the general science journal

PRODUCT OF DISTRIBUTIONS AND ZETA REGULARIZATION OF DIVERGENT INTEGRALS $\int_a^\infty x^{m-s}dx$ AND FOURIER TRANSFORMS

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- ABSTRACT: Using the theory of distributions and Zeta regularization we manage to give a definition of product for Dirac delta distributions, we show how the fact of one can be define a coherent and finite product of dDirac delta distributions is related to the regularization of divergent integrals $\int_a^\infty x^{m-s}dx$ a >0 and Fourier series, for a Fourier series making a Taylor substraction we can define a regular part $F_{reg}(u)$ defined as a function for every 'u' plus a dirac delta series $\sum_{i=0}^{N} c_i \delta^{(i)}(u)$, which is divergent for u=0 , we show then how $\delta^{(i)}(0)$ can be regularized using a combination of Euler-Mclaurin formula and analytic continuation for the series $\sum_{i=0}^{\infty} i^k = \zeta(-k)$

PRODUCT OF DIRAC DELTA DISTRIBUTIONS $\delta^{(m)}(x) \times \delta^{(n)}(x)$

One of the problems with distributions , as proved by Schartz (see ref [1]) is that we can not (in general) define a coherent product of distributions, for example

$$\left(\delta \times x\right) \times P\left(\frac{1}{x}\right) = 0 \qquad \delta \times \left(\delta \times P\left(\frac{1}{x}\right)\right) = 0 \qquad P\left(\frac{1}{x}\right)[\phi] = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\phi(x)}{x} dx \quad (1)$$

For the case of the product of a Heaviside step function H(x) with the derivatives of the Delta function (and its derivatives) we have to deal with the problem of divergent quantities, for example according to [2] we can define the product $H \times \delta^{(m)}$, with the aid of a test function $\phi(x) \in C^{\infty}(R)$ as the recurrence

$$\int_{-\infty}^{\infty} dx H(x)\delta^{(m)}(x)\phi(x) = \int_{0}^{\infty} dx \delta^{(m)}(x)\phi(x) = -\delta^{(m-1)}(0)\phi(0) - \int_{0}^{\infty} dx \delta^{(m-1)}(x)\phi'(x) \quad (2)$$

The case m=0 is just $H \times \delta = \frac{1}{2}\delta$, and comes from considering the Heaviside fucntion H(x) to be the derivative of $\delta(x)$, so $\int_{-\infty}^{\infty} dx H(x)\delta(x) = \frac{1}{2}\left(H^2(\infty) - H^2(-\infty)\right) = \frac{1}{2}$

If we use the 'Convolution theorem' [5] in a formal sense, so it can be regarded as valid even for the case that the Fourier transform are defined ONLY as distributions

$$(2\pi)^2 i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = F_{\omega}\left(x^m * x^n\right) = A F_{\omega}\left\{\int_{-\infty}^{\infty} dt t^m (x-t)^n\right\} \quad (3)$$

Here 'A' is a normalization (finite) constant that depends on the definition you take for the Fourier transform, but it can not be dependent on m or n and $D = \frac{d}{dx}$. Unfortunately (3) makes no sense (even using distribution theory) since the integral over 't' is DIVERGENT and needs to be regularized, if we use the Binomial theorem on $t^m (x-t)^n$ for m and n integres

$$i^{m+n} D^m \delta(\omega) D^n \delta(\omega) = \sum_{k=0}^{n} \binom{n}{k} i^{m+k} A D^{n-k} \delta(\omega) (-1)^k i^{n-k} D^{m+k} \delta(0) \quad (4)$$

The problem here is that $D^{m+k}\delta(0) = \frac{i^{m+k}}{2\pi}\int_{-\infty}^{\infty} x^{m+k} dx$ is infinite and would need to be regularizad in order to make sense inside (3) or (4) , for m+k being an Odd integer, using Cauchy's principal value definition $P.v\left(\int_{-\infty}^{\infty} x^{2n+1} dx\right) = 0 \quad n \in N$ (this imposes the condition that only +1 or -1 can appear inside (4) as the 'i' on both sides cancel) , the problem is that $2\int_{0}^{\infty} x^{2n} dx$ is still divergent , the same problem happened inside (2) where one needs to to regularize expressions $\delta^{(m-1)}(0)$ in order to define a coherente product of distributions involving Heaviside step-function and Dirac delta and its derivatives. In general (4) will be non-commutative so we can in general expect $\delta^{(m)}(u) \times \delta^{(n)}(u) \neq \delta^{(n)}(u) \times \delta^{(m)}(u)$ example

$$\delta(u) \times \delta^{(1)}(u) = \delta_{reg}(0)\delta^{(1)}(u) \quad \text{but} \quad \delta^{(1)}(u) \times \delta(u) = \delta(u)\delta_{reg}{}^{(1)}(0) = 0 \quad (5)$$

