Two Remarkable Ortho-Homological Triangles

Prof. Ion Pătraşcu - The National College "Fraţii Buzeşti", Craiova, Romania
Prof. Florentin Smarandache – University of New Mexico, U.S.A.

In a previous paper [5] we have introduced the ortho-homological triangles, which are triangles that are orthological and homological simultaneously.

In this article we call attention to two remarkable ortho-homological triangles (the given triangle ABC and its first Brocard's triangle), and using the Sondat's theorem relative to orthological triangles, we emphasize on four important collinear points in the geometry of the triangle.

Definition 1

The first Brocard triangle of a given triangle ABC is the triangle formed by the projections of the symmedian center of the triangle ABC on its perpendicular bisectors.

Observation

In figure 1 we note with K the symmedian center, OA', OB', OC' the perpendicular bisectors of the triangle ABC and $A_1B_1C_1$ the first Brocard's triangle.

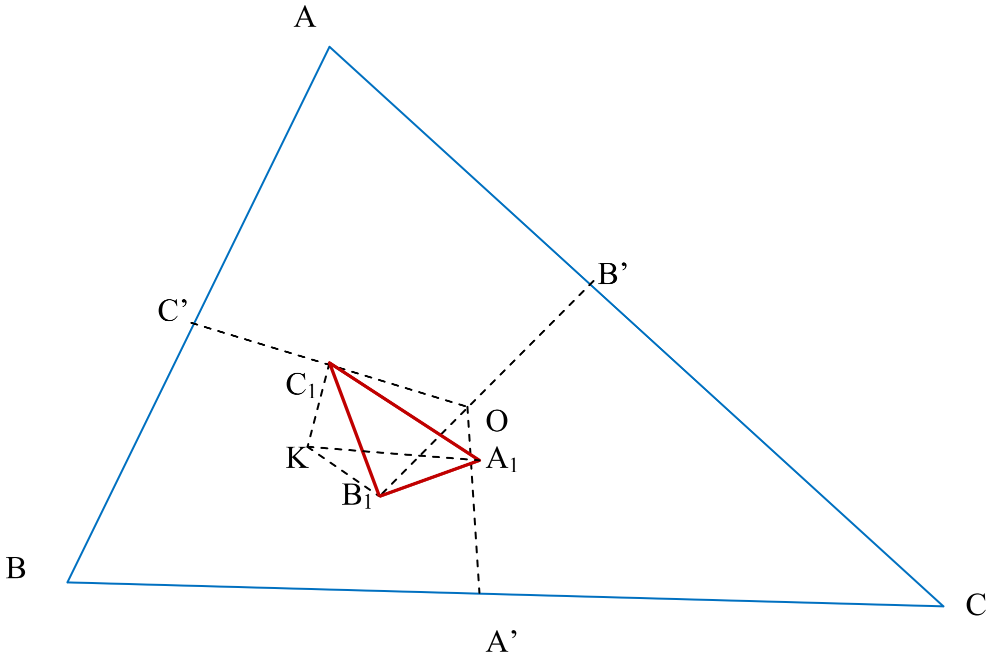

Fig. 1

Theorem 1

If ABC is a given triangle and $A_1B_1C_1$ is its first triangle Brocard, then the triangles ABC and $A_1B_1C_1$ are ortho-homological.

We'll perform the proof of this theorem in two stages.

I. We prove that the triangles $A_1B_1C_1$ and ABC are orthological.

The perpendiculars from A_1, B_1, C_1 on BC, CA respective AB are perpendicular bisectors in the triangle ABC, therefore are concurrent in O, the center of the circumscribed circle of triangle ABC which is the orthological center for triangles $A_1B_1C_1$ and ABC.

II. We prove that the triangles $A_1B_1C_1$ and ABC are homological, that is the lines AA_1, BB_1, CC_1 are concurrent.

To continue with these proves we need to refresh some knowledge and some helpful results.

