Proof Wolstenholme-Lenhard ciclic inequality for real numbers and L.Fejes Tóth conjecture

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Abstract: In this paper an elementary proof of the Wolstenholme-Lenhard ciclic inequality for real numbers and L.Fejes Tóth conjecture(equivalent by Erdős-Mordell inequality for polygon) is given, using a remarcable identity

We give the following:

Theorem 1.*For every* $x_k \in R,\ 1 \le k \le n,\ x_{n+1} = x_1$ *and* $\alpha_k \in R$ *with* $\sum_{k=1}^{n} \alpha_k = (2r+1)\pi, r \in N,$ *the following inequality holds true*:

$$\cos\tfrac{\pi}{n} \sum_{k=1}^{n} x_k^2 \ge \sum_{k=1}^{n} x_k x_{k+1} \cos\alpha_k \qquad (1)\ \ (\textit{Wolstenholme-Lenhard})$$

Theorem 2. *For every* $a_k \in R$ *the following identity holds true* :

$$\cos\frac{\pi}{n} \sum_{k=1}^{n} a_k^2 - \sum_{k=1}^{n-1} a_k a_{k+1} + a_1 a_n =$$

$$= \sum_{k=1}^{n-2} \frac{1}{2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}} \left(\sin\frac{(k+1)\pi}{n} a_k - \sin\frac{k\pi}{n} a_{k+1} + \sin\frac{\pi}{n} a_n\right)^2$$

by comparing the coefficients of a_k^2 and $a_k a_{k+1}$. For example ,the coefficient of a_k^2 is

$$\frac{\sin\frac{(k+1)\pi}{n}}{2\sin\frac{k\pi}{n}} + \frac{\sin\frac{(k-1)\pi}{n}}{2\sin\frac{k\pi}{n}} = \frac{2\sin\frac{k\pi}{n}\cos\frac{\pi}{n}}{2\sin\frac{k\pi}{n}} = \cos\frac{\pi}{n}$$

for $k = 2,3,..n-2$ and the coefficient of a_1^2 is

$$\frac{\sin\frac{2\pi}{n}}{2\sin\frac{\pi}{n}} = \cos\frac{\pi}{n},$$

the coefficient of a_{n-1}^2 is

$$\frac{\sin\frac{(n-2)\pi}{n}}{2\sin\frac{(n-1)\pi}{n}} = \frac{\sin\frac{2\pi}{n}}{2\sin\frac{\pi}{n}} = \cos\frac{\pi}{n}$$

and the coeficient of a_n^2 is

$$\sum_{k=1}^{n-2} \frac{\sin^2\frac{\pi}{n}}{2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}} = \sum_{k=1}^{n-2} \frac{\sin\frac{\pi}{n}}{2}\left(\cot\frac{k\pi}{n} - \cot\frac{(k+1)\pi}{n}\right) =$$

$$= \frac{\sin\frac{\pi}{n}}{2}\left(\cot\frac{\pi}{n} - \cot\frac{(n-1)\pi}{n}\right) = \cos\frac{\pi}{n}.$$

The coefficient of $a_k a_{k+1}$ is

$$\frac{-2\sin\frac{(k+1)\pi}{n}\sin\frac{k\pi}{n}}{2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}} = -1$$

for $k = 1,2,...,n-2$ and the coefficient of $a_{n-1}a_n$ is

$$\frac{-2\sin\frac{(n-2)\pi}{n}\sin\frac{\pi}{n}}{2\sin\frac{(n-2)\pi}{n}\sin\frac{(n-1)\pi}{n}} = \frac{-\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}} = -1$$

The coefficient of $a_1 a_n$ is

$$\frac{2\sin\frac{2\pi}{n}\sin\frac{\pi}{n}}{2\sin\frac{\pi}{n}\sin\frac{2\pi}{n}} = 1$$

and the coeficient of $a_k a_n$ for $k = 2,3,..n-2$ is

$$\sum_{k=2}^{n-2}\frac{2\sin\frac{(k+1)\pi}{n}\sin\frac{\pi}{n}}{2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}-\sum_{k=1}^{n-3}\frac{2\sin\frac{k\pi}{n}\sin\frac{\pi}{n}}{2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}=\sum_{k=2}^{n-2}\frac{\sin\frac{\pi}{n}}{\sin\frac{k\pi}{n}}-\sum_{k=1}^{n-3}\frac{\sin\frac{\pi}{n}}{\sin\frac{(k+1)\pi}{n}}=0$$

