THE THEORY OF DISTRIBUTIONS APPLIED TO DIVERGENT INTEGRALS OF THE FORM $\int_a^c \frac{dx\varphi(x)}{(x\pm b)^u}$

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ABSTRACT: In this paper we review some results on the regularization of divergent integrals of the form $\int_a^c \frac{\varphi(x)dx}{x-b}$ and $\int_a^\infty \frac{dx}{x+a}$ in the context of distribution theory

- *Keywords:* Regularization, distribution theory, fractional calculus

Regularization of divergent integrals:

In QFT (Quantum field theroy) there are mainly two kinds of divergent integral , the UV (ultraviolet) divergence, that happens whenever an integral is divergent when $x\to\infty$ and the IR (infrared) one that occurs if the integral is divergent as $x\to 0$,a few examples of these integrals are

$$\int_0^\infty \frac{dxx^3}{x+a} \qquad \int_0^\infty \frac{dxx^2}{(x+a)^2} \qquad \int_0^\infty \frac{dx}{x^2(x+a)} \qquad \int_0^\infty \frac{dx}{x^3} \qquad a\in R^+ \quad \text{or} \quad a\in Z^+ \qquad (1)$$

The first two integrals have an UV divergence, whereas the last ones have an IR divergence,the names IR and UV divergences come from the fact that if we use the Wave-particle Duality in QM $\lambda=\frac{h}{|p|}$ setting x=p we find that UV divergence is equivalent to have small wavelengths λ (ulra violet) and the IR divergence happens for big wavelengths (infrared) so the name is not casual, for more details [5] and [9]

To deal with UV divergences on an integral of the form $\int_{R^d} dk^d F(k)$ on R^d we simply make a change of variable to d- dimensional polar coordinates , to rewrite the integral

$$\int_{R^d} dk^d F(k) = \int_{\Omega} d\Omega \int_0^c dr r^{d-1} F(r,\Omega) + \int_{\Omega} d\Omega \int_c^{\infty} dr \left(\sum_{i=0}^{k} a_i(\Omega) r^i + \frac{a_{-1}(\Omega)}{r} \right) - \int_{\Omega} d\Omega \sum_{n=2}^{\infty} \frac{a_{-n}}{c^{n-1}(1-n)}(\Omega) \qquad (2)$$

With the constants $a_n(\Omega) = \frac{1}{2\pi i}\oint_{\gamma} \frac{dzF(z,\Omega)}{(z-1)}$, here Ω means that we must integrate over the angular variables , in case F is invariant under rotations on the d-dimensional plane the angular part of the integral can be calculated exactly as $\frac{2\pi^{d/2}}{\Gamma(d/2)} = \int_{\Omega} d\Omega$, the idea of expanding our function into a Laurent series , is to isolate the UV divergencies of the form $\int_0^{\infty} dr r^k$ so they can be 'cured' using the zeta regularization algorithm. Choosing any c >1 , since the integral has a UV divergence , after expanding the integrand $F(r,\Omega)r^{d-1}$ into a convergent Laurent series for |r| >1 there will be only a finite number of divergent integrals of the form $\int_0^{\infty} dr r^k$ plus a logarithmic divergent integral (this is another example of UV divergence) $\int_0^{\infty} \frac{dr}{r+c}$, now to get finite results from the divergent integrals we will apply the recurrence deduced in our previous paper [4] when discussing the zeta regularization applied to integrals

$$\int_0^{\infty} x^m dx = \frac{m}{2}\int_0^{\infty} x^{m-1} dx + \zeta(-m) - \sum_{r=1}^{\infty} \frac{B_{2r} m!(m-2r+1)}{(2r)!(m-2r+1)!}\int_0^{\infty} x^{m-2r} dx \qquad (3)$$