The last equality in (5) comes from the fact that $i\delta'(0) = (2\pi)^{-1}\int_{-\infty}^{\infty} xdx$ is 0 by using Cauchy's principal value , the case m=n=0 is just the square of delta function $\delta \times \delta = \delta^2 = \frac{A}{2\pi}\delta$, this can be obtained from the zeta regularization $\int_{-\infty}^{\infty} dx = 2\int_{0}^{\infty} dx = \left(2\sum_{n=0}^{\infty} 1\right)_{reg} = 1$ as we will see in the next section

- *Zeta regularization for divergent integrals:*

In our previosu paper [4] we used the Euler-Maclaurin summation formula with $f(x) = x^{m-s}$ in order to stablish

$$\begin{aligned}&\int_{a}^{\infty} x^{m-s}dx = \frac{m-s}{2}\int_{a}^{\infty} x^{m-1-s}dx + \zeta(s-m) + a^{m-s} - \sum_{k=1}^{a} k^{m-s} \\ &-\sum_{r=1}^{\infty} \frac{B_{2r}\Gamma(m-s+1)}{(2r)!\Gamma(m-2r+2-s)}(m-2r+1-s)\int_{a}^{\infty} x^{m-2r-s}dx\end{aligned} \qquad a \in Z^{+} - \{0\} \quad (6)$$

The idea is , given a fixed 'm' we define an s sufficiently large so the integral $\int_{a}^{\infty} x^{m-s}dx$ and the series $\zeta(s-m) = \sum_{i=0}^{\infty} i^{m-s}$ both converge , and then use the analytic continuation to extend the definition of the sum as the negative value of the Riemann Zeta $\zeta(-m) = \sum_{i=0}^{\infty} i^{m}$, in order to regularize , using (5) the divergent integrals, if 'm' is an integer we can set a=0 and (5) becomes an easier expression in the limit $s \to 0$

$$\int_{0}^{\infty} x^{m}dx = \frac{m}{2}\int_{0}^{\infty} x^{m-1}dx + \zeta(-m) - \sum_{r=1}^{\infty} \frac{B_{2r}\Gamma(m+1)}{(2r)!\Gamma(m-2r+2)}(m-2r+1)\int_{0}^{\infty} x^{m-2r}dx \quad (7)$$

The case m-s=-1 inside (6) can not be regularized inmediatly due to the pole $\zeta(1) = \sum_{i=0}^{\infty} i^{-1} = \infty$, hence to regularize $\int_{0}^{\infty} \frac{dx}{x+a}$ we integrate with respect to 'a' to find $C_a + \int_{0}^{\infty} dx \log(x+a)$, using Euler-Maclaurin summation formula plus the regularization of Hurwitz Zeta function $\sum_{n=0}^{\infty} \log(x+a) = -\partial_s \zeta(0,a)$ and taking the derivative respect to 'a'

$$\int_{0}^{\infty} \frac{dx}{x+a} = -\frac{1}{2a} - \frac{\partial^2 \zeta(0,a)}{\partial s \partial a} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!}\frac{d^{2r-1}}{dx^{2r-1}}\left(\frac{1}{x+a}\right)_{x=0} \qquad (8)$$

The first two terms of the recurrence (7) are

$$1/2 = 1+\zeta(0) = \int_0^\infty dx \qquad\qquad \frac{1}{2}\zeta(0)+\frac{1}{2}+\zeta(-1) = \int_0^\infty xdx \qquad (9)$$