Similarly we can prove the identity for variabile b_k

$$\cos\frac{\pi}{n}\sum_{k=1}^{n}b_k^2-\sum_{k=1}^{n-1}b_kb_{k+1}+b_1b_n=$$

$$=\sum_{k=1}^{n-2}\frac{1}{2\sin\frac{k\pi}{n}\sin\frac{(k+1)\pi}{n}}\left(\sin\frac{(k+1)\pi}{n}b_k-\sin\frac{k\pi}{n}b_{k+1}+\sin\frac{\pi}{n}b_n\right)^2$$

For $a_k=R_k\cos\beta_k$ and $b_k=R_k\sin\beta_k$ where $R_k\in R$ and $\beta_k\in R$ and sumation the twe identity to obtain:

$$\cos\frac{\pi}{n}\sum_{k=1}^{n}R_k^2-\sum_{k=1}^{n-1}R_kR_{k+1}[\cos\beta_k\cos\beta_{k+1}+\sin\beta_k\sin\beta_{k+1}]+R_1R_n[\cos\beta_1\cos\beta_n+\sin\beta_1\sin\beta_n]\geq$$

or

$$\cos\frac{\pi}{n}\sum_{k=1}^{n}R_k^2-\sum_{k=1}^{n-1}R_kR_{k+1}\cos(\beta_k-\beta_{k+1})+R_1R_n(\beta_1-\beta_n)\geq 0 \qquad (2)$$

let $\beta_k-\beta_{k+1}=\alpha_k$ for k=1,2,...,n-1 and $\beta_1-\beta_n=(2r+1)\pi-\alpha_n, r\in N, r=natural\ number$ to obtain the inequality Wolstenholme-Lenhard for

$R_k\in R$ and $\alpha_k\in R$ and $\sum_{k=1}^{n}\alpha_k=(2r+1)\pi$

We give the L.Fejes Tóth conjecture:

Let $A_1A_2,...,A_n$ be the vertices of a convex n-gon and P and internal point. Let $R_k=PA_k$ and let r_k be the distance from P to the side A_kA_{k+1} the following inequality:

$$\cos\frac{\pi}{n}\sum_{k=1}^{n}R_k\geq\sum_{k=1}^{n}r_k \qquad (3)$$

Later ,Lenhard proves complicated (1) and deduced the following stronger inequality:

$$\cos\frac{\pi}{n}\sum_{k=1}^{n}R_k\geq\sum_{k=1}^{n}w_k \qquad (4)$$

where w_k is the segment of the bisector of the angle $A_kPA_{k+1}=2\alpha_k$ from P to its intersection with the side A_kA_{k+1}.

For proved (3) and (4) ,let in (1) the $x_k=\sqrt{R_k}$, because

$\sqrt{R_k}\bullet\sqrt{R_{k+1}}\bullet\cos\alpha_k\geq\frac{2R_kR_{k+1}}{R_k+R_{k+1}}\bullet\cos\alpha_k=w_k\geq r_k$

to obtain : $\cos\frac{\pi}{n}\sum_{k=1}^{n}R_k\geq\sum_{k=1}^{n}\sqrt{R_k}\bullet\sqrt{R_{k+1}}\bullet\cos\alpha_k\geq w_k\geq r_k$

We give now the following generalization.