Equation (3) is a 'regularization' in the sense that if we had the upper limit N-1 instead of ∞ the expression (3) would give the well-known result $\int_0^N dx x^k = \frac{N^{k+1}}{k+1}$ k >0 or k=0 , for the case $\int_0^{\infty} dx x^k$, the series $\sum_{n=0}^{\infty} n^k$ is no longer convergent and must be given a finite value in the spirit of Zeta function regularization [4] so $\sum_{n=0}^{\infty} n^k \to \zeta_R(-k)$, this is why the term $\zeta_R(-m) = \zeta(-m)$ apperas inside (3) , note that (3) is a recurrence formula with initial term $\int_0^{\infty} dx = 1+1+1+1+1+1+1 \to \zeta(0) = -1/2$ and allows to calculate or assign a finite value to every divergent integral of the form $\int_0^\infty drr^k$ for 'k' a positive integer , (the main problem for the logarithmic divergence is the pole of the Zeta funciton at s=1) the first terms are

$$I(0,\Lambda)=\zeta(0)=-1/2=\int_0^\infty dx$$

$$I(1,\Lambda)=\frac{I(0,\Lambda)}{2}+\zeta(-1)=\int_0^\infty xdx$$

$$I(2,\Lambda)=\left(I(0,\Lambda)\frac{1}{2}+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)=\int_0^\infty x^2dx \tag{4}$$

$$I(3,\Lambda)=\frac{3}{2}\left(\frac{1}{2}\left(I(0,\Lambda)+\zeta(-1)\right)-\frac{B_2}{2}a_{21}I(0,\Lambda)\right)+\zeta(-3)-B_2a_{31}I(0,\Lambda)=\int_0^\infty x^3dx$$

Notation $I(m,\Lambda)$ stands for $\lim_{\Lambda\to\infty}\left(\int_0^\Lambda dxx^k\right)$, being 'Lambda' a cut-off with certain physical meaning.

This example is valid for UV divergences but what would happen to the case of 'singular integrals' $\int_a^c \frac{dx}{x-b}$ a <b<c or to the logarithmic divergence $\int_0^\infty \frac{dr}{r+c}$?

- *Regularization of divergent integrals using Distribution theory:*

As a first example, let us suppose we want to calculate the integral $\int_{-\infty}^\infty dxf(x)D^n\delta(x-a)$, for n=0 we know that in spite of the 'pole' of the delta function at x=a we have that $\int_{-\infty}^\infty dxf(x)\delta(x-a)=f(a)$ for any test function f(x) , then we can use two method

- We perform integration by parts avoiding the singular point at x=a so $\int_{-\infty}^\infty dx(-D)^n f(x)\delta(x-a)=\int_{-\infty}^\infty dxf(x)D^n\delta(x-a)=(-D)^n f(a)$ (5)

- We integrate n-times (no matter if n is non-integer) with respect to 'a' $I(a)=\int_{-\infty}^\infty dxf(x)D^n\delta(x-a)\to D_a^{-n}I(a)=\int_{-\infty}^\infty dxf(x)(-1)^n\delta(x-a)=(-1)^n f(a)$ (6)

Both methods are equivalent since if we recall the identity $D_a^{-n} D_a^n I(a) = I(a)$ we yield to the same result, no matter if we integrate/differentiate with respect to the parameter 'a' or if we integrate by parts.

A similar idea can be applied to integrals of the form $\int_a^c \frac{dx}{x-b}$ and $\int_a^\infty \frac{dx}{x+a}$ first we define the distribution $T(b,x) = \begin{cases} (x-b)^{-s} & \text{for a<x<c} \\ 0 \text{ otherwise} \end{cases}$, for a regular enough function $\varphi(x)$ we can consider (in the sense of distribution) our divergent integral to be the linear operator $T[\varphi] = \int_a^c \frac{dx}{(x-b)^s} \varphi(x)$ for any real 's' , the idea is to perform a differ-integration μ-times so $s + \mu = 1/2$. First we must introduce the concept of differ-integral or fractional derivative of any order, there are mainly [6] 3 definitions

$$D^{-n} f(x) = \frac{1}{\Gamma(n)} \int_0^x dt (x-t)^{n-1} f(t) dt \qquad D^q f(x) = \lim_{h \to 0} \frac{1}{h^q} \sum_{m=0}^{\infty} (-1)^m \binom{q}{m} f(x + (q-m)h) \quad (7)$$

