The Big Boing Theory

(The exam at the end of all exams)

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0. The Problem

Note: The narrative here is based on really fictitious events!

Two cosmonauts had been living in an obscure compartment of the Mir Orbiter for seven years, ever since Ground Control inadvertently omitted them from a list of return passengers.

As usual, they charged their ion self-propulsion units from Mir's main batteries, and drew a little fluid from the air conditioning unit for impulse.

They then jetted away into space, far from any nearby object, taking with them a large, compressed, coiled spring and a long bungee cord, each of finite, but vanishingly small, mass.

One cosmonaut, with all equipment, weighed half as much as the other. They each tied themselves to one end of the bungee cord and took up positions on opposite ends of the spring. They synchronized their watches to the Big Ivan clock on the Mir steeple; and, exactly at 12:00, released the latch on the spring.

The spring expanded in negligible time, sending the two cosmonauts off in exactly opposite directions.

The smaller cosmonaut determined by spectral shift of the fixed stars that her speed was 0.9c. After an hour had lapsed by her watch, the bungee cord's slack was used up, and the cord rapidly decelerated her and pulled her back in the opposite direction.

The two cosmonauts were returned to opposite ends of the spring, compressing it and resetting the latch.

By the smaller cosmonaut's watch, it was exactly 14:00.

Questions:

1. What time was it by the larger cosmonaut's watch?

   Hint: What time would Big Ivan say it was?

2. Mir having been deorbitted because of repeated battery and air conditioning problems, how will they return to Earth?

3. Should they argue about the time?

4. Final Exam:

   A. Describe the stress on the bungee cord during the above.

   B. If you can't describe it, at least explain it.

   C. If you can't explain it, then at least please excuse it.

I. The First Pass

First of all, this really happened!

In 1993, Boris Yeltsin was filmed on CNN, walking out on the tarmac to greet the cosmonauts on their return from Mir. He shook their hands and congratulated them, briefly speaking to each one individually. As he was getting into his car after the ceremony, he was reported to remark (in Russian),

"Huh. That's funny. I thought there were more of them."

Any way, here is an answer to Question 1:

The problem is amenable to reasonable additional assumptions. The only error would be to assign a specific mass either to the spring or the cord--for example, to compute a mechanical resonant frequency or something along those lines.

The behavior of the spring is irrelevant to the problem. You may allocate it a tiny bit of momentum and have it sit in the frame of Big Ivan; or, let it follow one of the cosmonauts or bounce around between them.

Similarly, you may have the bungee cord sit in the rest frame of Big Ivan and pay out slack to the cosmonauts at both ends; or, you may let one of the cosmonauts carry the slack and pay it out to the other.

The first problem is very straightforward: One solution would be by defining the lab frame time, t_F , as the time in the frame of the spring before it was released. This would be the frame of Big Ivan. Then, call the proper time of the lighter and heavier cosmonauts, t_L and t_H , respectively.

By Lorentz time dilation,

$$t_F = t_L \gamma_L = t_H \gamma_H \, , \tag{1}$$

in which $\gamma_j = 1 \big/ \sqrt{1-(v_j/c)^2}$.

So, from (1), the heavier cosmonaut's time would be

$$t_H = t_L \frac{\gamma_L}{\gamma_H} \, . \tag{2}$$

Incidentally, before continuing, notice that γ_L was about $1\big/\sqrt{1-0.81} \cong 2.25$; so, the t_F duration of the excursion must have been about 4.5 hours: It was 16:30 when they met again, according to Big Ivan time.

They missed Mir, but maybe they can catch ISS?

To solve Eq. (2), the quickest way would be simply to approximate the heavier cosmonaut's speed by assuming the doubled mass put that speed twice as far away from c

as that of the lighter cosmonaut's: So, if $v_H \cong 0.8c$, then $\gamma_H \cong 1.67$; and so, from Eq. (2), we would have,

$$t_H = 2 \cdot \frac{2.25}{1.67} \cong 2.7 \text{ hours,} \tag{3}$$

making it about 14:40 by the heavier cosmonaut's watch.

Call this COARSE ANSWER 1.

More precisely, we know momentum is conserved and may be set equally to 0 for the two cosmonauts in the frame of the spring just before release. The relativistic definition of momentum of a massive body is,

$$p = m\gamma v\text{; and so,} \tag{4}$$

$$m_L \gamma_L v_L = m_H \gamma_H v_H\,, \tag{5}$$

in the frame of Big Ivan. We need v_H. Solving Eq. (5) for $\gamma_H v_H$, and using γ_L from *ca*. Eq. (2) above, we find that

$$2.25 \cdot \frac{0.9c}{2} = \frac{v_H}{\sqrt{1-\left(\frac{v_H}{c}\right)^2}}\,. \tag{6}$$

After some algebra, we find that $v_H \cong 0.7c$, making γ_H about 1.4. Substituting into Eq. (2) above,

$$t_H = 2(2.25/1.4) \cong 3.2 \text{ hours,} \tag{7}$$

making it about 15:10 by the heavier cosmonaut's watch.

Call this FINE ANSWER 1.