Definition 2

In any triangle ABC there exist the points Ω and Ω' and the angle ω such that:

$$m(\sphericalangle \Omega AB) = \sphericalangle \Omega BC = \sphericalangle \Omega CA = \omega$$

$$m(\sphericalangle \Omega' BA) = \sphericalangle \Omega' CA = \sphericalangle \Omega' AB = \omega$$

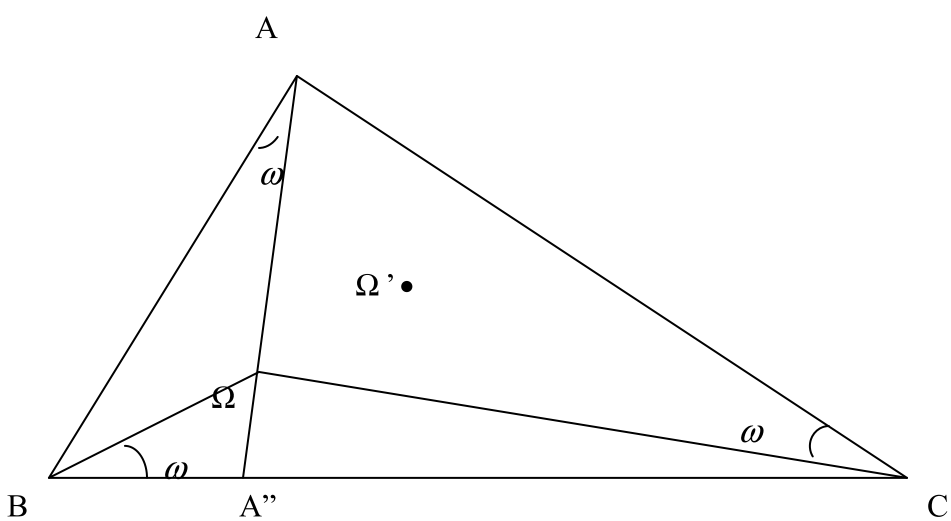

Fig. 2

The points Ω and Ω' are called the first, respectively the second point of Brocard and ω is called the Brocard's angle.

Lemma 1

In the triangle ABC let Ω the first point of Brocard and $\{A''\}$ $\{A''\} = A\Omega \cap BC$, then:

$$\frac{BA''}{CA''} = \frac{c^2}{a^2}$$

Proof

$$Aria_{\triangle}ABA'' = \frac{1}{2} AB \cdot AA'' \sin\omega \tag{1}$$

$$Aria_{\triangle}ACA'' = \frac{1}{2} AC \cdot AA'' \sin(A-\omega) \tag{2}$$

From (1) and (2) we find:

$$\frac{Aria_{\triangle}ABA''}{Aria_{\triangle}ACA''} = \frac{AB \cdot \sin\omega}{AC \cdot \sin(A-\omega)} \tag{3}$$

On the other side, the mentioned triangles have the same height built from A, therefore:

$$\frac{Aria_{\triangle}ABA''}{Aria_{\triangle}ACA''} = \frac{BA''}{CA''} \tag{4}$$

From (3) and (4) we have:

$$\frac{BA''}{CA''} = \frac{AB \cdot \sin\omega}{AC \cdot \sin(A-\omega)} \tag{5}$$

Applying the sinus theorem in the triangle $A\Omega C$ and in the triangle $B\Omega C$, it results:

$$\frac{C\Omega}{\sin(A-\omega)} = \frac{AC}{\sin A\Omega C} \tag{6}$$

$$\frac{C\Omega}{\sin\omega} = \frac{BC}{\sin B\Omega C} \tag{7}$$

Because

$$m(\sphericalangle A\Omega C) = 180^o - A$$

$$m(\sphericalangle B\Omega C) = 180^o - C$$

From the relations (6) and (7) we find:

$$\frac{\sin\omega}{\sin(A-\omega)} = \frac{AC}{BC} \cdot \frac{\sin C}{\sin A} \tag{8}$$

Applying the sinus theorem in the triangle ABC leads to:

$$\frac{\sin C}{\sin A} = \frac{AB}{BC} \tag{9}$$

The relations (5), (8), (9) provide us the relation:

$$\frac{BA''}{CA''} = \frac{c^2}{a^2}$$

Remark 1

By making the notations: $\{B''\} = B\Omega C \cap AC$ and $\{C''\} = C\Omega A \cap AB$ we obtain also the relations:

$$\frac{CB''}{AB''} = \frac{a^2}{b^2} \text{ and } \frac{AC''}{BC''} = \frac{b^2}{c^2}$$

Lemma 2

In a triangle ABC, the Brocard's Cevian $B\Omega$, symmedian from C and the median from A are concurrent.

