Covariant isolation from an Abelian gauge field of its nondynamical potential, which, when fed back, can transform into a "confining Yukawa"

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Abstract

For Abelian gauge theory a properly relativistic gauge is developed by supplementing the Lorentz condition with *causal* determination of the time component of the four-vector potential by retarded Coulomb transformation of the charge density. This causal Lorentz gauge *agrees* with the Coulomb gauge for *static* charge densities, but allows the four-vector potential to have a *longitudinal* component that is determined by the time derivative of the four-vector potential's time component. Just as in Coulomb gauge, the two *transverse* components of the four-vector potential are its sole *dynamical* part. The four-vector potential in this gauge covariantly *separates* into a dynamical transverse four-vector potential and a nondynamical timelike/longitudinal four-vector potential, where *each* of these two satisfies the Lorentz condition. In fact, analogous partition of the conserved four-current shows *each* to satisfy a Lorentz-condition Maxwell-equation system with its *own* conserved four-current. Because of this complete separation, either of these four-vector potentials can be tinkered with *without affecting its counterpart*. Since it satisfies the Lorentz condition, the nondynamical four-vector potential times a constant with dimension of inverse length squared is itself a conserved four-current, and so can be fed back into its own source current, which transforms its time component into an extended Yukawa, with *both* exponentially decaying *and* exponentially *growing* components. The latter might be the mechanism of quark-gluon confinement: in non-Abelian color gauge theory the Yukawa mixture ratio ought to be tied to color, with palpable consequences for "colorful" hot quark-gluon plasmas.

Introduction

Gauge theories have a quintessential *dual nature*. They *simultaneously* encompass *dynamical* transverse waves that travel at the speed of light, which phenomenon can be *independently* quantized, and *nondynamical* potential/force fields whose *source* is appropriately "charged" matter, and which, in turn, *affect the behavior* of such matter. The *nondynamical pure potentials/forces*, which are *part* of what gauge theories encompass, are *not* subject to *independent* quantization: they *merely conveniently isolate and abstract* a certain *intermediate mathematical aspect* of physical behavior that is *inherent* to the "charged" matter *itself*—useful mathematical abstractions, however suggestive or extremely convenient, are *still* not *physical degrees of freedom* that can be *independently* quantized; any *quantum* characteristics of such nondynamical pure potential/force fields are merely the *secondary consequences* of the *quantum character* of their "charged" matter *source*.

In view of the *dual* nature of gauge theories, their typically *seamless*-appearing *mathematical* formulations present a conundrum and challenge to the theoretical physicist. Part of the formal smoothness of gauge theories is attributable to the physics which they describe: the selfsame "charged" matter which inherently interacts with *itself* in a manner that can be conveniently mathematically analyzed using the intermediate constructs of source-determined pure potential/force fields, *also* emits, absorbs and scatters the *dynamical* transverse waves/radiation/massless quanta—whose *interactions* with that "charged" matter are made *part and parcel* of the *physics* treated by that *same* gauge theory. This naturally promotes the *formal similarity* of fields that *do* have *dynamical content* to fields which are merely extremely convenient *intermediate mathematical constructs* for the analysis of "charged" matter's *intrinsic* interaction. Another technical aspect of gauge theories which can contribute to such a lack of formal distinction between actual *dynamical* fields and *nondynamical* "intermediate mathematical construct" pure *potential* fields is the gauge invariance ambiguity of those theories: the *physically nonexistent degree of freedom* often serves to *enhance* misleading *surface* manifestation of *physically nonexistent symmetry*.

There are a number of reasons to seek to, in a formally neat and relativistically covariant fashion, "pull apart" gauge theory into two physically natural parts, one encompassing the *purely dynamical transverse gauge fields* and the other the *purely nondynamical gauge potential fields*. Obviously, pushing the nondynamical potential fields out of the way in a covariant fashion might conceivably be a boon to covariant quantization of the dynamical transverse gauge fields. More intriguingly, having a clean such separation in hand raises the theoretical possibility of tinkering with one of these parts without affecting the other, or perhaps even trying to discard one of the parts altogether. Certainly the issue of quark confinement lends itself to thoughts of somehow relativistically drastically reshaping "gluon" gauge *potential fields* without upending altogether the notion of the gluon as a gauge particle. The last half of this paper tentatively explores just this matter, albeit only in the theoretically insufficient context of purely Abelian gauge theory.