The question about the bungee cord tension will be answered soon. Hint: If the cord is not to break, what would have to be the speed of sound in it?

II. The Second Pass

The final question, the stress in the bungee cord, again allows for reasonable assumptions.

Let's start by looking at the length of the cord. We can't handle a relativistic problem validly by assuming length and time to be independent, but we can calculate the maximum length possible at any time, in any frame: This is just the length with no Lorentz contraction, the length to which the bungee cord would have to have been manufactured.

In the frame of the coiled spring just before release, we know the two cosmonauts travelled in opposite directions for about $4.5/2 = 2.25$ hours.

In that frame, then, using the previous answer, the cord unwrapped to a length L_F given by their combined speed of .9c + .7c = 1.6c. So,

$$L_F = 1.6 \cdot (3 \cdot 10^8 \text{ m/s}) \cdot 2.25\text{h} \cdot (60^2 \text{ s/h}); \text{ or,} \tag{8}$$

$$L_F \cong 4 \cdot 10^9 \text{km}. \tag{9}$$

This is slightly less than the average radius of the orbit of the planet Neptune.

Wow! The cord must be quite stiff, to reverse a velocity of .9c in a time negligible on the scale of an hour. It was, of course, made of nanotubules precipitated in a face-centered cubic array of the newly discovered particle, the strongerino.

Come to think of it, I wonder what happened to the spring? It should be stored safely on Mir somewhere, where some kid couldn't get hold of it and accidentally unlatch it . . ..

Let's assume for brevity that the slack in the bungee cord was carried by the heavier cosmonaut and paid out to the lighter. Looked at in the frame of the lighter cosmonaut, the cord then would have to be able to survive being unwrapped at a speed equal in that frame to what we shall call v_{LH}, the combined speed of separation.

To add the two velocities, we have to realize that the speeds .9c and .7c were given in the frame of the coiled spring just before release. So, time dilation will apply to both of them: Time passing in the larger cosmonaut's frame will be dilated relative to the dilation in the lighter ones. A clock in the frame of Big Ivan merely copying ticks of, say, the heavier cosmonaut's clock will seem to run slow by a factor of γ_H in the Big Ivan frame. This copy-clock, if viewed from the frame of the lighter cosmonaut, will run more slowly yet, this time by a factor of γ_L. Using the answers previously calculated, the gamma of the uncoiling point on the bungee cord therefore will be $\gamma_{LH} = \gamma_L \gamma_H = 2.25 \cdot 1.4 = 3.15$. Thus, we may write,

$$3.15 = 1 \Big/ \sqrt{1 - \left(\frac{v_{LH}}{c}\right)^2}\text{; and,} \tag{10}$$

$$v_{LH} \cong 0.95c\,. \tag{11}$$

In the frame of the lighter cosmonaut, the cord will be lengthening and propagating a wave of slight tension in the direction of the heavier cosmonaut at a speed of $0.95c$. Therefore, the speed of sound in the cord must exceed $0.95c$; otherwise, the cord would separate as it was being unwrapped.

As we know, spring keeps returning. So, thinking again of our lost spring . . . while compressed, it must have stored energy enough to accelerate a combined mass of, say, 150 kg to $0.95c$. This stored energy may be considered equal to the kinetic energy of the two separating cosmonauts.

The relativistic total energy is given by $E = \sqrt{(pc)^2 + (mc^2)^2} = \gamma mc^2$. The kinetic energy E_K will be the difference between the total and the mc^2 energy residing in mass; so,

$$E_K = (\gamma - 1)mc^2 = 2.15 \cdot 150c^2 \cong 3 \cdot 10^{19}\,\text{joules.} \tag{12}$$

Starting at 20 C, it takes about $4\,\text{joule/cal} \cdot (540 + 80^2)\,\text{cal/g} \cdot 10^6\,\text{g/ton} \cong 3 \cdot 10^{10}$ joules to vaporize a metric ton of water; so, the kinetic energy stored in the compressed spring is enough to vaporize one billion metric tons of water. Luckily, it is safely out in space, where nothing ever happens.

But, Mir, ISS, and everything else actually is not in space but rather in the thinnest reaches of the Earth's upper atmosphere! So, things can happen.

What happens when the slack in the bungee cord is all paid out? An answer to this will be attempted next.

III. The Third Pass

I'd like to start by apologizing for panicking that way: I thought the compressed spring had been stolen; and, in all the excitement, I misunderstood what they were saying.

If there are any members of the press still present, I'd just like to say that the three third-grade kids I overheard talking about football now have been released into the custody of their parents.

Of course, they will remain under suspicion of terrorism for the rest of their lives; but, think of all the bother of changing the police records.

The bungee cord also clearly is a problem. We estimated its maximum length previously, but that may have to be changed after considering how it must work while being paid out.

In the frame of the lighter cosmonaut, the stress in the cord is constant in time during the first hour after spring release. Call this the *slack tension*. However small the mass per unit length of the cord, as it unwraps, the additional increment in mass per unit time accelerated by the lighter cosmonaut is proportional to the additional length, which is constant. In the frame of the heavier cosmonaut, a negligible and much smaller constant tension may be assumed exerted in the direction of the lighter one.

