On Smarandache Rings II

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Abstract

In this paper we show that a commutative semisimple ring is always a Smarandache ring. We will also give a necessary and sufficient condition for group algebra to be a Smarandache ring. Examples are provided for justification.

Key Words: Group, ring, field, semisimple ring, group algebra, nilpotent element, Smarandache ring

1. Introduction

Smarandache notions which can be undoubtedly characterized as revolutionary mathematics, have the capacity of being utilized to analyze, study and introduce, naturally, the concepts of seven structures by means of extension or identification as a substructures. A particular case of Smarandache notions, an excellent means to study local properties in rings, is Smarandache ring [11].

The notion of semisimple ring is introduced in [7] and [8]. Semisimple rings have played prominent role in the development of structure theory of rings. There are several open questions concerning semisimplicity and group algebra of any group over a field.

Group algebras were introduced by G.Frobenius and I.Schur[4] in connection with the study of group representations, since studying the representations of a group G over a field K; is equivalent to studying modules over the group algebra KG. Thus, Maschke's theorem about semisimplicity (stated in this paper) is formulated in the language of group algebras. In the early 1950s group algebras of infinite groups were studied in the context of integer group algebras in the algebraic topology and for investigation of structure groups. This was also promoted by a number of problems on group algebras, the best known of which is Kaplansky's problem.

The purpose of this paper is to show that a commutative semisimple ring is always a Smarandache ring. Then the Theorem 3.9 in [10] which states that the ring R in which for every element $x \in R$ there exists a (and hence the smallest) natural number $n(x)>1$ such that $x^{n(x)} = x$ is always a Smarandache ring, is a corollary of our result. We will also give a necessary and sufficient condition for group algebra to be a Smarandache ring.

In section 2 we give basic concepts, definitions and theorems of structure theory of rings. In section 3 we give our results. In section 4 we give examples to justify our results. For more details about fundamental concepts please refer [7],[8]and [9]. In this paper ring means ring with unit.

2. Preliminaries

In this section we give some definitions and theorems of structure theory of rings.

Definition 2.1.([5]): The intersection of all prime ideals of a ring R is called a prime radical of R.

Definition 2.2([5]). An element r in the ring R is called nilpotent if $r^n = 0$, for some natural number n.

Proposition 2.3([5]). A prime radical of a commutative ring R consists of all nilpotent elements of R.

Definition 2.4([5]). A ring R is said to be Semisimple if its prime radical is 0, that is , if it has no nonzero nilpotent element.

In other words, for every $x \in R$, $x^2 = 0$ if and only if $x = 0$. A commutative ring R which is semisimple is called Commutative semisimple ring.

The simple example of semisimple ring is the ring J of integers, Since, if p is a prime the principal ideal (p) is primitive in J and $\cap_p (p)=\{0\}$. More examples of commutative semisimple rings can be found in [5],[7]and [8].

Definition 2.5. Let $G=\{g_i / i \in I\}$ be any multiplicative group and let K be any field. Let KG be the set of all formal sums $\Sigma a_i g_i$, where $i \in I$ for $a_i \in K$ and $g_i \in G$, where all but a finite number of the a_i are 0. Define the sum of two elements of KG by

$(\Sigma a_i g_i) + (\Sigma b_i g_i) = \Sigma (a_i+b_i) g_i$, where $i \in I$; $a_i+b_i = 0$ except for a finite number of indices i. Multiplication of two elements of KG is defined by the use of the multiplication in G and K as

$(\Sigma a_i g_i)(\Sigma b_i g_i) = (\Sigma (\Sigma a_j b_k) g_i$, where $i \in I$; $g_j g_k = g_i$, and at most a finite number of the sums $(\Sigma a_j b_k)$, where $g_j g_k = g_i$, are nonzero. If KG is a ring then KG is called a group algebra of G over K.

From the definition, it follows that the additive identity element in KG is $\Sigma 0 g_i$, where $i \in I$, and multiplicative identity in KG is $1e_G$ (see [9])) where 1 is the identity of K and e is the identity of G. Clearly, the identity element of G is the unit of KG and KG is commutative if and only if G is an Abelian group. In the group algebra KG the elements of G form a basis for this algebra as $G \subset KG$. For examples of group algebras please see [6],[8] and [9].

