, $21\ominus 70=?$, $3\ominus 5=?$, $28\ominus 44=?$

#4. $a+b=c$ implies $ac\oplus bc = ab$. Use this to create three reductions involving integers only. (E.g. $1+2=3$, so $3\oplus 6=2$.)

#5. Prove: $(a+b)(c\oplus (bc/a)) = (a\oplus b)(c+(bc/a))$

#6. Assuming that their work rates add, how long would Alice and Bob, working together, take to trim the hedges, if separately:

a) Alice takes 45 minutes, and Bob takes 36 minutes.

b) Alice takes 17 minutes, and Bob takes *infinite* time.

c) Alice takes 23 minutes, and Bob takes no time at all.

#7. Two ships cross the same ocean, starting at the same time and heading for each other's home port. Let F be the time that the first ship took to go from port A to port B, let S be the time that the second ship took to go from port B to port A, and let M be the time it took for them to meet in mid-ocean.

What is M, if:

a) $T = 60$ and $F = 84$

b) $T = 60$ and $F = \infty$ (This ship is stuck in port.)

c) $T = 60$ and $F = 0$ (This ship is a *starship*.)

#8. You drive a mile at velocity v_1, and then another mile at velocity v_2; on average you go at velocity $\underline{V}$.

a) What is $\underline{V}$ if $v_1 = 55$ and $v_2 = 66$?

b) What is v_2 if $\underline{V} = 70$ and $v_1 = 60$?

#9. Two racecars, when driving towards each other from a distance of a mile, pass each other in 12 seconds; when the faster car pursues the slower car from one mile behind, it takes 60 seconds to catch up. How quickly does each car run a mile?

#10. Resistances R_1 and R_2 are wired in parallel, and their combined resistance is R_{12P}.

a) What is R_{12P} if $R_1 = 90$ and $R_2 = 10$?

b) What is R_{12P} if $R_1 = 2$ and $R_2 = \infty$? (A break.)

c) What is R_{12P} if $R_1 = 100$ and $R_2 = 0$? (A short.)

d) What is R_1 if $R_{12P} = 20$ and $R_2 = 45$?

#11. Graph: $y = 2 \oplus x$

#12. Graph: $y = 2 \ominus x$

#13.

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  |\   /|
  | \ / |
x |  *  | y
  | /|\ |
  |/ |z\|
  *--*--*
```

a) What is z when x=680 and y=920?

b) What is x when z=546 and y=2310?

#14. Solve:

a) $7x\oplus2 = 5$

b) $4x\ominus4 = 5x\oplus8$

#15. Solve:

$$33x \oplus 42y = 110$$
$$48x \oplus 21y = 80$$

#16. Solve:

$$x\oplus y = 12$$
$$x\ominus y = 84$$

#17. Solve: $x^2 \oplus 6x \oplus 150 = \infty$.

#18. Solve:

$$x + y = 144$$
$$x \oplus y = 35$$

#19. Solve: $(3 \oplus x) + x = 8$

#20. $(x\oplus1)^3 = ?$ (Expand using $\oplus$ only, no +)

Elementary Reduction Answer Key

#1. 24/29, 11/7, 12/77, 18/11

#2. 21/10, 36, 45, 1

#3. 15/2, 30, 15/2, 77

#4. This is up to the student.

#5. $(a+b)(c\oplus(bc/a)) = (a+b)(a\oplus b)(c/a) = ab(c/a) = bc$
$(a\oplus b)(c+(bc/a)) = (a\oplus b)(a+b)(c/a) = ab(c/a) = bc$

#6. a: $45 \oplus 36 = 20$ minutes
b: $17 \oplus \infty = 17$ minutes
c: $23 \oplus 0 = 0$ minutes

#7. a) M = 35
b) M = 60
c) M = 0

#8. a) 60
b) 84

#9. 20 seconds and 30 seconds.

#10. $R_{12P} = 9$
$R_{12P} = 2$
$R_{12P} = 0$
$R_1 = 36$

#11.

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                   .*     |
                  . *     |
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#12.

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                             |    .*
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#13. a) 391
     b) 715

#14. a) $x = (5 \ominus 2)/7 = -10/21$
     b) $x = (8 \oplus 4)/(4 \ominus 5) = 2/15$

#15. $X = 7$, $y = 5$

#16. $x = 2(12 \oplus 84) = 21$
     $y = 2(12 \ominus 84) = 28$

#17. $x = -15$ or -10

#18. $x = 84$, $y = 60$

#19. $x = 6$ or -4

#20. $x^3 \oplus x^2/3 \oplus x/3 \oplus 1$