Of course, during slack tension, the tension must be small enough that it does not noticeably reduce the combined speed of $v_{LH} = 0.95c$.

When the slack has been consumed, a sudden increase in tension will occur, first felt by the heavier cosmonaut, who suddenly has run out of cord to pay. Call this the *taut tension*. The taut tension will propagate as a wave toward the lighter cosmonaut at the speed of sound in the cord, say, $0.98c$.

We might at this point assume (*a*) that the cord must stretch; or, we might disallow this and accelerate the heavier cosmonaut (*b*) immediately to the velocity of the lighter; (*c*) immediately to the negative of the heavier velocity; or, perhaps (*d*) gradually to some other intermediate velocity. The immediate choices assume virtualization of energy over a space-like interval.

We decide here to use the first alternative, stretching, because we do not wish to assume a specific mass for the cord. We also will assume immediate reversal of the velocity of the heavier cosmonaut at some *unspecified* time in the frame of the lighter. In reality, the mass and elastic constants of the cord would determine the dynamics of the restoring force in all frames.

Because we know that the lighter cosmonaut must continue at $0.9c$ (Big Ivan frame) until the taut tension reaches her, we permit the cord to stretch freely for the duration of the propagation of the taut tension wave.

So, using the reversed frame of the heavier cosmonaut at the point of initiation of taut tension, we may define an origin of distance x by the location 0 of the heavier cosmonaut at the instant of initiation of the taut tension wave. Then, in the bungee cord, we have a region of taut tension (TT) in the frame of the heavier cosmonaut propagating in the direction of the lighter cosmonaut and bounded by a point x_{TT} obeying,

$$x^{H}{}_{TT} = 0.98c \cdot t_H \,, \tag{13}$$

the taut segment of the cord at $0 < x << x_{TT}$ now being assumed very close to at rest in the frame of the heavier cosmonaut and therefore moving toward the lighter cosmonaut at speed $v_H = 0.7c$. Other, more repetitive assumptions might be made, but here we take the easy way out and assume the cord to apply enough tension to reverse the momentum of the heavier cosmonaut in negligible time *at some time*. The rest of the analysis will be in the frame after reversal. Otherwise, we run afoul of the twine paradox.

How fast is x_{TT} changing in the frame of Big Ivan? Well, the TT wave must cover a distance equal to the entire extent of the fully extended bungee cord; it propagates on a segment of the cord moving at $0.9c$ toward the lighter cosmonaut, and its trailing extent is in the reversed frame of the heavier cosmonaut, which is moving in the same direction at $0.7c$. Call the former segment the *taut relaxing* segment, and the latter the *slack relaxing* segment:

For the speed $v^{Ft}{}_{TT}$ in the **t**aut relaxing segment in the frame of Big Ivan, we may use the formula for composition of velocities, with v_j a velocity known in the frame of v_F :

$$v^{Ft}{}_{j} = \frac{v_j + v_F}{1 + v_j v_F / c^2} \,. \tag{14}$$

Using this, we find that

$$v^{Ft}{}_{TT} = \frac{0.98c + 0.7c}{1 + 0.98 \cdot 0.7} \cong 0.996c \,, \tag{15}$$

making $\gamma^{Ft}{}_{TT} \cong 11.2$.

For the speed $v^{Fs}{}_{TT}$ in the **s**lack relaxing segment in the frame of Big Ivan, we may use the previously calculated results *ca*. Eq. (7) for the heavier cosmonaut: $v^{Fs}{}_{TT} = 0.7c$, and $\gamma^{Fs}{}_{TT} = 1.4$.

Both velocities are in the frame of Big Ivan; so, as above *ca*. Eq. (10), their Lorentz factors compose multiplicatively, yielding, $\gamma^{Fst}{}_{TT} = \gamma^{Ft}{}_{TT} / \gamma^{Fs}{}_{TT} = 8.0$. Solving as in Eq. (10), we obtain a stretching speed of $v^{Fst}{}_{TT} \cong 0.992c$.

From Eq. (9), we know the total, fully extended length of the bungee cord in the frame of Big Ivan must be $L_F = 4 \cdot 10^{12}$ m. Thus, the stretching strain during TT wave propagation must have lasted for a time in the frame of Big Ivan equal to,

$$t^{Fst}{}_{TT} = \frac{L_F}{v^{Fst}{}_{TT}} = \frac{4 \cdot 10^{12}}{0.992c} \cong 0.47 \text{ hours}. \tag{16}$$

In the frame of Big Ivan, the fractional added length of the cord because of stretching then would be something like,

$$\text{stretch} = \frac{4 \cdot 10^{12} + 0.47 \cdot 60^2 \cdot 0.992c}{4 \cdot 10^{12}} \cong 1.125\,. \tag{17}$$

Thus, after adding a stretch of no more than about 12.5% in the frame of Big Ivan, the taut wave will reach the lighter cosmonaut and, we assume jerkily, will reverse her momentum.

With both cosmonaut momenta reversed, a slack wave now will propagate over the bungee cord until all tension has been relaxed. This concludes this answer to the final exam question.

This is as far out as I go with this theory. I hope readers have found it humorous and thought-provoking.