A HAMILTONIAN OPERATOR WHOSE ZEROS ARE THE ROOTS OF THE RIEMANN XI-FUNCTION $\xi\left(\frac{1}{2}+iz\right)$

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- ABSTRACT: We give a possible interpretation of the Xi-function of Riemann as the Functional determinant $\det(E-H)$ for a certain Hamiltonian quantum operator in one dimension $-\frac{d^2}{dx^2}+V(x)$ for a real-valued function V(x) , this potential V is related to the half-integral of the logarithmic derivative for the Riemann Xi-function, through the paper we will assume that the reduced Planck constant is defined in units where $\hbar=1$ and that the mass is $2m=1$
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RIEMANN FUNCTION AND SPECTRAL DETERMINANTS

The Riemann Hypothesis is one of the most important open problems in mathematics, Hilbert and Polya [4] gave the conjecture that would exists an operator $\frac{1}{2}+iL$ with $L=L^{\dagger}$ so the eigenvalues of this operator would yield to the non-trivial zeros for the Riemann zeta function, for the physicists one of the best candidates would be a Hamiltonian operator in one dimension $-\frac{d^2}{dx^2}+V(x)$, so when we apply the quantization rules the Eigenvalues (energies) of this operator would appear as the solution of the spectral determinant $\det(E-H)$, if we define the Xi-function by $\xi(s)=\frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2}$, then RH (Riemann Hypothesis) is equivalent to the fact that the function $\xi\left(\frac{1}{2}+iE\right)$ has REAL roots only , and then from the Hadamard product expansion [1] for the Xi-function , then $\frac{\xi\left(\frac{1}{2}+iE\right)}{\xi(1/2}=\det(E-H)$ is an spectral (Functional) determinant of the Hamiltonian operator, if we could give an expression for the potential V(x) so the eigenvalues are the non-trivial zeros of the zeta function, then RH would follow, we will try to use the semiclassical WKB analysis [8] to obtain an approximate expression for the inverse of the potential.

Trough this paper we will use the definition of the half-derivative $D_x^{1/2}f$ and the half integral $D_x^{-1/2}f$, this can be defined in terms of integrals and derivatives as

$$\frac{d^{1/2}f(x)}{dx^{1/2}}=\frac{1}{\Gamma(1/2)}\frac{d}{dx}\int_0^x\frac{dtf(t)}{\sqrt{x-t}} \qquad \frac{d^{-1/2}f(x)}{dx^{1/2}}=\frac{1}{\Gamma(1/2)}\int_0^x dt\frac{f(t)}{\sqrt{x-t}} \quad (1)$$

The case $D_x^{3/2}f$ we can simply use the identity $D_x^{3/2}f=\frac{d}{dx}\left(D_x^{1/2}f\right)$, these half-integral and derivative will be used further in the paper in order to relate the inverse of the potential V(x) to the density of states g(E) that 'counts' the energy levels of a one dimensional (x,t) quantum system.

- *Semiclassical evaluation of the potential V(x) :*

Unfortunately the potential V can not be exactly evaluated, a calculation of the potential can be made using the semiclassical WKB quantization of the Energy

$$2\pi n(E)=2\int_0^{a=a(E)}\sqrt{E-V(x)}dx\rightarrow 2\int_0^E\sqrt{E-V}\frac{dx}{dV}=\sqrt{\pi}D_x^{-3/2}\left(\frac{dV^{-1}(x)}{dx}\right) \quad (2)$$

Here we have introduced the fractional integral of order 3/2 , for a review about fractional Calculus we recommend the text by Oldham [11] for a good introduction to fractional calculus , a solution to equation (2) can be obtained by applying the inverse operator $D_x^{1/2}$ on the left side to get

$$V^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}n(x)}{dx^{1/2}} \qquad V^{-1}(x)=2\sqrt{\pi}\frac{d^{-1/2}g(x)}{dx^{-1/2}} \qquad \frac{dn}{dx}=g(x) \quad (3)$$

Here n(E) or N(E) is the function that counts how many energy levels are below the energy E , and g(E) is the density of states $g(E)=\sum_{n=0}^{\infty}\delta(E-E_n)$, for the case of Harmonic oscillator $N(E)=\frac{E}{\omega}$ so using formula (2) and taking the inverse function we recover the potential $V(x)=\frac{\omega^2x^2}{4}$,which is the usual Harmonic potential for a mass

$2m = 1$ a similar calculation can be made for the infinite potential well of length 'L' with boundary conditions on $[0, \infty)$ to check that our formula (3) can give coherent results