With $a_{mr} = \dfrac{\Gamma(m+1)}{\Gamma(m-2r+2)}$ and $\dfrac{x}{e^x-1} = \sum_{n=0}^{\infty}\dfrac{B_n x^n}{n!}$ being the Bernoulli numbers $B_{2n+1}=0$, from the definition of our product of Dirac delta distributions given in (4) and since we want the identity $H\times\delta = \frac{1}{2}\delta$ to be true for every test function , we can identify $\pi\delta_{reg}(0) = 1+\zeta(0)$, $\delta_{reg}{}'(0)=0$ and $-2\pi\delta_{reg}{}''(0) = \left(\zeta(0)+2\zeta(-1)\right) - B_2 a_{21}\zeta\left(0\right)$ from the point of view of Zeta regularization. Although we have used only a definition for distributions on R , it can be generalized to R^n by using the definition of Dirac delta function and Heaviside function in several variables $\prod_{j=1}^{n}\delta(x_j)$ $\prod_{j=1}^{n}H(x_j)$, in any case we have chosen the regularization $2\pi i^m \delta^{(m)}(0) = \int_{-\infty}^{\infty} x^m dx$ for 'm' integer odd or even, other definition for the Fourier transform can make a factor different to 2π appear in (4) for example $\int_{-\infty}^{\infty} dx e^{2\pi i u x} = \delta(u)$

REGULARIZATION OF FOURIER INTEGRAL USING DISTRIBUTIONS

Let be R^n , then we can regularize the Fourier transform $\int_{R^n} d^n k e^{i\vec{u}.\vec{k}} f(k) = F(u)$ via a taylor series substraction with the definition $\left\langle e^{i\vec{u}.\vec{k}} \mid f(k)\right\rangle = \int_{R^n} d^n k e^{i\vec{u}.\vec{k}} f(k)$ (see [6])

$$\left\langle e^{i\vec{u}.\vec{k}} \mid f(k) - \sum_{|\alpha|<N}\frac{k^\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\right\rangle + \left\langle e^{i\vec{u}.\vec{k}} \mid \sum_{|\alpha|<N}\frac{k^\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\right\rangle \qquad (10)$$

$|\alpha| = \alpha_1+\alpha_2+....+\alpha_n$ $\quad \alpha! = \alpha_1!.\alpha_2!....\alpha_n!$ $\quad \partial^\alpha = \partial^{\alpha_1}.\partial^{\alpha_2}......\partial^{\alpha_n}$ is the multi-index notation to write down the definition of Taylor series (9)

The Taylor series is finite and is truncated after a given N so $\int_{R^n} d^n k f(k) \approx \Lambda^{N+1}$ (ultraviolet divergence cut-off) , this allows us to write down a regular part of the Fourier transform plus a distributional part for the Fourier transform

$$F_{reg}(u) = \int_{R^n} d^n k e^{i\vec{u}.\vec{k}}\left(f(k) - \sum_{|\alpha|<N}\frac{k^\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\right) \qquad \sum_{|\alpha|<N}\frac{C_\alpha}{\alpha!}\left(\partial_\alpha f\right)(0)\left(-i\frac{\partial^\alpha}{\partial x^\alpha}\right)\delta(u) \quad (11)$$

(regularized part = function) (singular part = distribution)

The problem with (10) comes whenever the integral is divergent and we set u=0 , in this case we should have to evaluate $\delta^{(m)}(0)$ and other divergent quantities, also since two distributions can not in general be multiplied then $F(u)\times G(u)$ can NOT be defined, only the 'regular' parts of both F and G $F_{reg}(u)\times G_{reg}(u)$ or $F_{reg}(u)\times \delta^{(m)}(u)$, $G_{reg}(u)\times \delta^{(m)}(u)$ can be defined, here we find the problem of giving a regularized definition to $\delta^{(n)}(u)\times \delta^{(m)}(u)$ for integers (m,n) , this was discussed in (4) (5) (6) and (7) and (8) formulae including on how to deal with with the infinite terms $\delta^{(m)}(0)$ via Zeta-regularization , an small problem we find here is that depending on the definition of the Dirac delta function via Fourier transform an extra term proportional to 2π or similar could appear, this happens because usually the definition of the Fourier transform is not universal (up to a factor proprtional to 2π or square root of 2π) . So in general depending on the definition for the Fourier transform we should make the replacement $u \to 2\pi u$ to get the correct results.

