A New Proof of an Inequality of Oppenheim

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Abstract

In this short note a new proof of a classical inequality involving the areas of a pair of triangles is presented.

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1 Introduction

In 1974 Oppenheim [1] published a generalization of the well-known Finsler-Hadwiger inequality (see [2], [3]). Namely,

Theorem 1 *If ABC is a triangle of sides a, b, c and area F, there exists a triangle of sides $a^{1/p}, b^{1/p}, c^{1/p}$, $(p > 1)$ and area F_p such that*

$$(4F_p/\sqrt{3})^p \geq 4F/\sqrt{3}$$

Equality holds only if $a = b = c$.

Our goal in this paper is to give a new proof of the preceding statement using elementary inequalities. Moreover, an open problem that is a generalization of the about result involving the areas of a pair of polygons is also posed.

2 Proof of Oppenheim's Inequality

In the following a new proof of Theorem 1 is given. First, we write it in the most convenient form: *If ABC is a triangle of sides a, b, c and area F, there exists a triangle of sides $a^{1/p}, b^{1/p}, c^{1/p}$, $(p > 1)$ and area F_p such that*

$$16F_p^2 \geq 3^{p-1}F^2 \qquad (1)$$

Equality holds only if $a = b = c$.

To prove (1) we need the following results.

Lemma 1 *Let* Δ_p *be the triangle of sides* $a^{1/p}, b^{1/p}, c^{1/p}$, $(p > 1)$ *with angles* α, β, γ *measured in radians and let* Δ *be the triangle* ABC *with its angles* A, B, C *also measured in radians. Then*

$$(\cos\gamma)^p \leq \left(\cos\frac{\pi}{3}\right)^{p-1} \cos C \tag{2}$$

Proof. Taking into account the Law of Cosine the preceding inequality can be written as

$$\left(\frac{a^{2/p} + b^{2/p} - c^{2/p}}{2a^{1/}b^{1/p}}\right)^p \leq \left(\frac{1}{2}\right)^{p-1} \frac{a^2 + b^2 - c^2}{2ab}$$

which is equivalent to

$$\left(a^{2/p} + b^{2/p} - c^{2/p}\right)^p \leq a^2 + b^2 - c^2$$

or

$$\left(a^2 + b^2 - c^2\right)^{1/p} + c^{2/p} \geq a^{2/p} + b^{2/p}$$

To prove the last inequality we assume without loss of generality that $a \geq c \geq b$. Now we consider the function $f : [0, +\infty) \to \mathbb{R}$ defined by $f(x) = -x^{1/p}$ which is convex for all $p \geq 1$. Next we will apply Karamata's inequality [4]. Namely, if $(x_1; x_2) \succ (y_1; y_2)$ and f is convex, then $f(x_1) + f(x_2) \geq f(y_1) + f(y_2)$. Setting $(x_1; x_2) = (a^2; b^2)$ and $(y_1; y_2) = (\max(a^2 + b^2 - c^2, c^2) ; \min(a^2 + b^2 - c^2, c^2))$ we have $x_1 \geq y_1$ and $x_1 + x_2 = y_1 + y_2$. That is, $(x_1; x_2) \succ (y_1; y_2)$. Then, by Karamata's inequality, we get

$$f(a^2) + f(b^2) \geq f\left(\max(a^2 + b^2 - c^2, c^2)\right) + f\left(\min(a^2 + b^2 - c^2, c^2)\right)$$

from which immediately follows

$$\left(a^2 + b^2 - c^2\right)^{1/p} + c^{2/p} \geq a^{2/p} + b^{2/p}$$

and the proof is complete. □

Lemma 2 *Let* $0 < a_k \leq 1$, $1 \leq k \leq n$, *be real numbers and let* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be positive numbers such that* $\lambda_1 + \lambda_2 + \ldots + \lambda_n = s$. *Then*

$$\prod_{k=1}^{n} (1 - a_k)^{\lambda_k} \leq \left(\frac{1}{s}\sum_{k=1}^{n} \lambda_k (1 - a_k)\right)^s \leq \left(1 - \prod_{k=1}^{n} a_k^{\lambda_k/s}\right)^s \tag{3}$$

Proof. Statement (3) follows immediately applying Jensen's inequality to the function $f(x) = \ln(1 - e^x)$ which is concave for $x < 0$ and setting $x_k = \ln a_k$. □

Proof of Theorem 1. Taking into account the usual expressions for the area of triangles Δ_p and Δ respectively, equation (1) reads

$$\left(a^{1/p} b^{1/p} \sin\gamma\right)^p \geq \left(\frac{3}{4}\right)^{p-1} \left(ab \sin C\right)^2$$

which after simplification reduces to

$$(\sin\gamma)^p \geq \left(\sin\frac{\pi}{3}\right)^{p-1} \sin C \tag{4}$$

Particularizing Lemma 2 to the case when $n = 2$ and putting $a_1 = \sin^2\dfrac{\pi}{3}$, $a_2 = \sin^2 C$, $\lambda_1 = p - 1$ and $\lambda_2 = 1$ we obtain

$$\left(1 - \sin^2\frac{\pi}{3}\right)^{p-1} \left(1 - \sin^2 C\right) \leq \left(1 - \sin^{(2p-2)/p}\frac{\pi}{3}\,\sin^{2/p} C\right)^p$$

Combining the above result with (2) yields

$$\left(1 - \sin^2\gamma\right)^p \leq \left(1 - \sin^2\frac{\pi}{3}\right)^{p-1} \left(1 - \sin^2 C\right) \leq \left(1 - \sin^{(2p-2)/p}\frac{\pi}{3}\,\sin^{2/p} C\right)^p$$

and $1 - \sin^2\gamma \leq 1 - \sin^{(2p-2)/p}\dfrac{\pi}{3}\,\sin^{2/p} C$. Rearranging terms, we get

$$\sin\gamma \geq \left(\sin\frac{\pi}{3}\right)^{(p-1)/p} \sin^{1/p} C$$

from which (4) immediately follows. This completes the proof. □

Finally, we state the following open question.

Theorem 2 *Let $a_1, a_2, \ldots, a_n$ be the sides of a polygon $A_1A_2\ldots A_n$ inscribed in a circle $\mathcal{C}_1(O, R_1)$, and let $a_1^{1/p}, a_2^{1/p}, \ldots, a_n^{1/p}$ be the sides of a polygon $B_1B_2\ldots B_n$ inscribed in a circle $\mathcal{C}_2(O, R_2)$. Then, for all $p \geq 1$,*

$$[\mathcal{A}(B_1B_2\ldots B_n)]^p \geq \frac{n^{p-1}}{2^{2p-2}} \left(\cot\frac{\pi}{n}\right)^{p-1} \mathcal{A}(A_1A_2\ldots A_n),$$

where $\mathcal{A}(P_1P_2\ldots P_n)$ represents the area of a polygon with vertices $P_1, P_2, \ldots, P_n$.

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