Proof

It is known that the symmedian CK of triangle ABC intersects AB in the point C_2 such that $\frac{AC_2}{BC_2}=\frac{b^2}{c^2}$. We had that the Cevian $B\Omega$ intersects AC in $B"$ such that $\frac{BC"}{B"A}=\frac{a^2}{b^2}$.
The median from A intersects BC in A' and $BA'=CA'$.

Because $\frac{A'B}{A'C}\cdot\frac{B"C}{B"A}\cdot\frac{C_2A}{C_2B}=1$, the reciprocal of Ceva's theorem ensures the concurrency of the lines $B\Omega$, CK and AA'.

Lemma 3

Give a triangle ABC and ω the Brocard's angle, then

$$ctg\,\omega = ctgA + ctgB + ctgC \tag{9}$$

Proof

From the relation (8) we find:

$$\sin\left(A-\omega\right)=\frac{a}{b}\cdot\frac{\sin A}{\sin C}\cdot\sin\omega \tag{10}$$

From the sinus' theorem in the triangle ABC we have that

$$\frac{a}{b}=\frac{\sin A}{\sin B}$$

Substituting it in (10) it results: $\sin\left(A-\omega\right)=\frac{\sin^2 A\cdot\sin\omega}{\sin B\cdot\sin C}$

Furthermore we have:

$$\sin(A-\omega)=\sin A\cdot\cos\omega-\sin\omega\cdot\cos A$$

$$\sin A\cdot\cos\omega-\sin\omega\cdot\cos A=\frac{\sin^2 A\cdot\sin\omega}{\sin B\cdot\sin C} \tag{11}$$

Dividing relation (11) by $\sin A\cdot\sin\omega$ and taking into account that $\sin A=\sin(B+C)$, and $\sin(B+C)=\sin B\cdot\cos C+\sin C\cdot\cos B$ we obtain relation (5)

Lemma 4

If in the triangle ABC, K is the symmedian center and K_1, K_2, K_3 are its projections on the sides BC, CA, AB, then:

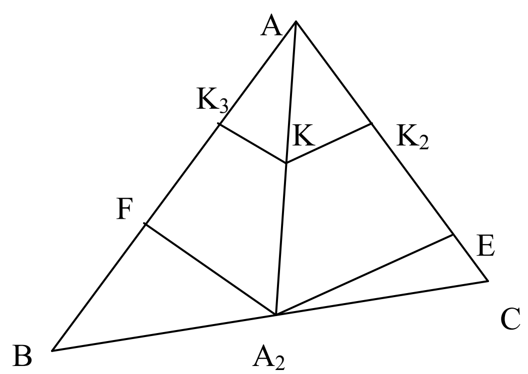

Fig. 3

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} = \frac{1}{2} tg\omega$$

Proof:

Let AA_2 the symmedian in the triangle ABC, we have:

$$\frac{BA_2}{CA_2} = \frac{Aria\triangle BAA_2}{Aria\triangle CAA_2},$$

where E and F are the projection of A_2 on AC respectively AB.

It results that $\frac{A_2F}{A_2E} = \frac{c}{b}$

From the fact that $\triangle AKK_3 \sim \triangle AA_2F$ and $\triangle AKK_2 \sim \triangle AA_2E$ we find that $\frac{KK_3}{KK_2} = \frac{A_2F}{A_2E}$

Also: $\frac{KK_2}{b} = \frac{KK_3}{c}$, and similarly: $\frac{KK_1}{a} = \frac{KK_2}{b}$, consequently:

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} \qquad (12)$$

The relation (12) is equivalent to:

$$\frac{aKK_1}{a^2} = \frac{bKK_2}{b^2} = \frac{cKK_3}{c^2} = \frac{aKK_1 + bKK_2 + cKK_3}{a^2 + b^2 + c^2}$$

Because

$$aKK_1 + bKK_2 + cKK_3 = 2Aria\triangle ABC = 2S,$$

we have:

$$\frac{KK_1}{a} = \frac{KK_2}{b} = \frac{KK_3}{c} = \frac{2S}{a^2 + b^2 + c^2}$$