PRODUCT OF DISTRIBUTIONS $\delta(u)\times P\left(\frac{1}{u}\right)$, $P\left(\frac{1}{u}\right)\times P\left(\frac{1}{u}\right), \delta'(u)\times P\left(\frac{1}{u}\right)$

Applying the convolution plus the zeta regularization algorithm and the Fourier transform for the Heaviside function $\int_{-\infty}^{\infty} dxH(x)e^{-iux} = \pi\delta(u) + iP\left(\frac{1}{u}\right)$ we can extend our definition of (regularized) product of distribution to include the Principal value distribution $P\left(\frac{1}{u}\right)$ related to Cauchy's principal value of the integral $P\left(\frac{1}{u}\right)[\phi] = pv\int_{-\infty}^{\infty}\frac{\phi(x)}{x}$, using again the Fourier transform convolution theorem

- Product of $\delta(u)\times P\left(\frac{1}{u}\right)$: in this case using the convolution definition

$$\delta(u)\times\left(\pi\delta(u) + iP\left(\frac{1}{u}\right)\right) = AF\left\{\int_{-\infty}^{\infty} dtH(x-t)\right\} = iA\delta'(u) + (\zeta(0)+1)A\delta(u) \quad (12)$$

- $P\left(\frac{1}{u}\right)\times\delta(u)$: using again the Fourier transform for H(x)

$$\left(\pi\delta(u) + iP\left(\frac{1}{u}\right)\right)\times\delta(u) = AF\left\{\int_{-\infty}^{\infty} dtH(t)\right\} = \frac{1+\zeta(0)}{2\pi}A\delta(u) \quad (13)$$

- $P\left(\frac{1}{u}\right)\times P\left(\frac{1}{u}\right)$: this case is a far bit more complicated to obtain this product we need the identity $\int_{-\infty}^{\infty} dtH(t)H(x-t)=H(0)xH(x)$ H(0)= 1/2

$$\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)\times\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)=AH(0)i\pi\delta'(u)+AH(0)P\left(\frac{1}{u^2}\right) \quad (14)$$

- $i\delta'(u)\times\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)$ and $\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)\times i\delta'(u)$, again using the appropiate form of the convolution theorem

$$i\delta'(u)\times\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)=-AI_1\delta(u)-\frac{\delta''(u)}{2}A \quad (15)$$

$$\left(\pi\delta(u)+iP\left(\frac{1}{u}\right)\right)\times i\delta'(u)=iA\zeta(0)\delta'(u)-I_1\delta(u)A \quad (16)$$

- $\delta'(u)\times P\left(\frac{1}{u^2}\right)$ and $P\left(\frac{1}{u^2}\right)\times\delta'(u)$: using $\int_{-\infty}^{\infty} dxe^{-iux}H(x)x=\pi i\delta'(u)+P(u^{-2})$ and the convolution theorem we can write down

$$\left(\pi\delta'(u)+iP\left(\frac{1}{u^2}\right)\right)\times i\delta'(u)=i\delta'(u)\zeta(0)A-I_1\delta(u) \quad (17)$$

$$i\delta'(u)\times\left(i\pi\delta'(u)+P\left(\frac{1}{u^2}\right)\right)=-\frac{1}{2}\delta''(u)A-AI_1\delta(u) \quad (18)$$

- $P\left(\frac{1}{u^2}\right)\times P\left(\frac{1}{u^2}\right)$: using (14) (17) (18) and the product

$$\left(\pi i\delta'(u)+P\left(\frac{1}{u^2}\right)\right)\times\left(\pi i\delta'(u)+P\left(\frac{1}{u^2}\right)\right)=AF\left\{\int_{-\infty}^{\infty} dtH(x-t)H(t)(x-t)t\right\} \quad (19)$$

The last expression in (19) is just $\frac{-i\pi A\delta'''(u)}{6}+AP\left(\frac{1}{u^4}\right)$,again we have used the identity $\int_{-\infty}^{\infty} dtH(t)H(x-t)=H(0)xH(x)$ together with (4) and (5) in order to give a finite meaning for the product $P\left(\frac{1}{u^2}\right)\times P\left(\frac{1}{u^2}\right)$, note that in expressions (12-18) we need to evaluate products of the form $\delta^{(m)}(u)\times\delta^{(n)}(u)$ which need to be regularized by (4)

Depending on the order in which convolution is taken we may find H(x-t) or H(t) (x-t) or simply 't' inside (12-18) , here as always A is a number introduced by the definition taken for the convolution and $\frac{1+\zeta(0)}{2}+\zeta(-1)=\left(\int_0^\infty xdx\right)_{reg}=I_1$, $\zeta(0)+1=\left(\int_0^\infty dx\right)_{reg}$ are finite corrections (regularizations) for the divergent integrals that appear when we try to define a correct product of distributions , from these formulae above together with the Lebiniz formula (considered to be valid at least in a formal sense) $\frac{d\left(A\times B\right)}{du}=\frac{dA}{du}\times B+A\times\frac{dB}{du}$, we can define also $\delta\left(u\right)\times P\left(\frac{1}{u^2}\right)$ or similar products