Definition 2.6([1]). A Smarandache ring (in short S-ring) is defined to be a ring A such that a proper subset of A is a field with respect to the operations induced. By a proper subset we understand a set included in A different from empty set, from the unit element , if any, and from A. For examples of Smarandache rings please see[11].

Theorem 2.7(Maschke[8]) . Let G be a finite group of order n and let K be a field whose Characteristic does not divide n, then the group algebra KG is semisimple.

Theorem 2.8. If R is a finite ring and has no nonzero nilpotent element then R is commutative.

3. Proofs of the Theorems.

In this section we show that a commutative semisimple ring R is always a Smarandache ring. We will also give a necessary and sufficient condition for a group algebra to be a Smarandache ring. For completeness, we write some definitions and lemmas from [2].

Lemma 3.1. Let R be a commutative semisimple ring. The ring R is partially ordered by $\leq$ where for every element x and y of R ,

$x \leq y$ if and only if $xy = x^2$ ……..(i)

Proof : Since $xx = x^2$, it follows from (i) that $x \leq x$. Thus , $\leq$ is reflexive

Moreover, if $x \leq y$ and $y \leq x$ then $xy=x^2$ and $yx = y^2$ so that $x^2 – xy – yx + y^2 = (x – y)^2 = 0$. But, then $x – y = 0$ as R has no nonzero nilpotent element. Thus, $x = y$ and therefore , $\leq$ is antisymmetric .

Further more, if $x \leq y$ and $y \leq z$ then

$xy = x^2$ and $yz = y^2$ so that $x^2 z = xyz = xy^2 = x^2y = x^3$. Thus , $x^2z^2=x^3z$ and $x^3z = x^4$ so that $x^2z^2 - x^3 z - x^3z + x^4 = 0$ or $(xz – x^2)^2 = 0$. But, then $xz - x^2 = 0$ or $xz = x^2$ as R has no nonzero nilpotent element. Hence, $x \leq z$ by (i) and therefore, $\leq$ is transitive. Thus, $\leq$ is a partial order and further $(R , \leq)$ is a partially ordered (p.o) set.

Let us observe that from (i) it follows immediately that for every element x, y of a commutative semisimple ring R $x \leq y$ implies $xz \leq yz$ ….(ii)

and $x^2 = x$ implies $xy \leq y$ ……(iii)

Definition 3.2: A nonzero element a of a commutative semisimple ring R is called a hyperatom of R if and only if

for every element x of R $x \leq a$ implies $x = 0$ or $x = a$ …..(iv) and

$ax \neq 0$ implies $a x s = a$ for some element s of R ……..(v)

Next , we prove

Lemma 3.3: Let ‘a’ be a hyperatom of a commutative semisimple ring R. For every element r of R if

$ar \neq 0$ then ar is a hyperatom of R.

Proof : Let $ar \neq 0$. We show that ar is a hyperatom according to the Definition 3.2 .

Since $ar \neq 0$, by (v) we have $ars = a$ for some element s of R. Now, let $x \leq ar$ then from (ii) it follows that $xs \leq ars$. Hence, $xs \leq a$ and in view of (iv) we have $xs = 0$ or $xs = a$

However , $x \leq ar$ so that $arx = x^2$ and therefore $rsx^2 = rs\,(arx) = (rsa)\,rx = arx = x^2$. Consequently, $(rsx - x)^2 = (rs)^2x^2 - 2rsx^2 + x^2 = 0$. Thus, $rsx = x$ as R has no nilpotent element and in view of the above implies that $x = 0$ or $x = ar$. Hence, ar satisfies (iv).

On the other hand, if $arx \neq 0$ then there exists an element t of R such that $arxt = a$. Thus, $(arx)\,tr = ar$, so that ar satisfies (v). In view of the above two cases, we see that ar is a hyperatom of R as desired.

Lemma 3.4 . Let ‘a’ be a hyperatom of commutative semisimple ring R, then there exists an element s of R such that as is an idempotent hyperatom of R.