If we note H_1, H_2, H_3 the projections of A, B, C on BC, CA, AB, we have

$$ctgA = \frac{H_2A}{BH_2} = \frac{bc\cos A}{2S}$$

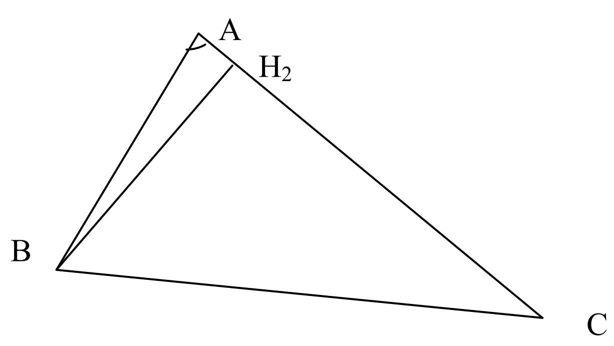

Fig. 4

From the cosine's theorem it results that : $b \cdot c \cdot \cos A = \frac{b^2 + c^2 - a^2}{2}$, and therefore

$$ctgA = \frac{b^2 + c^2 - a^2}{4S}$$

Taking into account the relation (9), we find:

$$ctg\omega = \frac{a^2 + b^2 + c^2}{4S},$$

then

$$tg\omega = \frac{4S}{a^2 + b^2 + c^2}$$

and then

$$\frac{KK_1}{a} = \frac{2S}{a^2 + b^2 + c^2} = \frac{1}{2} tg\omega.$$

Lemma 5

The Cevians AA_1, BB_1, CC_1 are the isotomics of the symmedians AA_2, BB_2, CC_2 in the triangle ABC.

Proof:

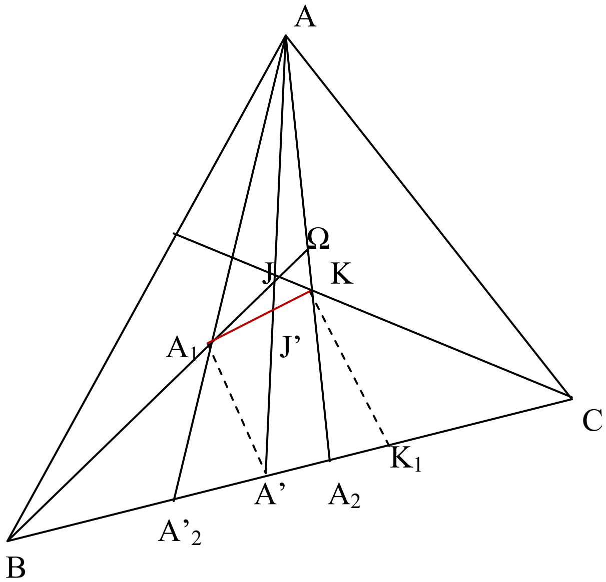

Fig. 5

In figure 5 we note J the intersection point of the Cevians from the Lemma 2.

Because $KA_1 \parallel BC$, we have that $A_1A' = KK_1 = \frac{1}{2} atg\omega$. On the other side from the right triangle $A'A_1B$ we have: $tg\sphericalangle A_1BA' = \frac{A_1A'}{BA'} = tg\omega$, consequently the point A_1, the vertex of the first triangle of Brocard belongs to the Cevians $B\Omega$.

We note $\{J'\} = A_1K \cap AA'$, and evidently from $A_1K \parallel BC$ it results that JJ' is the median in the triangle JA_1K, therefore $A_1J' = J'K$.

We note with A'_2 the intersection of the Cevians AA_1 with BC, because $A_1K \parallel A'_2 A_2$ and AJ' is a median in the triangle AA_1K it results that AA' is a median in triangle $AA'_2 A_2$ therefore the points A'_2 and A_2 are isometric.

Similarly it can be shown that BB_2' and CC_2' are the isometrics of the symmedians BB_2 and CC_2.

The second part of this proof: Indeed it is known that the isometric Cevians of certain concurrent Cevians are concurrent and from Lemma 5 along with the fact that the symmedians of a triangle are concurrent, it results the concurrency of the Cevians AA_1, BB_1, CC_1 and therefore the triangle ABC and the first triangle of Brocard are homological. The homology's center (the concurrency point) of these Cevians is marked in some works with Ω'' with and it is called the third point of Brocard.

From the previous proof, it results that Ω'' is the isotomic conjugate of the symmedian center K.