In general, g(E) is difficult to calculate and we can only give semiclassical approximations to it via the Gutzwiller Trace formula [8] , for the case of the Riemann Zeta function , N(E) can be defined by the equation

$$N(E) = \frac{1}{\pi} Arg\xi\left(\frac{1}{2} + iE\right) \qquad \xi\left(s\right) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{-s/2} \qquad (4)$$

So, in this case the Potential V(x) inside the one dimensional Hamiltonian operator whose energies are precisely the imaginary part of the Riemann zeros is given implicitly by the functional equation

$$V^{-1}(x) = \frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\xi'}{\xi}\left(\frac{1}{2} + ix\right)\right) + \frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\xi'}{\xi}\left(\frac{1}{2} - ix\right)\right) \qquad (5)$$

$V^{-1}(x)$ is the inverse of V(x) , taking the inverse function of formula (5) we could recover the potential (at least numerically).

Using the asymptotic calculation of the smooth density of states , we could separate formula (4) into an oscillating part defined by the logarithmic derivative of the Riemann zeta function and a smooth part whose behaviour is well-known for big 'x'

$$\frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2} + ix\right)\right) + \frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2} - ix\right)\right) + \frac{1}{\sqrt{\pi}}\frac{d^{1/2}}{dx^{1/2}}\left(x\log x - x + c\right) \quad (6)$$

$c = \frac{7\pi}{4}$, Using Zeta regularization , as we did in our previous paper [6] we can expand the oscillating part of formula (6) into the divergent series

$$\frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2} + ix\right)\right) + \frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2} - ix\right)\right) = 2\sum_{n=2}^{\infty}\frac{\Lambda(n)\cos(x\log n + \pi/4)}{\sqrt{n\pi}\log n} \quad (7)$$

$\Lambda(n)$ is the Von-Mangoldt function that takes the value log(p) if $n = p^m$ for some positive integer 'm' and a prime p and 0 otherwise , so the last sum inside (7) involves a sum over primes and prime powers.

Then , using (3) we have found a relationship between a classical quantity , the potential V(x), and the density of states $g(E) = \sum_{n=0}^{\infty}\delta(E - E_n)$ of a one dimensional dynamical system, the problem here is that g(E) can not be determined exactly unless for trivial Hamiltonians (Harmonic oscillator, potential well) the best evaluation for g(E) would come from the Gutzwiller Trace [8] , some people believe [4] that a possible proof for the Riemann Hypothesis would follow from the quantization of an hypothetical dynamical system whose dynamical zeta function is proportional to $\zeta\left(\frac{1}{2}+iE\right)$ to the spectral determinant of this dynamical system is $\det\left(E-H\right)=e^{-iN(E)}\zeta\left(\frac{1}{2}-iE\right)$, in this simple case the periodic orbits of the dynamical system are proportional to $\log\left(p^{m}\right)$ for m positive integer and 'p' a prime number, in this case the Quantization of the Hamiltonian 'H' would yield to the imaginary part of the non-trivial zeros, these zeros then would appear to be eigenvalues (energies) of H , since H is self-adjoint /Hermitean this energies would be all REAL and all the non-trivial zeros would be of the form $\frac{1}{2}+it \quad t\in R$, in this case the approximate Gutzwiller Trace would be of the form

$$g(E)\approx g_{smooth}(E)+\frac{1}{\pi}\Im m\left(\frac{\partial}{\partial E}\log\zeta\left(\frac{1}{2}+iE\right)\right) \quad (8)$$

Here $g_{smooth}(E)=\frac{1}{2\pi}Arg\Gamma\left(\frac{s}{2}\right)s(s-1)\pi^{-s/2}$, $s=\frac{1}{2}+iE$ this contribution is well-known , for big energies E this is the main contribution to the density of states g(E) , the part involving the logarithmic derivative inside (8) is the oscillating part of the potential giving the zeros, if we combine (5) and (8) we can obtain an expression for the inverse of the potential V(x) , then solving the Hamiltonian $H=-\frac{d^2}{dx^2}+V(x)$ with the potential given by formulae (5) and (6) we could obtain approximately the imaginary parts of the non-trivial zeros.