The first definition inside (7) is the expression for fractional integral (not valid for negative 'n') , the second one is the 'Grunwald-Letnikov' differintegral valid for positive 'q' , the third alternative for the derivative comes from the definition of Cauchy's integral formula $D^q f(x) = \frac{\Gamma(q+1)}{2\pi i} \int_\gamma \frac{dz}{(z-x)^{q+1}} f(z)$ for any rectifiable curve γ on the complex plane that includes the point z=x

Example: Let be the singular integral $I(b) = \int_a^c \frac{dx}{(x-b)^s} \varphi(x)$, in order to give it a finite value first we differentiate μ-times with respect to 'b' so $s + \mu = 1/2$ hence $D_b^\mu I(b) = \frac{\Gamma(1-s)}{\Gamma(1-s-\mu)} \int_a^c \frac{dx}{\sqrt{x-b}} \varphi(x)$, we make then the change of variable $x = b + u^2$ so our integral becomes $D_b^\mu I(b) = \frac{2\Gamma(1-s)}{\Gamma(1-s-\mu)} \int_{\sqrt{a-b}}^{\sqrt{c-b}} du \varphi(b+u^2)$, then we define the function F so $\frac{dF}{du} = \varphi(b+u^2)$ this implies $D_b^\mu I(b) = \frac{2\Gamma(1-s)}{\Gamma(1-s-\mu)} \left(F(\sqrt{c-b}) - F(\sqrt{a-b}) \right)$ finally taking the inverse operator we can set

$$I(b) = \frac{2\Gamma(1-s)}{\Gamma(1-s-\mu)} \left(D_b^{-\mu} F(\sqrt{c-b}) - D_b^{-\mu} F(\sqrt{a-b}) \right) + \Psi(b,\mu) \qquad (8)$$

Here 'mu' is a real number and $D_b^{\pm\mu} f = \frac{d^{\pm\mu} f}{dx^{\pm\mu}}$ exists in the sense of a fractional derivative/integral and $D_b^\mu \Psi(b,\mu) = 0$. We set the condition $s + \mu = 1/2$ because with a change of variable we can avoid the pole $(x-b)^{-1/2}$ at x=b.

The same strategy can be applied to singular integral equation, for example if we wished to solve the following equation

$$f(x)+g(x)=\int_a^\infty \frac{dt}{(t-x)^n}f(t) \quad \text{so} \quad D_x^\mu f(x)+D_x^\mu g(x)=\frac{\Gamma(1-n)}{\Gamma(1-n-\mu)}\int_a^\infty \frac{dt}{\sqrt{t-x}}f(t) \quad (9)$$

Where $n+\mu=1/2$, making the change of variable $t=x+u^2$ so (9) becomes

$$D_x^\mu f(x)+D_x^\mu g(x)=\frac{2\Gamma(1-n)}{\Gamma(1-n-\mu)}\int_{\sqrt{a-x}}^\infty du\, f(x+u^2) \quad \Gamma=\text{Gamma function} \quad (10)$$

Now, equation (10) has NO poles at t=x , to solve this integral equation without singularity we could use an iterative process

$$f_0(x)=g(x) \qquad D_x^\mu f_n(x)+D_x^\mu g_n(x)=\frac{2\Gamma(1-n)}{\Gamma(1-n-\mu)}\int_{\sqrt{a-x}}^\infty du\, f_{n-1}(x+u^2) \quad (11)$$

Where we have supposed that $D_x^\mu \Psi(x,\mu)=0=\Psi(x,\mu)$ in order to simplify the calculations of the solution of the integral.

Then one could ask, what happens in the limit $b\to\infty$, in this case the quantity $(x-b)^{-1}$ becomes zero for every x , so the I(b) must tend to 0 for ‘b’ big hence we should choose the function $\Psi(b,\mu)$ with the conditions $D_b^\mu \Psi(b,\mu)=0$ and

$$\lim_{b\to\infty}\left(\frac{2\Gamma(1-s)}{\Gamma(1-s-\mu)}\left(D_b^{-\mu}F(\sqrt{c-b})-D_b^{-\mu}F(\sqrt{a-b})\right)+\Psi(b,\mu)\right)=0 \qquad (12)$$