Theorem 3. *If* $x=(x_1,x_2,...,x_N)$ *and* $y=(y_1,y_2,...,y_N)$ *are vectors in* $\mathbf{R}^N$, *endowed with the inner product*

$$\langle x,y\rangle=\|x\|\|y\|\cos\theta=\sum_{k=1}^{N}x_ky_k,$$

where θ *is the angle between the vectors x and y, while*

$$\|x\|=\sqrt{x_1^2+x_2^2+...+x_N^2}$$

is the corresponding norm. Then the following inequality of Erdos-Mordell type holds:

$$cos\frac{\pi}{n}\sum_{k=1}^{n}\|X_k\|^2\geq\sum_{k=1}^{n}\langle X_k,X_{k+1}\rangle=\sum_{k=1}^{n}\|X_k\|\|X_{k+1}\|\cos\theta_k,$$

where $X_k=(x_{1k},x_{2k},...,x_{Nk})$, $k=1,2,...,n$ *and* $\sum_{k=1}^{n}\theta_k=\pi$.

In the case in which P is an exterior point, we can give the Erdös-Mordell inequality for convex polygons and exterior point P :

Theorem 4 *Let $A_1A_2...A_n$ be a convex polygon and let P be an exterior point and* $\sphericalangle A_kPA_{k+1} = \alpha_k$ $,k = 1,...,n$. *Let* $\max(\sphericalangle A_iPA_j) = \alpha,\ i \neq j \in \{1,2,...n\}$ *and* $\sum_{k=1}^{n}\alpha_k = 2\alpha < 2\pi$ *and*

$$r_k \leq w_k \leq \sqrt{R_kR}_{k+1}\cos\frac{\alpha_k}{2}.$$

Then

$$\cos\frac{\alpha}{n}\sum_{k=1}^{n}R_k - \sum_{k=1}^{n}r_k \geq \cos\frac{\pi}{n}\sum_{k=1}^{n}R_k - \sum_{k=1}^{n}r_k \geq$$

$$\geq \cos\frac{\pi}{n}\sum_{k=1}^{n}R_k - \sum_{k=1}^{n}w_k \geq \cos\frac{\pi}{n}\sum_{k=1}^{n}R_k - \sum_{k=1}^{n}\sqrt{R_kR_{k+1}}\cos\frac{\alpha_k}{2}.$$

Theorem 5 *Let $A_1,A_2,...,A_p$ be p arbitrary points in N-dimensional euclidian space* $\mathbf{R}^N$ *and M an arebitrary point in the same spaceand,* $\sphericalangle A_kMA_{k+1} = \alpha_k$ $,k = 1,...,p$. *Let* $\max(\sphericalangle A_iMA_j) = \alpha,\ i \neq j \in \{1,2,...p\}$ *and* $\sum_{k=1}^{p}\alpha_k = 2\alpha < 2\pi$, w_k is the segment of the angle bisector $\sphericalangle A_kMA_{k+1}$.We consider $A_{p+1} = A_1$.Then we obtain the following inequality:

$$\cos\frac{\alpha}{p}\sum_{k=1}^{p}R_k - \sum_{k=1}^{p}r_k \geq \cos\frac{\pi}{p}\sum_{k=1}^{p}R_k - \sum_{k=1}^{p}r_k \geq$$

$$\geq \cos\frac{\pi}{p}\sum_{k=1}^{p}R_k - \sum_{k=1}^{p}w_k \geq \cos\frac{\pi}{p}\sum_{k=1}^{p}R_k - \sum_{k=1}^{p}\sqrt{R_kR_{k+1}}\cos\frac{\alpha_k}{2} \geq 0.$$

where R_k is the norm of A_kM and r_k is the distance from M to the segment $A_kA_{k+1}, k \in \{1,2,...,p\}$ and w_k is the length of the segment of the angle bisector $\sphericalangle A_kPA_{k+1}$.

Remark: Obtain the egality in (1) for

$\sin\frac{(k+1)\pi}{n}R_k\cos\beta_k - \sin\frac{k\pi}{n}R_{k+1}\cos\beta_{k+1} + \sin\frac{\pi}{n}R_n\cos\beta_n = 0$ and

$\sin\frac{(k+1)\pi}{n}R_k\sin\beta_k - \sin\frac{k\pi}{n}R_{k+1}\sin\beta_{k+1} + \sin\frac{\pi}{n}R_n\sin\beta_n = 0, k = 1,2,...,n-2$

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