- *Numerical calculations of functional determinants using the Gelfand-Yaglom formula :*

In the semiclassical approach to Quantum mechanics we must calculate path integrals of the form $\int_V D\left[\phi\right]e^{-\langle\phi|H|\phi\rangle}=\frac{1}{\sqrt{\det H}}$ and hence compute a Functional determinant, one of the fastest and easiest way is the approach by Gelfand and Yaglom [2] , this technique is valid for one dimensional potential and allows you calculate the functional determinant of a certain operator 'H' without needing to compute any eigenvalue, for example if we assume Dirichlet boundary conditions on the interval $[0,\infty)$

$$\frac{\det\left(H+z^2\right)}{\det(H)}=\frac{\prod_{n=0}^{\infty}(\lambda_n+z^2)}{\prod_{n=0}^{\infty}\lambda_n}=\prod_{n=0}^{\infty}\left(1+\frac{z^2}{\lambda_n}\right)=\frac{\Psi^{(z)}(L)}{\Psi^{(0)}(L)} \qquad L\to\infty \qquad (9)$$

Here the function $\Psi^{(z)}(L)$ is the solution of the Cauchy initial value problem

$$\left(-\frac{d^2}{dx^2}+V_{RH}(x)+z^2\right)\Psi^{(z)}(x)=0 \qquad \Psi^{(z)}(0)=0 \qquad \frac{d\Psi^{(z)}(0)}{dx}=1 \qquad (10)$$

For our Hilbert-Polya Hamiltonian , the imaginary part of the non-trivial zeros would appear as the solution of the eigenvalue problem $H\phi = E_n\phi$ so $E_n=\gamma_n$ (imaginary part of the Riemann zeros) with the boundary condition

$$\phi(0)=\phi(L)=0 \quad L\to\infty \qquad V_{RH}{}^{-1}(x)=\frac{2}{\sqrt{\pi}}\Re e\left(\frac{1}{\sqrt{i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\xi'}{\xi}\left(\frac{1}{2}+ix\right)\right)\right) \qquad (11)$$

Since $N(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$ then $N(0)=0$, also the Riemann Xi-function is an even function because $\xi(s)=\xi(1-s)$, $s=\frac{1}{2}+iz$, another possible Dirichlet boundary conditions are $\phi(-L)=\phi(L)=0$ as $L\to\infty$, this is equivalent to the assertion that $\phi\in L^2(R)$, in QM the eigenfunctions must be square-integrable $\int_{-\infty}^{\infty}dx\left|\phi(x)\right|^2<\infty$, then we could use the Gelfand-Yaglom theorem to evaluate the spectral determinants so for our Hamiltonian operator H we find the formula

$$\frac{\det(H-z)}{\det(H)}\cdot\frac{\det(H+z)}{\det(H)}=\frac{\xi\left(\frac{1}{2}+iz\right)}{\xi(1/2)}=\frac{\phi^{(z)}(L)}{\phi^{(0)}(L)}\cdot\frac{\phi^{(-z)}(L)}{\phi^{(0)}(L)}\cdot \quad L\to\infty \quad (12)$$

the functions $\phi^{(\pm z)}(x)$ defined in (12) satisfy the initial value problem

$$(H\pm z)\phi^{(\pm z)}(x)=\left(-\frac{d^2}{dx^2}+V(x)\pm z\right)\phi^{(\pm z)}(x)=0 \qquad \phi^{(\pm z)}(0)=0 \qquad \frac{d\phi^{(\pm z)}(0)}{dx}=1 \qquad (13)$$

If we take the logarithm inside the Gelfand-Yaglom expression for the functional determinants [2] we can also get an expression for the spectral zeta function of eigenvalues for integer values of 's'

$\log\frac{\det(H-z)}{\det(H)}+\log\frac{\det(H+z)}{\det(H)}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\zeta_H(n)z^{2n}$ with $\sum_{n=0}^{\infty}\frac{1}{\lambda_n^s}=\zeta_H(s)$, for the case of the Riemann Xi-function , if RH is true then we should have that the Taylor expansion of $\log\xi\left(\frac{1}{2}+z\right)-\log\xi\left(\frac{1}{2}\right)$ can be used to extract information about the sums $\sum_{n=0}^{\infty}\frac{1}{\gamma_n^k}$ involving the imaginary parts of the Riemann zeros , in general these sums $\sum_{n=0}^{\infty}\frac{1}{\gamma_n^k}$ can be evaluated by numerical methods so we can compare the Taylor series of the logarithm of Xi function near x=0 and these sums to check the validity (at least numerically) of Riemann Hypothesis. The condition for the functional determinant of the self-adjoint operator $(H+1im)(H-i1m)$ to be proportional to the function $\xi\left(\frac{1}{2}+x\right)=\prod_{n=1}^{\infty}\left(1+\frac{x^2}{\gamma_n^2}\right)$ must be imposed in order to ensure that ALL the zeros of the Riemann Zeta function are real , for example for the hyperbolic sine $\sinh(\sqrt{x})=\sqrt{x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n^2\pi^2}\right)$, $E_n=n^2\pi^2$ Energies of the infinite potential well of length 1 , all the roots are purely imaginary , this can be viewed as a Riemann Hypothesis for the hyperbolic sine function , another example is the cosine function $d(x)=\cos\left(\frac{\pi x}{\omega}\right)$, whose roots are precisely the energies of the Quantum harmonic oscillator $E_n=\left(n+\frac{1}{2}\right)\omega$, and the density of states is defined by the Poisson summation formula $\sum_{n=-\infty}^{\infty}\delta\left(x-n\omega-\frac{\omega}{2}\right)=g(x)=\sum_{n=-\infty}^{\infty}(-1)^n e^{\frac{2\pi inx}{\omega}}$