- $\delta^{(m)}(u)\times P\left(\frac{1}{u}\right)$ and $P\left(\frac{1}{u}\right)\times\delta^{(m)}(u)$ for arbitrary 'm' , $H(u)\times P\left(\frac{1}{u}\right)$

$$i^m\delta^{(m)}(u)\times P\left(\frac{1}{u}\right)=AF\left\{\int_{-\infty}^{\infty} dtH(x-t)t^m\right\}=\delta(u)(-1)^m I_m+\frac{\delta^{(m+1)}(u)i^{m+1}}{m+1} \quad (20)$$

$$P\left(\frac{1}{u}\right)\times i^m\delta^{(m)}(u)=AF\left\{\int_{-\infty}^{\infty} dt(x-t)^m H(t)\right\}=A\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}I_{m-k}i^k\delta^{(k)}(u) \quad (21)$$

Here $I_m=\left(\int_{-\infty}^{\infty} t^m dt\right)_{reg}$ these integrals can be regularized via formula (6) or (7) However if we put m=-1 in order to evaluate $H\left(u\right)\times P\left(\frac{1}{u}\right)$ inside (20) and (21) we will find several oddities that prevent us from defining a coherent expression , however the derivative of this product of distribution satisfy

$$\frac{d}{dx}\left(H\times P\left(\frac{1}{u}\right)+P\left(\frac{1}{u}\right)\times H\right)=\delta\times P\left(\frac{1}{u}\right)-H\times P\left(\frac{1}{u^2}\right)+P\left(\frac{1}{u}\right)\times\delta-P\left(\frac{1}{u^2}\right)\times H \quad (22)$$

Another possibility is to define $H\left(u-a\right)\times P\left(\frac{1}{u}\right)=T_a(u)$ so its derivative $-\delta\left(u-a\right)\times P\left(\frac{1}{u}\right)=\frac{dT_a(u)}{dx}$, using the Taylor distributional series given in [2] $\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}\delta^{(n)}(u)=\delta(u-a)$, using formulae (21) and (22) and integration with respect to 'a' we can get $H\left(u-a\right)\times P\left(\frac{1}{u}\right)=T_a(u)$ up to some constant C_a. Also if we knew how to multiply $H\left(u-a\right)\times P\left(\frac{1}{u}\right)=T_a(u)$ for some a >0 , (to avoid the singular point u=0) , using the Taylor distributional series $\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}\delta^{(n-1)}(u)+H(u)=H(u-a)$ and then using (21) (22) . Although we have only considered the 1-D case, the Convolution theorem, Binomial theorem and similar can be defined also in R^n , also we must take into account that in general for divergent integrals a change of variable could not work $\int_{-\infty}^{\infty} x^m dx \int_{-\infty}^{\infty} y^n dy \neq \int_0^{\infty} dr \int_0^{2\pi} r^{m+n+1} d\alpha \sin^n \alpha \cos^m \alpha$, the best method would be to use a Feynmann parametrization to define the product of n integrals

$$\frac{1}{A_1 A_2 ....A_n} = (n-1)! \int_0^1 du_1 \int_0^1 du_2 ......\int_0^1 du_n \frac{\delta(u_1 + ...u_n - 1)}{(u_1 A_1 + ...u_n A_n)} \qquad (23)$$

With $A_n(\mu) = \int_0^{\infty} x^{\mu} dx$ being a divergent integral that can be regularized ($\mu > 0$) via Zeta-regularization

CONCLUSIONS AND FINAL REMARKS

Using the zeta regularization algorithm (6) (7) we have managed to give a finite (Non-commutative) product of dirac delta distributions $\delta^{(m)}(u) \times \delta^{(n)}(u)$, and $\delta^{(m)}(u) \times H(u)$, with 'H' being the Heaviside step-function , since the product is non-commutative we should also take care when taking the products $(\delta^{(m)} \times \delta^{(n)}) \times \delta^{(k)} \neq \delta^{(m)} \times \left(\delta^{(n)} \times \delta^{(k)}\right)$ so associativity will not always hold , using the Convolution theorem plus the use of Fourier transform, with the m-th and n-th powers of 'x' $F\left(x^m * x^n\right) = AF\left(\int_{-\infty}^{\infty} dt(x-t)^n t^m\right)$ A = normalization constant , will allow us to compute the product $\delta^{(m)} \times \delta^{(n)}$ up to several divergent quantities $\delta^{(m)}(0)$, which are proportional to the divergent integral $\int_{-\infty}^{\infty} x^m dx$, this integral can be regularized [4] using the zeta regularization algorithm in order to 'substract' finite quantities proportionals to $\zeta(-m)$ m=0,1,2,3,........ . Although we have only examinated the case of dirac delta and its derivatives , in several cases it could appear the distribution