- *Logarithmic divergences*

The case of the logarithmic integral is a bit different, since formula (4) can not handle it ,due to the pole of zeta function at s=1, one of the ideas to apply [4] is just to replace the divergent integral $\int_0^\infty \frac{dx}{x+a}$ by a divergent series $\sum_{n=0}^\infty \frac{1}{n+a/h}$ with ‘h’ being an step (we use Rectangle method) ,this series is still divergent but can be assigned a finite value via ‘Ramanujan resummation’ equal to the logarithmic derivative of Gamma function $\frac{\Gamma'}{\Gamma}\left(\frac{a}{h}\right)$,however this kind of method depends on the value of the step ‘h’ given.

From Fourier analysis one can interpretate , the integral $\int_0^\infty \frac{dx}{x+a}$ as the following convolution $\left(H * x^{-1}\right) = \int_{-\infty}^{\infty} dx \frac{H(x)}{x+a} = \int_a^{\infty} dx \frac{1}{x} = \int_{-\infty}^{\infty} dx \frac{H(x-a)}{x}$ with H(x) the 'Heaviside step function' , using the property of the Fourier transform and the convolution theorem

$$\left(H * x^{-1}\right) = \int_{-\infty}^{\infty} dx \frac{H(x)}{x+a} = -\frac{1}{2i} \int_{-\infty}^{\infty} d\left|u\right| \left(\frac{i}{u} + \pi\delta(u)\right) e^{iua} = I(a) \qquad (13)$$

Here |x| is the absolute value function that takes the value x or –x depending on if 'x' is either positive or negative, solving Fourier transform (13) we can solve the logarithmic UV divergence. If we can solve (13) for a fixed a then for another value 'b' so b is different from 0 or inifinite we have $\int_0^\infty \frac{dx}{x+a} = \log\left(\frac{b}{a}\right) + \int_0^\infty \frac{dx}{x+b} = I(a)$

Another simpler method is that if we have $I(a) = \int_0^\infty dx \frac{1}{x+a}$ divergent integral , we differentiate with respect to 'a' $I'(a) = -\int_0^\infty dx \frac{1}{(x+a)^2} = -\frac{1}{a}$ so integrating again with respect to 'a' $I(a) = -\log(a) + c(a) + c_1$, with c_1 an universal divergent

$$D_a^\mu I(a) = \int_0^\infty dx \frac{\Gamma(\mu+1)}{(x+a)^{\mu+1}} (-1)^\mu = (-1)^\mu \Gamma(\mu+1) \frac{1}{a^\mu} \qquad \mu > 0 \qquad (14)$$

So $I(a) = (-1)^\mu \Gamma(\mu+1) D_a^{-\mu} a^{-\mu} + \psi(a,\mu)$, with $\mu > 0$ and $D_a^\mu \Psi(a,\mu) = 0$, one of the problems is the apparent absurdity since due to the term $\left(-1\right)^\mu$ there is a complex contribution to an integral with real-valued integrand. An alternative formulation based on Hurwitz Zeta function is the following

$$\frac{\partial}{\partial a} \int_0^\infty dx \log(x+a) = \int_0^\infty dx \frac{1}{x+a} \quad \text{and} \quad \frac{\partial \zeta_H(0,a)}{\partial a} = \log\Gamma(a) - \log\sqrt{2\pi} \qquad a \neq 0 \ (15)$$

For the sum $\sum_{n=0}^{\infty} \log(n+a) = -\partial_s \zeta_H(0,a)$, this is the Zeta-regularized definition for the Determinant of an operator , a combination of the 2 expressions inside (15) gives the regularized value for the Harmonic sum $\int_0^\infty dx \frac{1}{x+a} = -\frac{\Gamma'}{\Gamma}(a) + R$ (in case a=1 we get the Euler-Mascheroni constant) , this is the analogue result to simply using 'Ramanujan resummation' for the series $\sum_{n=0}^{\infty} n^s$ s >0 , and s=-1 , 'R' stands for the Remainder term inside Euler-Maclaurin summation formula and is $R \approx \frac{1}{2a} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial x^{2r-1}} \left(\frac{1}{x+a}\right)_{x=0}$

Appendix: Convolution theorem

In this paper we have introduced and used the 'Convolution theorem' , if we have the convolution of 2 functions or distributions f(x) and g(x) defined as

$h(z)=\int_{-\infty}^{\infty} dxf(x)g(x-z)$ then $H(u)=-F(u)G(-u)$ with F, G and H the fourier transform respectively of h(z) f(x) and g(x).