- *Why this method works ?:*

Using the semiclassical approach we have stablished that the inverse of potential V(x) is related to the half-derivative of the eigenvalues counting function N(E) , for the case of the infinite potential well (V=0 and L=1) the linear potential and the Harmonic oscillator, using the semiclassical WKB approach together with $V^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}n(x)}{dx^{1/2}}$

Harmonic oscillator $V=\frac{(\omega x)^2}{4}$ $\quad N(E)=\frac{E}{\omega}$ $\quad V^{-1}(x)=\frac{2\sqrt{E}}{\omega}$ (14)

Linear potential $V=kx$ $\quad N(E)=\frac{2E^{3/2}}{3\pi k}$ $\quad V^{-1}(x)=\frac{x}{k}$ (15)

Infinite potential well $V=0$ $\quad N(E)=\frac{\sqrt{E}}{\pi}$ $\quad V^{-1}(x)=1$ (16)

In all cases and for simplicity we have used the notation $\hbar=2m=1=L$, here 'L' is the length of the well inside (14) , (15) and (16) are correct results that one can obtain using the exact Quantum theory , (16) gives 1 instead of the expected result $V=0$, in order to calculate the fractional derivatives for powers of E we have used the identity $\frac{d^{1/2}E^k}{dE^{1/2}}=\frac{\Gamma(k+1)}{\Gamma(k+1/2)}E^{k-1/2}$ [11] , a similar formal result can be applied to Bohr's atomic model for the quantization of Energies inside Hidrogen atom $E=-\frac{13.6}{n^2}$.
For the general case of the potentials $V=Cx^m$ with m being a Natural number our formula , $V^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}n(x)}{dx^{1/2}}$ predicts that the approximate number of energy levels

below a certain Energy E will be (approximately) $N(E)=\frac{1}{\sqrt{4\pi}}\frac{\Gamma\left(\frac{1}{m}+1\right)}{\Gamma\left(\frac{1}{m}+\frac{3}{2}\right)}E^{\frac{1}{m}+\frac{1}{2}}$, see [11] for the definition of the half-integral for powers of 'x' . It was prof. Mussardo [10] who gave a similar interpretation to our formula $V^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}n(x)}{dx^{1/2}}$ in order to calculate the Quantum potential for prime numbers, he reached to the conclussion that the inverse of the potential inside the Quantum Hamiltonian $-\frac{d^2}{dx^2}+V(x)=H$ giving the prime numbers as Eigenvalues/Energies of H , should satisfy the equation $V^{-1}(x)=2\sqrt{\pi}\frac{d^{1/2}\pi(x)}{dx^{1/2}}$, here $\pi(x)=\sum_{p\le x}1$ is the Prime counting function that tells us how many primes are below a given real number x , there is no EXACT formula for $\pi(x)=\sum_{p\le x}1$ so Mussardo used the approximate expression for the derivative given by the Ramanujan formula $\frac{d\pi(x)}{dx}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{x^{1/n}}{\log x}\right)$ [10] , where $\mu(n)$ is the Mobius function , a number-theoretical function that may take the values -1,0, 1 (see Apostol [1] for further information).