$$\int_{-\infty}^{\infty} \frac{e^{iux}}{x} dx = \pi \sin g(u) = \pi \frac{d|u|}{du} = \pi H(u) - \pi H(-u) \qquad (24)$$

Although we have not mentioned the case $\int_0^{\infty} dx f(x) e^{iux}$, this integral can be reduced to the calculation of a Fourier integral by setting

$$f(x)H(x) + f(-x)H(-x) = f_+(x) + f_-(x) = g(x) \quad \frac{1}{2}\int_{-\infty}^{\infty} dx g(x) e^{iux} = \int_0^{\infty} dx f(x) e^{iux} \quad (25)$$

In this case we will encounter divergent terms $\delta^{(m)}(0)$, when using the Leibniz's formula to perform the Taylor substraction near x=0 $\frac{d^n}{dx^n}(f.H)=\sum_{k=0}^{n}\binom{n}{k}\frac{d^k f}{dx^k}\cdot\frac{d^{n-k}H}{dx^{n-k}}$ since the derivative of an step funciton involves a dirac delta , again we will need formula (5) (6) and (7) to get some finite results.

If the integral of f(x) has some logarithmic divergence so $\int_0^{\Lambda} dx f(x) \approx \log \Lambda$, then we may have to regularize the distribution $H(x)x^{-1}$ as

$$\left\langle P.f\left(\frac{H}{x}\right) | \phi \right\rangle = -\phi(0)\log\varepsilon + \int_{\varepsilon}^{1}\frac{\phi(x)-\phi(0)}{x}dx + \int_{1}^{\infty}\frac{\phi(x)}{x}dx \qquad (26)$$

And then ignoring all the divergent terms proportional to $\log\varepsilon$ (via counterterms) inside (12) so only finite contributions will appear inside $\int_0^{\infty} dx f(x)$.

Why this method should work ? , the justification would be the following , let $\varphi_n(x)$ and $\phi_n(x)$ a family of smooth function depending on the parameter 'n' let us assume that in the limit as n tends to infinity $\lim_{n\to\infty}\varphi_n(x)=S(x)$, $\lim_{n\to\infty}\phi_n(x)=T(x)$ with S(x) and T(x) being distributions, for n finite the convolution theorem will hold so

$$(\varphi_n x \phi_n)(u) = AF\left\{\int_{-\infty}^{\infty} dt F^{-1}(\varphi_n)(t) F^{-1}(\phi_n)(x-t)\right\} \qquad (27)$$

With $F(f(x))$ being the Fourier transform of the function f(x) , for finite 'n' (27) will give finite results, in the limit $n\to\infty$, the integrals inside the convolution will be divergent and will need to be regularized, for example in the case of the Dirac delta distributions and its derivatives $\delta^{(k)}(x)$, for 'k' a natural number , these integrals appear in the form $\lim_{n\to\infty}\left(\int_{-n}^{n} dx x^k\right)$, Zeta regularization can be used to handle the difficulty of divergent integrals and is a powerful method (together with Convolution theorem) to define an acdequate definition of the product of 2 distributions, as we have shown in equations (12-22) . For the case of the Zeta regularization algorithm and how it can be applied , we strongly recommend [3] ,anyway Zeta-regularization can be easily explained in this way, in the limit $n\to\infty$ for the series $\sum_{i=1}^{n} i^k$ we make the replacement

$$\sum_{i=1}^{n} i^k = \zeta(-k)+\zeta(-k,n+1)\to\zeta(-k) \quad \text{so} \quad \zeta(-k,n+1)=0 \quad \forall k\in R \qquad (28)$$