Proof : Since $a \neq 0$, it follows that $a^2 \neq 0$ as R has no nilpotent element. Thus, by (v) there exists an element s of R such that $a^2 s = a$. Clearly, $as \neq 0$ and therefore as is an hyperatom by Lemma 3.3. But, also, $(as)^2=(a^2s)s = as$. Thus, as is an idempotent hyperatom of R.

Definition 3.5 : A subset S of a commutative semisimple ring R is called orthogonal if and only if xy = 0 for every two distinct elements x and y of S.

Lemma 3.6 : The set (e_i), $i \in I$ of all idempotent hyperatoms of a commutative semisimple ring R is an orthogonal set.

Proof : Let e_i and e_j be idempotent hyperatoms of a commutative semisimple ring R. From (iii) it follows that $e_ie_j \leq e_i$ and $e_ie_j \leq e_j$ so that $e_ie_j= e_i=e_j$ or $e_ie_j = 0$.

Lemma 3.7 Let (e_i), $i\in I$, be the set of all idempotent hyperatoms of a commutative semisimple ring R. Then for every i of I the ideal $F_i =\{re_i / r \in R\}$ is a subfield of R

Proof : Since $e_i^2 = e_i$, it follows that e_i is an element of F_i and for every element r of R, we have $(re_i)e_i=re_i$ so that e_i is the unit of F_i. Further, if $re_i \neq 0$, then since e_i is a hyperatom there exists an element s of R such that $(se_i)(re_i) = sre_i = e_i$. Thus, each non zero element of F_i has an inverse in F_i so that F_i is a field.

Now, we are ready to prove our result.

Theorem 3.8. A commutative semisimple ring R is always a Smarandache ring.

Proof : Let (e_i), $i \in I$ be the set of all idempotent hyperatoms of R. Then, in view of the preceding Lemma 3.7, for every i of I, the ideal $F_i =\{re_i / r \in R\}$ is a field of R. Hence, the ring R is a Smarandache ring.

Corollary 3.9 (Theorem 3.9 in [10]) . The ring R is which for every element $x \in R$ there exists a (and hence the smallest) natural number n (x) > 1 such that $x^{n(x)} = x$ is always a Smarandache ring.

Proof : In [7] , it is known that the ring R in which for every element $x \in R$ there exists a (and hence the smallest) natural number n(x) > 1 such that $x^{n(x)} = x$ is commutative and has no nonzero nilpotent element. Hence, this ring R is a commutative semisimple ring. Thus, in view of Theorem 3.8 we get that this ring R is a Smarandache ring as desired.

Corollary 3.10. If R is a finite ring with no nonzero nilpotent element then R is a Smarandache ring.

Proof : In structure theory of rings we have an elegant Theorem 2.8 which states that if R is a finite ring with no nonzero nilpotent element then R is commutative. Hence, R is a commutative semisimple ring. Thus, in view of the Theorem 3.8 , we get that ring R is Smarandache ring.

Next , we give a necessary and sufficient condition for a group algebra to be a Smarandache ring.

Theorem 3.11 . Let KG be a group algebra, where K is a field with multiplicative identity 1 and G is a multiplicative group with identity e. Then, KG is a Smarandache ring if and only if $| K | \geq 3$, where | K | denotes the cardinality of the field K.

Proof . If KG is a Smarandache ring then there exists a proper subset F = { 0 , 1e } of KG such that F is a field under the operations defined on the group algebra KG. In view of Definition 2.5 , F must contain an element a_1e , where $a_1 \in K$, other than 0 and 1e. so that F is a field of KG . Thus , $| K | \geq 3$.

On the other hand, since | K | $\geq$ 3 , let K = { 0,1,$a_{1,}$ … }be the field.

Then, take F = { 0e, 1e, a_1e , . . . } = { 0,1e,a_1, . . . } , clearly , F is a proper subset of KG and F is field under the operations of the group algebra KG. Hence, the group algebra KG is a Smarandache ring.

Observation 3.12. In view the Theorem 3.11 it is evident that the statement and proof of Theorem 3.1.7 , the proof of Theorem 3.1.9 in [11] should be revised. We will also provide an example for justification in the following section.