A formal justification of why the density of states is related to the imaginary part of the logarithmic derivative of $\xi\left(\frac{1}{2}+iz\right)$ can be given as the following, let us suppose that the Xi-function has only real roots , then in the sense of distribution we can write

$$\frac{\xi'}{\xi}\left(\frac{1}{2}+iz\right)=\sum_{n=-\infty}^{\infty}\frac{a_n}{x+i\varepsilon-\gamma_n} \qquad a_n=\mathrm{Re}\,s\left(z=\gamma_n,\frac{\xi'}{\xi}\right) \qquad (17)$$

Here, $\varepsilon\to 0$ is an small quantity to avoid the poles of (16) at the Riemann Non-trivial zeroes $\{\gamma_n\}$,taking the imaginary part inside the distributional Sokhotsky's formula $\frac{1}{x-a+i\varepsilon}=-i\pi\delta(x-a)+P\left(\frac{1}{x-a}\right)$ one gets the density of states

$$g(E)=\frac{1}{\pi}\Im m\partial_E\log\xi\left(\frac{1}{2}+iE\right)=-\sum_{n=-\infty}^{\infty}\delta(E-\gamma_n) \qquad (18)$$

Integration with respect to E will give the known equation $N(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$, a similar expression can be obtained via the 'argument principle' of complex integration $N(E)=\frac{1}{2\pi i}\int_{D(E)}\frac{\xi'}{\xi}(z)\,dz$, with D a contour that includes all the non-trivial zeros below a given quantity E , the density of states can be used to calculate sums over the Riemann zeta function (nontrivial) zeros, for example let be the identities

$$\sum_{\gamma} f(\gamma) = -\frac{1}{\pi}\int_0^{\infty} dsf'(s)Arg\xi\left(\frac{1}{2}+is\right) \qquad -\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iz\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}e^{is\log n} \qquad (19)$$

Combining these both [6] we can prove the Riemann-Weyl summation formula

$$\sum_{\gamma} f(\gamma) = 2f\left(\frac{i}{2}\right) - g(0)\log\pi - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dsf(s)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right) \qquad (20)$$

With $f(x) = f(-x)$ and $g(x) = g(-x)$ and $g(y) = \frac{1}{\pi}\int_0^{\infty} dx\cos(yx)f(x)$, if we are allowed to put $f = \cos(ax)$ into (20) ,then the Riemann-Weyl formula can be regarded as an exact Gutzwiller trace for a dynamical system with Hamilton equations

$$2p = \dot{x} \qquad \dot{p} = -\frac{\partial V}{\partial x} \qquad n(E) = \frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right) \qquad V^{-1}(x) = 2\sqrt{\pi}\frac{d^{1/2}n(x)}{dx^{1/2}} \qquad (21)$$

Then the Gutzwiller trace for this dynamical one dimensional system (x,t) is $g(E) = g_{smooth}(E) + \sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\cos(E\log n)$, for big E the smooth part can be approximated by $g_{smooth}(E) \approx \frac{\log E}{2\pi}$. The sum involving the Mangold function $\Lambda(n)$ is divergent, however it can be regularized in order to give the real part of the logarithmic derivative of Riemann Zeta $-\frac{\zeta'}{\zeta}\left(\frac{1}{2}+iE\right)$

- *Numerical solution of Schröedinguer equation:*

In order to solve our operator $-\frac{d^2}{dx^2} + V(x) = H$ with boundary conditions $y(0) = y(L) = 0 \quad L = 10^6$ we need to calculate the potential V(x). We plot the functions $\frac{\xi'}{\xi}\left(\frac{1}{2} \pm iz\right)$ on the intervals over the Real line, $\left(\gamma_j, \gamma_{j+1}\right)$ $\gamma_j > 0$ $\xi\left(\frac{1}{2} \pm i\gamma_j\right) = 0 \quad \forall j$, the problem here is that the logarithmic derivative of the Xi-function is singular on the Riemann Zeros , in this case we should make a piecewise Lagrange Polynomial fitting $L_m^j(x)$ for every interval involving 2 zeros of Riemann Xi-function , in order to calculate the fractional integral of order ½ for our fitting Polynomials, we can use the formula of fractional calculus $\frac{d^{-1/2}x^n}{dx^{-1/2}} = \frac{\Gamma(n+1)}{\Gamma(n+3/2)}x^{n+1/2}$. Now we plot a graph of the fractional integral for our fitting Polynomials and make the reflection of every point acrros the line $y = x$, in order to obtain the plot of the potential V(x) , since no analytic form of V(x) is known we will have to get another polynomial to fit the points , so $L_{2,m}(x) \approx V(x)$ and we solve the Sturm-Liouville problem on the interval (0,L) to get the Eigenvalues.