However (28) is only true for Re (k) >1 otherwise the last expression is divergent , the idea of Zeta regularization is to set $\zeta(-k,n+1)=0$ by imposing analytic continuation of the Hurwitz Zeta function as $n \to \infty$, a simpler example is the following , let be $f(s) = 0^s$, this function is not continuous (pole) at s=0 , however we can avoid this singularity by imposing $f(0) = 1$ and the functional equation $f(s) = f(-s)$, so for every 's' we would have $f_{reg}(s) = 0$. If we simply put down all the positive powers of n as $n \to \infty$, then we would find the trivial result $\sum_{i=1}^{\infty} i^k \to \zeta(-k) + \frac{B_{k+1}}{k+1} = 0$, which clearly contradcit the spirit of the Zeta regularization algorithm that gives a finite but nonzero value to the divergent sums of integer powers. In general the regularized value of the integral $\int_0^{\infty} x^m dx$ will depend on a linear combination of the values $\zeta(-k)$ for k=0,1,2,3,...,m if we replace the value $\zeta(-k)$ by the discrete sum $\sum_{i=1}^{n-1} i^k$ and use the Euler-Maclaurin summation formula, then the result $\int_0^n x^k dx = \frac{n^{k+1}}{k+1}$ for , positive or k=0 is recovered , in case we need to evaluate more complicate integrals like the following $\int_0^{\infty} (x+a)^{-s} f(x) dx = F(s)$ as $s \to 0$, we simply may expand it into a convergent Laurent series for x >0 , $\sum_{i=-\infty}^{m} c_i (x+a)^{i-s}$ the value i=1 is just the logarithmic divergence of the integral and can be regularized by setting the value –log(a) , the coefficients $i \in [0, m]$ are the UV (ultraviolet) divergences of the integral, which can be regularized by means of (6) and (7) and the term $\sum_{i=-\infty}^{-2} c_i (x+a)^{i-s}$ is a finite part of the integral as $s \to 0$, the idea is to separate the divergent part from the finite part in order to regularize every divergence . As an example let be the Laurent series for $\left(x^2 + a^2\right)^{\alpha}$

$$\left(x^2 + a^2\right)^{\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1) a^{2\alpha}}{\Gamma(n+1)\Gamma(\alpha-n+1)} \left(\frac{x}{a}\right)^{2\alpha-2n} \quad \text{valid for } |x^2| > a \quad (29)$$

Formula (29) can be used to evaluate the integral $\int_{a+1}^{\infty} x^{m-s} \left(a^2 \pm x^2\right)^{\alpha} dx$ (m positive) , using expansion (29) valid for x > a then one can isolate the divergent parts $\int_{a+1}^{\infty} x^{m-s+2\alpha-2n} dx$ and then use (6) to express these integrals in terms of the values of the Riemann Zeta function $\zeta(m-s+2\alpha-2n)$ and the logarithm $-\log(a+1)$, the limit $s \to 0$ is then taken afterwards . In case $\alpha < 0$ there is an IR (infrared) divergence inside the integral $\int_0^{a+1} x^{m-s} \left(a^2 - x^2\right)^{\alpha} dx$, using again the Euler-maclaurin summation formula this divergence may be avoided and we would find the (approximate) finite value

$$\sum_{i=0}^{a+1(*)} \left(a^2 - i^2 + i\varepsilon\right)^{\alpha} i^m - \frac{(a+1)^m \left(-1-2a+i\varepsilon\right)^{\alpha}}{2} - \sum_{r=1}^{N} \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial x^{2r-1}} \left(x^m (a^2 + i\varepsilon - x^2)^{\alpha} \right) \Big|_0^{a+1}$$

(30)

Here the asterisk (*) means that we have excluded the value i=a from the summation inside (30) (in case a is a non-integer this problem would not appear) , if 'm' were negative then there is a pole at the point x=0 , and in order to regularize the integral $\int_0^{a+1} \frac{dx}{x^m}$ we make a change of variable $x \to \frac{1}{q}$ so it becomes $\int_{(a+1)^{-1}}^{\infty} q^{m-2} dq \quad m \neq 1$, which is now UV divergent.

APPENDIX A: HURWITZ ZETA FUNCTION AND THE SUMS $\sum_{n=0}^{\infty} \frac{\log^k (n+a)}{n+a}$:

We have studied the regularization of the Harmonic sum $\sum_{n=0}^{\infty} \frac{1}{n+a} = -\frac{\Gamma'}{\Gamma}(a)$, the idea is to see if using the definition of Zeta-regularized determinant for the Hurwitz zeta function $\prod_{n=0}^{\infty} \left(n+a\right) = e^{-\zeta'_H(0,a)}$, from the definition of the Hurwitz Zeta , taking logarithm to both sides and taking derivatives we reach to the result

$$\Psi_{s-1}(z) = \frac{d^s}{dx^s} \left(\log \Gamma(z) \right) = (-1)^s \Gamma(s) \zeta_H(s,z) \qquad \text{(A.1)}$$