Remark 2

The triangles ABC and $A_1B_1C_1$ (first Brocard triangle) are *triple-homological*, since first time the Cevians AB_1, BC_1, CA_1 are concurrent (in a Brocard point), second time the Cevians AC_1, BA_1, CB_1 are also concurrent (in the second Brocard point), and third time the Cevians AA_1, BB_1, CC_1 are concurrent as well (in the third point of Brocard).

Definition 3

It is called the Tarry point of a triangle ABC, the concurrency point of the perpendiculars from A, B, C on the sides B_1C_1, C_1A_1, A_1B_1 of the Brocard's first triangle.

Remark 3

The fact that the perpendiculars from the above definition are concurrent results from the theorem 1 and from the theorem that states that the relation of triangles' orthology is symmetric.

We continue to prove the concurrency using another approach that will introduce supplementary information about the Tarry's point.

We'll use the following:

Lemma 6:

The first triangle Brocard of a triangle and the triangle itself are similar.

Proof

From $KA_1 \parallel BC$ and $OA' \perp BC$ it results that

$$m(\sphericalangle KA_1O) = 90°$$

(see Fig. 1), similarly

$$m(\sphericalangle KB_1O) = m(\sphericalangle KC_1O) = 90°$$

and therefore the first triangle of Brocard is inscribed in the circle with OK as diameter (this circle is called the Brocard circle).

Because

$$m\left(\sphericalangle A_1OC_1\right)=180°-B$$

and A_1, B_1, C_1, O are concyclic, it results that $\sphericalangle A_1B_1C_1=\sphericalangle B$, similarly

$$m\left(\sphericalangle B'OC'\right)=180°-A,$$

it results that

$$m\left(\sphericalangle B_1OC_1\right)=m(A)$$

but

$$\sphericalangle B_1OC_1\equiv\sphericalangle B_1A_1C_1,$$

therefore

$$\sphericalangle B_1A_1C_1=\sphericalangle A$$

and the triangle $A_1B_1C_1$ is similar wit the triangle ABC.

Theorem 2

The orthology center of the triangle ABC and of the first triangle of Brocard is the Tarry's point T of the triangle ABC, and T belongs to the circumscribed circle of the triangle ABC.

Proof

We mark with T the intersection of the perpendicular raised from B on A_1C_1 with the perpendicular raised from C on A_1B_1 and let