A comparison of our first 6 Eigenvalues obtained by the numerical solution of the Shcröedinguer equation with the tables of Zeros obtained by Ozdlyzko [12] gives

n	0	1	2	3	4	5
roots	14.1347	21.0220	25.0108	30.4248	32.9350	37.5861
Eigenvalues	14.0867	21.0169	25.0099	30.4194	32.9256	37.5856

- *Corrections to the Wu-Sprung potential:*

We can split our inverse potential into 2 terms , a first term is proportional to the real part of the Half integral of $\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)$, and an oscillating part that were known to Wu and Sprung [14] , that used the approximation $N(T)\approx\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}$ for big 'T'

$$V^{-1}(x)=\frac{1}{\sqrt{\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}+ix\right)\right)+\frac{1}{\sqrt{-\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}-ix\right)\right)-\frac{\sqrt{x}}{\pi^{3/2}}\log\pi+$$
$$\frac{1}{\sqrt{\pi x}}+\frac{1}{\sqrt{2\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ix}{2}\right)\right)+\frac{1}{\sqrt{-2\pi i}}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-\frac{ix}{2}\right)\right) \qquad (22)$$

The first part is just the oscillating contribution to the potential produced by the distribution of the prime numbers, and it is equal to the zeta-regularized value of the sum $\sum_{n=1}^{\infty}\frac{\Lambda(n)\cos\left(s\log n+\pi/4\right)}{\sqrt{n\log n}}$, the idea is that the potential V(x) must be compatible with the semiclassical approximation of Quantum mechanics , but also if the imaginary part of the zeros are the Eigenvalues of a certain operator, it must also obey the Rieman-Weyl trace formula relating primes and Riemann Zeta zeros

$$\sum_{\gamma}h(\gamma)=2h\left(\frac{i}{2}\right)-2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n)+\frac{1}{2\pi}\int_{-\infty}^{\infty}dr h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)-g(0)\log\pi \qquad (23)$$

If we combine (23) with the semiclassical approximation for the sums over eigenvalues

$$\sum_{n=-\infty}^{\infty}e^{iuE_n}\approx\sqrt{\frac{\pi}{u}}\int_{-\infty}^{\infty}dx e^{iux+i\pi/4}\frac{dV^{-1}(x)}{dx}=-i\sqrt{\pi u}\int_{-\infty}^{\infty}dx e^{iux+i\pi/4}V^{-1}(x) \qquad (24)$$

In order to obtain $V^{-1}(x)$ from (24) we take the inverse Fourier transform, this involves the sum over Rieman Zeros $\sum_{\gamma}H(x-\gamma)\left(x-\gamma\right)^{-1/2}$, if we put inside formula (23) $\frac{d^{1/2}H(x\pm u)}{dx^{1/2}}\approx H(x\pm u)\left(x\pm u\right)^{-1/2}$ with $H(x)\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x<0\end{cases}$ and use the integral $\int_{0}^{\infty}\frac{dx}{\sqrt{x}}e^{iux}=\sqrt{\frac{\pi}{u}}e^{i\pi/4}$ we find the desired result given in (22) , so our inverse potential

$V^{-1}(x)$ is in perfect agreement with the one given by Wu and Sprung, and also is compatible with the Riemann-Weyl expression (at least in distributional sense) relating Riemann Zeros and prime numbers. This inverse potential according to Riemann-Weyl formula plus the Zeta-regularized series (ignoring the divergent terms proportional to ε^{-k} as $\varepsilon \to 0$) has an oscillating and an smooth part

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)\cos\left(s\log n+\pi/4\right)}{\sqrt{n\log n}} \qquad \int_{-\infty}^{x} \frac{dt}{\sqrt{x-t}} \Re e \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}\right) \quad (25)$$

The first term can be regularized to give the Real part of $\frac{d^{-1/2}}{ds^{-1/2}} \frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)$ for any 's' the second one is just the real part of $\frac{d^{-1/2}}{ds^{-1/2}} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{is}{2}\right)$, in general since we are interested in the Riemann Zeros γ_n as $n \to \infty$ the smooth part can be approximated (for big 'x') very well by the term $\frac{d^{1/2}}{dx^{1/2}}\left(\frac{x}{2\pi}\log\frac{x}{2\pi}-\frac{x}{2\pi}\right)$, but in order our Hamiltonian fulfills Riemann Hypothesis , also the oscillating part must be included into the potential, this is a fact that is ignored by Wu and Sprung, the inverse of the potential $V^{-1}(x)$ can be simply obtained by imposing the Riemann-Weyl formula plus using the semiclassical approximation to relate a quantum mechanical Quantities (Energies, density of energies) to a pure classical quantity like the potential V(x). Also the formula for the Energies $\sum_n H(x-E_n) = \frac{1}{\pi}\arg\xi\left(\frac{1}{2}+ix\right)$, $\xi(s) = \frac{s(s-1)}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s)$ is just a consequence of the Riemann-Weyl formula , that stablishes a relationship between Riemann Zeros and prime numbers and that we have considered to be valid even in the distributional sense.