Taking derivatives with respect to 's' inside the Hurwitz Zeta function and using the definition given in (A.1) gives

$$\sum_{n=0}^{\infty} \frac{\log^k (n+a)}{(n+a)^s} = (-1)^{-k} \frac{d^k \zeta_H(s,a)}{ds^k} \qquad \sum_{n=0}^{\infty} \frac{\log^k (n+a)}{(n+a)^s} = (-1)^{-k} \frac{d^k}{ds^k} \left(\frac{(-1)^{-s}}{\Gamma(s)} \Psi_{s-1}(a) \right)$$

(A.2)

The main problem here is to define for every real 's' the function $\left(-1\right)^s = e^{i\pi s}$, a possibility is to take only the real part of the function $\cos\left(\pi s\right)$, in order to define for every 's' the function $\frac{d^s}{dx^s} \left(\log \Gamma(z) \right)$, we can use the definition of the Grunwald-Letnikov differintegral replacing the k-th derivative by the k-th difference

$$\frac{d^s}{dx^s} \left(\log \Gamma(z) \right) = \lim_{h \to 0} \frac{1}{h^s} \sum_{m=0}^{\infty} \left(-1\right)^m \frac{\Gamma(s+1)}{\Gamma(m+1)\Gamma(s-m+1)} \log \Gamma(z + (s-m)h) \qquad \text{(A.3)}$$

If we put an small 'h' we can compute an approximate s-th fractional derivative

Another possibility to define the regularization of the series $\sum_{n=0}^{\infty} \frac{\log^k(n+a)}{n+a}$ is to consider the regularized integral $\int_0^{\infty} \frac{dx \log^k(x+a)}{x+a} = -\frac{\log^{k+1}(a)}{(k+1)}$ and use the Euler-Maclaurin summation formula to obtain the finite (regularized) value a > 0

$$\sum_{n=0}^{\infty} \frac{\log^k(n+a)}{n+a} = \frac{\log^k(a)}{2a} - \frac{\log^{k+1}(a)}{(k+1)} - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial x^{2r-1}} \left(\frac{\log^k(x+a)}{x+a} \right)_{x=0} \qquad \text{(A.4)}$$

In general, for the logarithmic divergences, we must introduce an energy scale μ so the functional determinant of a differential operator A $\det A = \prod_{i=0}^{\infty} \frac{\lambda_n}{\mu}$ has no dimension , so for k =1 and for every real 'k' (either positive or negative)

$$\int_0^{\infty} \frac{dx \log^k(x+a)}{x+a} = -\frac{\log^{k+1}(a)}{(k+1)} + \frac{\log^{k+1}(\mu)}{(k+1)} \qquad \int_0^{\infty} \frac{dx}{(x+a)\log(x+a)} = \log\left(\frac{\log \mu}{\log a} \right) \quad \text{(A.5)}$$

For the case of the logarithmic integral , we can use another trick based on the Padé approximants for the square root , for example we write $\frac{1}{x+a} = \frac{1}{\sqrt{x+a}} . \frac{1}{\sqrt{x+a}}$ and use the identity $\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right) = x-1$ we can approximate the inverse square root by $\frac{1}{\sqrt{x}} \approx \frac{P(x)}{Q(x)}$ here P(x) and Q(x) are Polyonomials of degree 'm' and 'n' respectively , from this the logarithmic divergent integral becomes $\int_c^{\infty} \frac{P(x+a)dx}{\sqrt{x+a}Q(x+a)}$ here 'c' can be chosen so Q(x) has no roots on the interval $[c, \infty)$ expanding $\frac{P(x+a)}{Q(x+a)} = \sum_{j=-m+n}^{\infty} \frac{b_j}{x^j}$ into a Laurent series valid for x >c and using formula (6) to evaluate the divergent integrals $\int_c^{\infty} \frac{dx}{x^{j+1/2}}$ since 'j' is an integer, the pole of the Riemann Zeta at s=1 does not appear inside $\zeta\left(j+\frac{1}{2}\right)$, for other functions like $\log(x+a)$ a similar method of using Padé approximants could be used ,since $\frac{P(x+a)}{Q(x+a)}$ is a Rational function only a finite number of positive powers of 'x' will appear into its Laurent series.

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