Generalisation of the inequalities proposed I.M.O. Madrid 2008 and India-International Mathematical Olympiad Training Camp2010

by **Marian Dincă**

Abstract: *In the paper given generalisation inequalities using*

Lagrange identity.

The inequality proposed for I.M.O.2008 Madrid:

(i) If x,y and z are three real numbers,all different from 1,such that xyz=1, then prove that :

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1 \qquad (1)$$

(ii) Proof that egality achieved for infinitely many triples of rational numbers x,y and z

Author : Walther Janous ,Austria

For $x_k \in \mathbf{R}$ and $\lambda_k \in \mathbf{R}_+$ *following the Lagrange identity* :

$$\sum_{k=1}^{n}\lambda_k x_k^2 = \frac{\left(\sum_{k=1}^{n}\lambda_k x_k\right)^2}{\sum_{k=1}^{n}\lambda_k}+\frac{\sum_{1\leq i<j\leq n}\lambda_i\lambda_j(x_i-x_j)^2}{\sum_{k=1}^{n}\lambda_k} \qquad (2)$$

The identity (2) imply the inequality :

$$\sum_{k=1}^{n}\lambda_k x_k^2 \geq \frac{\sum_{1\leq i<j\leq n}\lambda_i\lambda_j(x_i-x_j)^2}{\sum_{k=1}^{n}\lambda_k} \geq \frac{\sum_{k=1}^{n}\lambda_k\lambda_{k+1}(x_k-x_{k+1})^2}{\sum_{k=1}^{n}\lambda_k},\ where\ \lambda_{n+1}=\lambda_1\ and\ x_{n+1}=x_1\ ;[3]$$

Obtaine the inequality: $\sum_{k=1}^{n}\lambda_k x_k^2 \geq \dfrac{\sum_{k=1}^{n}\lambda_k\lambda_{k+1}(x_k-x_{k+1})^2}{\sum_{k=1}^{n}\lambda_k}$; [4]

In the inequality [4] *for* $\lambda_k = \frac{1}{(x_k-x_{k+1})^2}$,k=1,2,...,n to obtain :

$$\sum_{k=1}^{n} \frac{x_k^2}{(x_k - x_{k+1})^2} \geq \frac{\sum_{k=1}^{n} \lambda_k \lambda_{k+1} \frac{1}{\lambda_k}}{\sum_{k=1}^{n} \lambda_k} = \frac{\sum_{k=1}^{n} \lambda_{k+1}}{\sum_{k=1}^{n} \lambda_k} = \frac{\sum_{k=2}^{n+1} \lambda_k}{\sum_{k=1}^{n} \lambda_k} = 1 \quad \text{because} \quad \lambda_{n+1} = \lambda_1 \qquad [5]$$

The inequality [5] write $\sum_{k=1}^{n} \frac{\left(\frac{x_k}{x_{k+1}}\right)^2}{\left(\frac{x_k}{x_{k+1}} - 1\right)^2} \geq 1$ let $\frac{x_k}{x_{k+1}} = y_k$ for k=1,2,...,n to obtain the inequality :$\sum_{k=1}^{n} \frac{y_k^2}{(y_k - 1)^2} \geq 1$ for $y_k \in \mathbf{R}$ and $\prod_{k=1}^{n} y_k = 1,\ y_k \neq 1$,*for* k=1,2,...,n

The inequality proposed India-International Mathematical Olympiad Training Camp 2010:

Let ABC be a triangle.Let Ω be the brocard point.Proof that:

$$\left(\frac{A\Omega}{BC}\right)^2 + \left(\frac{B\Omega}{CA}\right)^2 + \left(\frac{C\Omega}{AB}\right)^2 \geq 1 \qquad (6)$$

In the paper generalisation the inequality:

Let $A_1A_2...A_n$ be a convex polygon and $M \in Int(A_1A_2...A_n)$ then:

$$\left(\frac{MA_1}{A_pA_{p+1}}\right)^2 + \left(\frac{MA_2}{A_{p+1}A_{p+2}}\right)^2 + ... + \left(\frac{MA_n}{A_{p+n-1}A_{p+n}}\right)^2 \geq 1 \qquad (7)$$

where $p \in \{1,2,...,n\}$and $A_{n+k} = A_k,\ k = 1,2,...,n$

Following Lagrange relation in geometry;

$$\sum_{k=1}^{n} \lambda_k MA_k^2 = \sum_{k=1}^{n} \lambda_k \bullet MG^2 + \frac{1}{\sum_{k=1}^{n} \lambda_k} \bullet \sum_{i<j} \lambda_i \lambda_j (A_iA_j)^2 \qquad (8)$$

for permutation $\lambda_1 \to \lambda_{\sigma(1)}; \lambda_2 \to \lambda_{\sigma(2)}; ......\lambda_n \to \lambda\sigma(n)$

to obtain: $\sum_{k=1}^{n} \lambda_{\sigma(k)} MA_k^2 = \sum_{k=1}^{n} \lambda_k \bullet MG^2 + \frac{1}{\sum_{k=1}^{n} \lambda_k} \sum_{i \neq j} \lambda_{\sigma(i)} \lambda_{\sigma(j)} (A_{\sigma(i)} A_{\sigma(j)})^2 \geq$

$$\geq \frac{1}{\sum_{k=1}^{n} \lambda_k} \sum_{r=1}^{n} \lambda_r \lambda_{r+1} (A_rA_{r+1})^2$$

for $\lambda_r = \frac{1}{(A_rA_{r+1})^2}$; to obtain:

$$\frac{1}{\sum_{k=1}^{n} \lambda_k} \sum_{r=1} \lambda_r \lambda_{r+1} (A_rA_{r+1})^2 = \frac{1}{\sum_{k=1}^{n}} \sum_{r=1} \lambda_r \lambda_{r+1} \bullet \frac{1}{\lambda_r} = \frac{\sum_{r=1}^{n} \lambda_{r+1}}{\sum_{k=1}^{n} \lambda_k} = 1$$

because $\lambda_{n+1} = \lambda_1$

for $\sigma(k) = p + k - 1$, to obtain:$\lambda_{\sigma(k)} = \lambda_{p+k-1} = \frac{1}{(A_{p+k-1}A_{p+k})^2}$

and proved the inequality (7)

References:

[1] Constantin P. Niculescu:*Interferente intre Mecanica si Geometrie*,*Gazeta Matematica* ,**XVIII**(*XCVII*),*nr*.2 (2001).63-69

[2] T.F.Tokieda;*Macanical Ideas in Geometry*, American Math.Monthly,**105**(198),687-703