CONCLUSIONS AND FINAL REMARKS

Using the semiclassical analysis and the WKB quantization of energies, we have managed to prove that for one dimensional systems the inverse of the potential inside the Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$ (for simplicity through this paper we have used the notation $\hbar = 1 = 2m$ to simplify calculation) is related to the half-derivative of the energy counting function N(E) or the half-integral of the density of states $g(E) = \sum_{n=0}^{\infty}\delta(E-E_n)$ in the (approximate) form $V^{-1}(x) = \sqrt{\frac{2\pi\hbar^2}{m}\frac{d^{-1/2}g(x)}{dx^{-1/2}}}$. From the definition of the counting function N(T) for the nontrivial zeros of the Riemann Zeta function lying on the critical strip $N(E) = \frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$ and using our formula (2) with 2m=1 we have obtained the semiclassical approximation for the inverse of the potential V(x) (4) and have given some Numerical test that could be done in order to check that our definition for the inverse of the potential $V^{-1}(x)$ can be used to obtain a

Hilbert-Polya differential operator whose eigenvalues (Energies) are precisely the imaginary part of the non-trivial zeros of the Riemann Zeta function, in case our formula is correct and (4) is the potential of a Hilbert-Polya operator satisfying Riemann Hypothesis then imposing Dirichlet boundary conditions on $[0,\infty)$ then the Gelfand-Yaglom formula should give

$$\frac{\det(H-z)}{\det(H)}\cdot\frac{\det(H+z)}{\det(H)}=\frac{\xi\left(\frac{1}{2}+iz\right)}{\xi(1/2)}=\frac{\phi^{(z)}(L)}{\phi^{(0)}(L)}\cdot\frac{\phi^{(-z)}(L)}{\phi^{(0)}(L)}\cdot \quad L\to\infty$$ so the functional

determinant of $H=-\frac{d^2}{dx^2}+V(x)$ is proportional to the Riemann Xi-function , this potential V(x) is given implicitly in (5) and (6) . A possible better improvement of our formula (3) would be to write down $V^{-1}(x)=A\frac{d^{1/2}n(x)}{dx^{1/2}}$ for some constant 'A' to be fixed in order to obtain (if possible) the correct energies of our Hamiltonian at least when $n\to\infty$ E_n , for simplicity since the spatial term (derivative) inside Schröedinguer equation takes the form $\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ we chose $\hbar=2m=1$ to simplify calculations. The boundary conditions for the Eigenfunction y(x) in order to solve our proposed operator related to RH are of the form $y(\pm\infty)=0$ or $y(\infty)=0=y(0)$. As a summary of our paper we can say

- We look for a Hamiltonian operator $-\frac{d^2}{dx^2}+V(x)=H$, $\hbar=2m=1$ with certain Dirichlet boundary condition $y(\infty)=0=y(0)$ (infinite barrier at x= 0) or $y(\pm\infty)=0$ so its Eigenvalues /Energies are precisely the imaginary part of the Riemann Zeros $\zeta\left(\frac{1}{2}+iE_n\right)=0 \quad E_n=\gamma_n$
- The potential V(x) is REAL so the Hamiltonian is self-adjoint $H=H^{\dagger}$ so all the Energies will be Real and Riemann Hypothesis is TRUE
- In order to obtain the potential , we use the WKB approximation, in this approximation $V^{-1}(x)=\sqrt{\frac{2\pi\hbar^2}{m}}\frac{d^{1/2}n(x)}{dx^{1/2}}$, that is the inverse of the potential V(x) is proportional to the half derivative of the Eigenvalue counting function $n(E)=\sum_n H(E-E_n)$, with $H(x)$ being the Heaviside step function that takes the value 1 if x is positive and 0 otherwise , for the case of the Riemann Xi-function this n(E) can be evaluated exactly by the 'Argument principle' , $n(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$, the argument of a complex number is just the imaginary part of its logarithm
- Since $n(E)=\frac{1}{\pi}Arg\xi\left(\frac{1}{2}+iE\right)$ gives the number of zeros with imaginary part between 0 and E ,the energies of our Hamiltonian surely will give $E_n=\gamma_n\in R^+$, the positive imaginary part of the Riemann Zeros, so the

eigenvalues of $-H$ will be the negative imaginary part of the Riemann Zeros ,hence $-H^2 = \gamma^2$.