As a first step along this route, this paper deals only with the simplest case, namely Abelian gauge theory. Hopefully there will be other researchers who will be inspired to look at the more involved non-Abelian cases. The approach to carrying out covariant cleavage of Abelian gauge theory into two natural constituents will lean heavily on the *identification of a sensible potential field*. Since the potential field is an intermediate mathematical construct for facilitating the understanding of the *intrinsic* interaction of charged matter, it should be *entirely the creature* of its charged matter *source*; in other words, with the charged matter *source* in hand, the potential field should *follow uniquely*, and it should, of course, *vanish altogether* in the *absence* of the charged matter *source*. Since the equations of Abelian gauge theory are *linear*, it is clear that the potential field must be a *homogeneous functional* of its *source*. Furthermore, the *kernel* of the *appropriate homogeneous functional* is a *Green's function* of the linear equation that relates the potential field to its charged matter source. To ensure that a *relativistically sensible* Green's function is *available* for this purpose, we must take care that our *choice of gauge* for the Abelian gauge theory does *not* impose as the equation which links the potential field to its charged matter source one that is *incompatible with special relativity*. In other words, success of our envisioned project hinges on a relativistically compatible *choice of gauge*. It is to be noted that the physics of the *dynamical* gauge waves/radiation/massless particles in contrast does *not enslave* their transverse fields to a charged matter source; those fields can have a nonvanishing existence even in the *complete absence* of charged matter, albeit they *interact* with charged matter if it is present. But their fields are *not determined* by the charged matter, whereas the pure potential field *is* entirely so determined, *even* in a *fully quantum context*.

At first blush it might appear that selection of the Coulomb gauge of itself accomplishes the project just set out: at one stroke not only is gauge ambiguity abolished, but also potential effects are transparently assigned to the time component of the four-vector potential, dynamical effects are equally transparently assigned to the four-vector potential's spatial transverse components, and its spatial longitudinal component is set to *zero* by fiat. Everything, apparently, is now clear-cut and simple. The shock comes when examining the full consequences for that time-component potential: it responds *instantaneously throughout all of space* to any time variation of its charge-density source. Thus the penny drops: the marvelously simple and efficacious abolition of the longitudinal component of the spatial vector potential, which *is* the Coulomb gauge condition, spared no thought or concern for special relativity! Still, the Coulomb gauge's treatment of the *static limit* of the charge density is absolute bedrock; any proposed alternative gauge won't be worth its salt if it deviates from that.

The obvious alternative to the Coulomb gauge is the celebrated Lorentz condition. No slighting of relativity there, and an immediate payoff with a marvelous automatic formal simplification of the Maxwell equation system. But hold on, doesn't that equation system now look *too* symmetrical in the *components* of the four-vector potential? Is there the *slightest indication* of just *where* to seek the nondynamical potential versus the dynamical radiation fields? And what about the troubling restricted gauge invariance ambiguity which still remains?

In fact, it happens that the Lorentz condition *itself*, with *no reference* to the Maxwell equations, *completely determines* the the spatial longitudinal component of the four-vector potential in terms of the time derivative of its time component. Thus although the Lorentz condition *does not flatly abolish* the longitudinal component of the vector potential as the Coulomb gauge does, it *does completely subordinate it* to the time derivative of the four-potential's time component, which has a similar effect. A puzzling feature of this consequence of the Lorentz condition is that it is far from obvious how it could be demonstrated by manifestly covariant operations on four-vectors. The obvious approach is to note the precise mathematical analogy between the Lorentz condition and the Coulomb-law Maxwell equation, and then to recall that the latter determines the longitudinal component of the electric field in terms of the charge density. The determination of the longitudinal component of the spatial vector potential in terms of the time derivative of its time component is then worked out in strict analogy.

Though very helpful and enlightening, this consequence of the Lorentz condition leaves a "loose screw" in its wake; the restricted gauge invariance ambiguity that the Lorentz condition permits must still be addressed. Since the Lorentz condition ties the longitudinal space component of the four-vector potential to its time component, it is apparent that if the relation of the time component of the four- vector potential to the *charge density* could have all the slack *removed* from it (*without* offending special relativity, of course), then that time component would be the full-fledged potential field, and therefore the *dynamical* field would be *compelled* to occupy *only* the two *transverse spatial* components of the four-vector potential, being that the four-vector potential's *longitudinal spatial* component has *already* been *fully determined* by the Lorentz condition to be *entirely* the creature of the time derivative of what is now the *potential field*.

The key to removing the slack from the relation of the time component of the four-vector potential to the charge density—which slack is a manifestation of the restricted gauge invariance ambiguity—of course lies with the Green's functions of the light-speed wave equation *which connects that time component to the charge density after imposition of the Lorentz condition.* The physically most appealing such light-speed wave-equation Green's function to select is clearly the celebrated traditional causal *retarded* one, which makes perfect intuitive relativistic sense. The result bears a considerable resemblance to the consequences of using Coulomb gauge, except that the potential field is now tied to the charge density in a relativistically causal *retarded* fashion rather than in a relativistally offensive *instantaneous spatially uniform* fashion, and the *time derivative* of that potential field now *fully determines* the *longitudinal* spatial component of the four-vector potential, *instead* of its being decreed to *vanish under all circumstances.* The upshot for the *dynamical* gauge field is *exactly the same* as it is in Coulomb gauge: that field is described by the *two remaining transverse* spatial components of the four-vector potential. In the charge density's static limit, the causal *retarded* potential field *as well* comes out to be *static*, and therefore the *longitudinal* spatial component of the four-vector potential *vanishes identically.* This last is, of course, the Coulomb gauge condition, so that in the charge density's *static limit*, there is *no difference* from the Coulomb gauge result. Obviously this selection of the causal *retarded* Green's function to specify a completely determined relation of the time component of the four-vector potential to the charge density, in *addition* to the imposition of the Lorentz condition, has *also* determined a *gauge*, which we dub the causal Lorentz gauge. This gauge is as close to the Coulomb gauge as one can get *without* contravening the tenets of special relativity.

