# A new proof Viorel Vîjîitu inequality

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*Abstract*: *In the paper given new proof the inequality using convex function*

Prove that for all positive integers $0<a_1<a_2<....<a_n$ the following inquality holds:

$$(\sum_{k=1}^{n} a_k)^2 \le \sum_{k=1}^{n} a_k^3 \qquad (\textit{Viorel Vîjîitu}) \quad (1)$$

Proof: From hypothesis to rezult $k \le a_k \le a_{k+1} - 1, k = 1,2,...,n$

and $\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2 \quad (2)$, let $a_k = k + b_k,\ \ b_k \ge 0$ and

$b_k \le b_{k+1}, k = 1,2,...,n-1$

The inequality equivalent by:

$$\sum_{k=1}^{n}(k+b_k)^3 \ge (\sum_{k=1}^{n}(k+b_k))^2 = (\tfrac{n(n+1)}{2} + \sum_{k=1}^{n} b_k)^2 \quad ; \quad (2)$$

Following Karamata inequality for proved:

$$\sum_{k=1}^{n}(k+b_k)^3 \ge \sum_{k=1}^{n}(k + \tfrac{b_1+b_2+...+b_n}{n})^3 \qquad ; \qquad (3)$$

because the function $f(x) = x^3$ there is convexe

the sequences $x_k = n-k+1+b_{n-k+1},\ y_k = n-k+1+\frac{\sum_{r=1}^{n} b_r}{n}$
$k = 1,2,...,n$

to obtain :$x_1 \ge x_2 \ge ... \ge x_n$ and $y_1 \ge y_2 \ge ... \ge y_n$

and $\sum_{k=1}^{p} x_k \ge \sum_{k=1}^{p} y_k$ for $p = 1,2,...,n-1$

and $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k$

to rezult $\sum_{k=1}^{n} f(x_k) \ge \sum_{k=1}^{n} f(y_k)$

let $\sum_{k=1}^{n} b_k = x$ and $g(x) = \sum_{k=1}^{n}(k+\frac{x}{n})^3 - (\frac{n(n+1)}{2}+x)^2$

$$g'(x) = \frac{3}{n}\sum_{k=1}^{n}(k+\frac{x}{n})^2 - 2(\frac{n(n+1)}{2}+x)$$

and $g''(x) = \frac{6}{n^2}\sum_{k=1}^{n}(k+\frac{x}{n}) - 2 = \frac{3(n+1)}{n} + \frac{6x}{n^2} - 2 = 1 + \frac{3}{n} + \frac{6x}{n^2} > 0$

to rezult $g'(x)$ it's increasing and

$$g'(x) \geq g'(0) = \frac{3}{n}\sum_{k=1}^{n} k^2 - n(n+1) =$$

$$= \frac{3}{n}\frac{n(n+1)(2n+1)}{6} - n(n+1) = \frac{(n+1)(2n+1)}{2} - n(n+1) = (n+1)(\frac{2n+1}{2} - n)$$

$= \frac{n+1}{2} > 0$ therefore $g'(x) > 0$ and $g(x)$ it's increasing

$g(x) \geq g(0) = \sum_{k=1}^{n} k^3 - (\sum_{k=1}^{n} k)^2 = 0$ and proved the inequality

References:
[1]*www.mathlinks.ro* ;Problem of the day 24 july 2010