- If our approach is valid and the Energies are just the imaginary part of the Riemann Non-trivial zeros $E_n = \gamma_n$, the Hadamard product representation for the Riemann Xi-function may be described as the product of 2 functional determinants $\frac{\det(H-z)}{\det(H)} \cdot \frac{\det(H+z)}{\det(H)} = \frac{\xi\left(\frac{1}{2}+iz\right)}{\xi(1/2)} = \frac{\phi^{(z)}(L)}{\phi^{(0)}(L)} \cdot \frac{\phi^{(-z)}(L)}{\phi^{(0)}(L)}$.
- The functions $\phi^{(\pm z)}(x)$ can be calculated by solving an initial value problem $\left(-\frac{d^2}{dx^2} + V(x) \pm z\right)\phi^{(\pm z)}(x) = 0$ with $\phi^{(\pm z)}(0) = 0$ and $\frac{d\phi^{(\pm z)}(0)}{dx} = 1$
- If the Energies of our Hamiltonian H include both cases, positive and negative imaginary parts of Riemann Zeros then the Riemann Xi-function $\frac{\xi\left(\frac{1}{2}+iz\right)}{\xi(1/2)}$ may be given by the quotient of functional determinant $\frac{\det(H-z)}{\det(H)}$ $z \in R$
- We have checked our formula $V^{-1}(x) = \sqrt{\frac{2\pi\hbar^2}{m}} \frac{d^{1/2}n(x)}{dx^{1/2}}$,(as pointed before in the paper this equation is a result of the Bohr-sommerfeld quantization rule for the one dimensional quantum problem with a potential V(x)), with $\hbar = 2m = 1$ for the case of the Infinite potential Well $V = 0$ the linear potential $V = kx$ and the Quadratic potential $V = \frac{m\omega^2 x^2}{2}$ for almost all cases we have got correct results , so we believe that this is a good reason to check that our approach to the Riemann Hypothesis using QM is valid.
- Using Lagrange-Bürmann formula, we can express the potential V as a taylor series on ‘x’ $V(x) = \sum_{n=0}^{\infty} c_n x^n$ $c_n = \lim_{u \to 0} \frac{d^{n-1}}{du^{n-1}} \left(\frac{u}{n(u)}\right)^n$ $n(E) = \frac{1}{\pi} Arg\xi\left(\frac{1}{2}+iE\right)$

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). "Riemann Zeta Function and Other Sums of Reciprocal Powers." §23.2 in “*Handbook of Mathematical Functions*”. New York: Dover, pp. 807-808, 1972.

[2] Apostol Tom “Introduction to Analytic Number theory” ED: Springuer-Verlag, (1976)

[3] Berndt B. “Ramanujan's Notebooks: Part II “ Springer; 1 edition (1993) ISBN-10: 0387941096

[4] Conrey, J. B. "The Riemann Hypothesis." Not. Amer. Math. Soc. 50, 341-353, 2003. available at http://www.ams.org/notices/200303/fea-conrey-web.pdf.

[5] Dunne G. And Hyunsoo Min "*A comment on the Gelfand–Yaglom theorem, zeta functions and heat kernels for PT-symmetric Hamiltonians*" J. Phys. A: Math. Theor. **42** No 27 (10 July 2009) 272001.

[6] Garcia J.J "*Zeta Regularization applied toRiemann hipótesis and the calculations of divergen integralst*" General Science Journal, e-print: http://wbabin.net/science/moreta23.pdf

[7] Griffiths, David J. (2004). "*Introduction to Quantum Mechanics*" Prentice Hall. ISBN 0-13-111892-7.

[8] Gutzwiller, M. "*Chaos in Classical and Quantum Mechanics*", (1990) Springer-Verlag, New York ISBN=0-387-97173-4.

[9] Hardy G.H "*Divergent series*" , Oxford, Clarendon Press (1949)

[10] Mussardo G. " *The Quantum Mechanical potential for the Prime Numbers*"
-------http://arxiv.org/abs/cond-mat/9712010

[11] Oldham, K. B. and Spanier, J. "The Fractional Calculus: Integrations and *Differentiations of Arbitrary Order*". New York: Academic Press, (1974).

[12] Odlyzko A. " *The first* 10^5 *zeros of the Riemann function, accurate to within* $3x10^{-9}$ " freely avaliable at http://www.dtc.umn.edu/~odlyzko/zeta_tables

[13] Pitkanen M "*An startegy for proving Riemann Hypothesis* " arXiv:math/0111262

[14] Wu H. and D. W. L. Sprung "Riemann zeros and a fractal potential" Phys. Rev. E 48, 2595–2598 (1993)