$$\{B_1'\}=BT\cap A_1C_1,\ A_1\{C_1'\}=A_1B_1\cap CT.$$

We have

$$m\left(\triangle B_1'TC_1'\right)=180°-m\left(\triangle C_1A_1B_1\right)$$

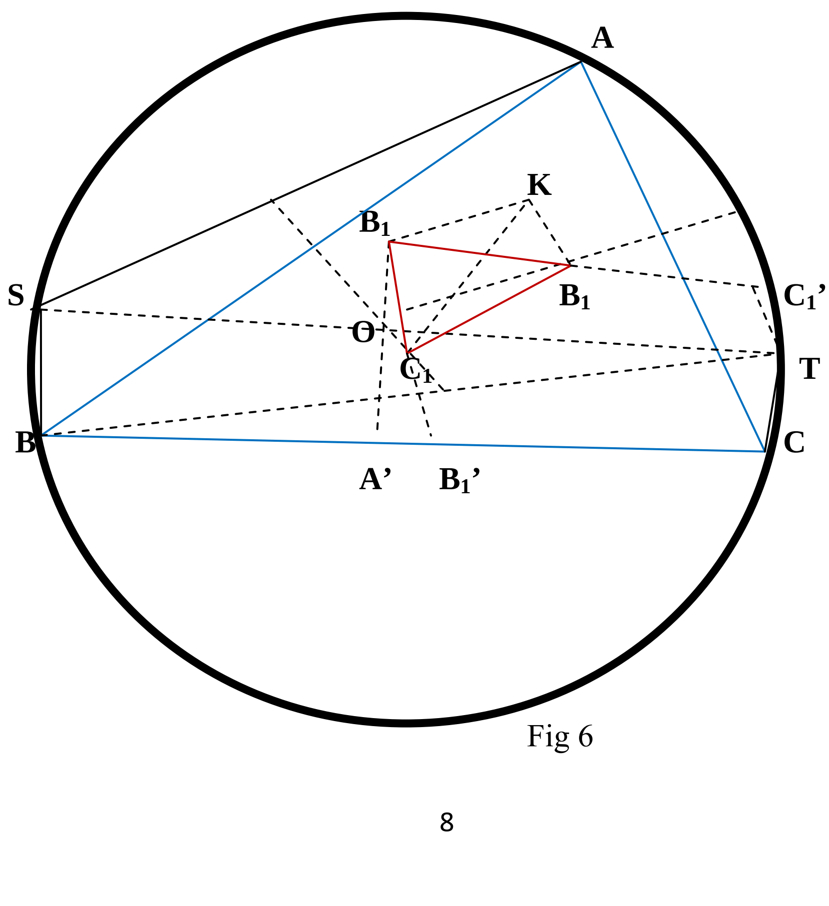

Fig 6

But because of Lemma 6 $\sphericalangle C_1A_1B_1 = \sphericalangle A$.

It results that $m(\sphericalangle B_1'TC_1') = 180° - A$, therefore

$$m(\sphericalangle BTC') + m(\sphericalangle BAC) = 180°$$

Therefore T belongs to the circumscribed circle of triangle ABC

If $\{A_1'\} = B_1C_1 \cap AT$ and if we note with T' the intersection of the perpendicular raised from A on B_1C_1 with the perpendicular raised from B on A_1C_1, we observe that

$$m(\sphericalangle B_1'T'A_1') = m(\sphericalangle A_1C_1B_1)$$

therefore

$$m(\sphericalangle BT'A) + m(\sphericalangle BCA)$$

and it results that T' belongs to the circumscribed triangle ABC.

Therefore $T = T'$ and the theorem is proved.

Theorem 3

If through the vertexes A, B, C of a triangle are constructed the parallels to the sides B_1C_1, C_1A_1 respectively A_1B_1 of the first triangle of Brocard of this triangle, then these lines are concurrent in a point S (the Steiner point of the triangle)

Proof

We note with S the polar intersection constructed through A to B_1C_1 with the polar constructed through B to A_1C_1 (see Fig. 6).

We have

$$m(\sphericalangle ASB) = 180° - m(\sphericalangle A_1C_1B_1) \text{ (angles with parallel sides)}$$

because

$$m(\sphericalangle A_1C_1B_1) = m\sphericalangle C,$$

we have

$$m(\sphericalangle ASB) = 180° - m\sphericalangle C,$$

therefore A_1SB_1C are concyclic.

Similarly, if we note with S' the intersection of the polar constructed through A to B_1C_1 with the parallel constructed through C to A_1B_1 we find that the points $A_1S_1'B_1C$ are concyclic.

Because the parallels from A to B_1C_1 contain the points A, S, S' and the points S, S', A are on the circumscribed circle of the triangle, it results that $S = S'$ and the theorem is proved.

Remark 4

Because $SA \| B_1C_1$ and $B_1C_1 \perp AT$, it results that

$$m(\sphericalangle SAT) = 90°,$$

but S and T belong to the circumscribed circle to the triangle ABC, consequently the Steiner's point and the Tarry point are diametric opposed.

Theorem 4

In a triangle ABC the Tarry point T , the center of the circumscribed circle O, the third point of Brocard Ω'' and Steiner's point S are collinear points

Proof

The P. Sondat's theorem relative to the orthological triangles (see [4]) says that the points T, O, Ω'' are collinear, therefore the points: T, O, Ω'', S are collinear.

References

[1] Roger A. Johnson – Advanced Euclidean Geometry, Dover Publications, Inc. Mineola, New York, 2007.

[2] F. Smarandache – Multispace & Multistructure, Neutrospheric Trandisciplinarity (100 Collected Papers of Sciences), Vol IV. North-European Scientific Publishers, Hanko, Findland, 2010.

[3] C. Barbu – Teoreme fundamentale din geometria triunghiului. Editura Unique, Bacău, 2008.

[4] I. Pătraşcu, F. Smarandache – http://www.scribd.com/doc/33733028 - An application of Sondat Theorem Regarding the Orthohomological Triangles.

[5] Ion Pătrașcu, Florentin Smarandache, A Theorem about Simulotaneously Orthological and Homological Triangles, http://arxiv.org/abs/1004.0347.