Proof:= if we define $\int_{-\infty}^{\infty} dze^{-iuz}h(z)=\int_{-\infty}^{\infty} dze^{-iuz}\int_{-\infty}^{\infty} dxf(x)g(x-z)$ $\int_{-\infty}^{\infty} dxe^{-iux}f(x)=F(u)$ and $\int_{-\infty}^{\infty} dxe^{-iux}g(x)=G(u)$, we make the change of variable $y=x-z$ so dx = -dz into the first integral so we have the following identities

$$\int_{-\infty}^{\infty} dze^{-iuz}\int_{-\infty}^{\infty} dxf(x)g(x-z)=-\int_{-\infty}^{\infty} dye^{-iu(x-y)}\int_{-\infty}^{\infty} dxf(x)g(y)=-\int_{-\infty}^{\infty} dye^{-iux}\int_{-\infty}^{\infty} dxf(x)g(y)e^{iuy} \quad (16)$$

The first integral on the left is just H(u) and using Fubini's theorem to interchange the order of integration we have been able to proof that $H(u)=-F(u)G(-u)$.

If we name $g(x)=H(x)$ and $f(x)=x^k$, then $\int_{a}^{\infty} dxx^k=\int_{-\infty}^{\infty} H(x-a)x^k dx$, hence applying convolution theorem and the Fourier transform

$$\int_{a}^{\infty} dxx^k=i^k\int_{-\infty}^{\infty} du\left(\pi\delta(u)+\frac{i}{u}\right)D^k\delta(u)e^{iua} \quad k>0 \qquad (17)$$

Unfortunately there are some oddities with defining product of distributions $D^k\delta\times\frac{1}{x}$, $D^k\delta\times\delta$, $\delta\times\delta$ within distribution theory, however if the integral $\int_{a}^{\infty} f(x)dx$ makes sense as a Riemann integral , if we define F(u) as the Fourier transform of f(x) then the Convolution theorem gives the regularized result $-\frac{F(0)}{2}-\int_{c}^{a} dtf(t)+f(c)$ for some Real 'c' , for the case of $f(x)=\frac{1}{x}$ with Fourier transform $\frac{d|x|}{dx}$ the integral is divergent in the Riemann sense but can be attached a finite value $-\frac{1}{2}\frac{d|x|}{dx}\Big|_{x=0}-\log(a)$, since |x| is even its derivative will be odd so the mean value of the derivative near x=0 will be 0 and we have $\int_a^{\infty} x^{-1}dx = -\log(a)$. The last method would be to use the Euler-Maclaurin summation formula to get

$$\int_0^{\infty} dx \frac{1}{x+a} = \sum_{n=0}^{\infty} \frac{1}{n+a} - \frac{1}{2a} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} \frac{d^{2r-1}}{dx^{2r-1}}\left(\frac{1}{x+a}\right)\Big|_{x=0} \qquad (18)$$

The problem is that $\sum_{n=0}^{\infty} \frac{1}{n+a} = \zeta_H(1,a)$ is still divergent , although using Ramanujan-summation we can attach this series the finite value $\sum_{n=0}^{\infty} \frac{1}{n+a} = \zeta_H(1,a) = -\frac{\Gamma}{\Gamma}(a)$ and plug this result into (17) , In both cases the approximation of the integral by a sum $\sum_{n=0}^{\infty} \frac{1}{n+a} = -\frac{\Gamma}{\Gamma}(a)$ and the result $\int_0^{\infty} \frac{dx}{x+a} = -\log(a) + c_a$ are equivalent for $a \to \infty$ (big a) since using the Stirling's approximation for $\log\Gamma(x)$ and taking the derivative we get the asymptotic result $\lim_{a\to\infty} \frac{\Gamma'(a)/\Gamma(a)}{\log(a)} = 1$

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