4. Examples

In this section we give examples to justify our results and observation 3.12

Example 4.1 . In view of the corollary 3.9 , the examples quoted in [10] to justify the Theorem 3.9 in [10] also justify our Theorem 3.8

Next , we show by example that the condition Semisimplicity on ring R in Theorem 3.8 is sufficient condition but not a necessary condition. In view of the following examples, it is, further , observed that , if R is a Commutative ring with nonzero nilpotent element then the ring R may or may not be a Smarandache ring.

Example 4.2 . Consider the ring Z_{12} = { 0,1,2,3,4,5,6,7,8,9,10,11}. This ring Z_{12} is a Smarandache ring as the proper subset A = {0,4,8,} is a field with 4 acting as a unit element. But, the ring Z_{12} has a nonzero nilpotent element 6 as $6^2 = 0$.

Example 4.3 .Let G = { e,a} be a cyclic group of order 2 and let Z_2 = { 0,1} be a field of characteristic 2. Z_2G ={0,a,e,e+a } is a group algebra with respect to the operations defined by table 1 and table 2

+	0	a	e	e + a
0	0	a	e	e + a
a	a	0	e + a	e
e	e	e + a	0	a
e + a	e + a	e	a	0

Table 1

.	0	a	e	e + a
0	0	0	0	0
a	0	e	e	e + a
e	0	a	e	e + a
e + a	0	e + a	e + a	0

Table 2

This example justifies our observation 3.12 as Z_2 is not contained in Z_2G , and also justifies our Theorem 3.11 as | Z_2 | = 2 . Observe that the element e+a is nonzero nilpotent element in Z_2G. But , this group algebra Z_2G is not a Smarandache ring as there is no proper subset of Z_2G which is a field under the operations defined by Table1 and Table 2 except the trivial field { 0,e} formed with identity elements

Example 4.3. Let Z_3={0,1,2 } be a prime field of characteristic 3 and let G={g: g^2 = 1 } be a group . Z_3G = { 0,1,2,g,2g,1+g,1+2g,2+2g } is a group algebra with respect to the operations defined by table 3 and table 4.

+	0	1	2	g	2g	1+g	2+g	1+2g	2+2g
0	0	1	2	g	2g	1+g	2+g	1+2g	2+2g
1	1	2	0	1+g	1+2g	2+g	G	2+2g	2g
2	2	0	1	2+g	2+2g	g	1+g	2g	1+2g
g	g	1+g	2+g	2g	0	1+2g	2+2g	1	2
2g	2g	1+2g	2+2g	0	g	1	2	1+g	2+g
1+g	1+g	2+g	g	1+2g	1	2+2g	2g	2	0
2+g	2+g	g	1+g	2+2g	2	2g	1+2g	0	1
1+2g	1+2g	2+2g	2g	1	1+g	2	0	2+g	g
2+2g	2+2g	2g	1+2g	2	2+g	0	1	g	1+g

Table 3

	0	1	2	g	2g	1+g	2+g	1+2g	2+2g
0	0	0	0	0	0	0	0	0	0
1	0	1	2	g	2g	1+g	2+g	1+2g	2+2g
2	0	2	1	2g	g	2+2g	1+2g	2+g	1+g
g	0	g	2g	1	2	1+g	1+2g	2+g	2+2g
2g	0	2g	g	2	1	2+2g	2+g	1+2g	1+g
1+g	0	1+g	2+2g	1+g	2+2g	2+2g	0	0	1+g
2+g	0	2+g	1+2g	1+2g	2+g	0	2+g	1+2g	0
1+2g	0	1+2g	2+g	2+g	1+2g	0	1+2g	2+g	0
2+2g	0	2+2g	1+g	2+2g	1+g	1+g	0	0	2+2g

Table 4

Observe the following :

(4.4.1) The group algebra Z_3G is a commutative ring and has no nonzero nilpotent element.

(4.4.2) In view of Theorem 2.7 due to Maschke , the group algebra Z_3G is semisimple

(4.4.3) The proper subset A = { 0,1e,2} is a field of Z_3G under the operations of the group algebra by table 3 and table 4.

Therefore, the group algebra Z_3G is a Smarandache ring. This example justifies Theorem 3.9 , Theorem 3.11 and observation 3.12.

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