The causal Lorentz gauge permits the full four-vector potential to be written as the sum of *two physically distinct* four-vector potentials, the *first* one consists of only the *time and spatial longitudinal components* of that full four-vector potential in this gauge, while the *second* one consists of just the *remaining two spatial transverse components* of that full four-vector potential in this gauge. This separation is covariant because *both* of these "pulled apart" four-vector potentials *satisfy the Lorentz condition.* Obviously the first, purely timelike/longitudinal four-vector potential in this causal Lorentz gauge is totally determined by the charge density, and is thus entirely in the nature of a collection of potential fields, whereas the second, remaining purely transverse four-vector potential in this causal Lorentz gauge is purely dynamical.

Now the four-current conservation constraint condition is highly analogous to the Lorentz condition, and permits the four-current itelf to be *similarly* partitioned into the sum of *two conserved four currents*, the first one encompassing the time component of the total four-current (i.e., the charge density) as well as the spatial longitudinal component of the total four-current, which turns out to be *completely determined by the charge density's time derivative.* It is readily shown that this first timelike/longitudinal four-current is *explicitly conserved.* The second four-current consists of just the two *remaining* purely *transverse* components of the total four-current, and is readily shown to be *conserved as well.* With this split up of the four-current into a conserved timelike/longitudinal part plus a conserved purely transverse part complementing the corresponding split up of the four-vector potential into two highly analogous parts which each satisfy the Lorentz condition, it turns out that Lorentz-condition light-speed wave-equation versions of the Maxwell equations neatly apply to *both* of the split-up four-vector potentials in causal Lorentz gauge.

We thus are now in a position to *deal entirely separately* with the pure potential and the pure dynamical parts of the Abelian gauge theory in causal Lorentz gauge. Being that *adherence to the Lorentz condition is a pervasive feature of this gauge*, the four-vector potentials *themselves* times a constant with the dimensions of inverse length squared qualify as *conserved currents.* In other words, in this gauge one can envision *feeding back* the four-potentials as partial contributors to their *own source current.* To do this to the purely transverse *dynamical* part of the four-vector potential seems tantamount to giving the gauge particles *mass*, which is physically questionable in view of their *transverse* character: such particles with *mass* are supposed to have *integer spin*, which is incompatible with the *two* transverse spin states of such particles when they are *massless.*

Feeding back *only* the timelike/longitudinal purely *potential* part of the four-vector potential in this causal Lorentz gauge does *not* affect the properties of the transverse dynamical massless gauge particles but does transform the *potential field* to one having *Yukawa character.* In fact, there is no reason to believe that such feedback would result in *only* the traditional exponentially *decaying* Yukawa potential; exponentially *growing* "Yukawa" potential components should make their presence felt as well. One can at least *speculate* that such exponentially *growing* feedback Yukawas might have a role in *quark confinement.* It turns out that the causal Lorentz gauge appears to be compatible with *every possible ratio* of exponentially growing Yukawa potential component to exponentially decaying Yukawa potential component. This result undoubtedly has to do with the limitations of considering Abelian gauge theory only. One might entertain the hope that these ratios of exponentially growing Yukawa potential component to exponentially decaying Yukawa potential component might eventually become tied to non-Abelian gauge *color* in such a way that *color singlet* states experience only the traditional exponentially *decaying* Yukawa potential. If the *confinement* of non-color-singlet quark/gluon configurations is indeed effected by exponentially *growing* Yukawa potentials, it seems rather obvious that quark-gluon "plasmas" which are too "hot" (thermally disturbed) to readily reorganize themselves into color singlets ("hot" enough to be "colorful") will *not* behave *at all* like a free gas.

We now proceed to the mathematical detail in Abelian gauge theory of the results we have been outlining above in words. We begin by considering the consequences of the imposition of the Lorentz condition on Abelian gauge theory, and continue to the development of the full causal Lorentz gauge and the consequence it has of allowing the four-vector potential to be covariantly cleaved into a timelike/longitudinal four-vector potential of purely potential field character and a remaining transverse four-vector potential of purely dynamical field character.

Isolation of the nondynamical four-vector potential in causal Lorentz gauge

The Lorentz-covariant four-vector Abelian gauge field A^μ,

$$A^\mu(\mathbf{r},t) = (\phi(\mathbf{r},t), \mathbf{A}(\mathbf{r},t)), \tag{1a}$$

whose source is the Lorentz-covariant four-vector current density j^μ,

$$j^\mu(\mathbf{r},t) = (c\rho(\mathbf{r},t), \mathbf{j}(\mathbf{r},t)), \tag{1b}$$

is governed by a "mixed bag" of constraint and dynamical equations [1] which are expressed in Lorentz-covariant notation as,

$$\partial_\lambda\partial^\lambda A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu/c, \tag{1c}$$

where the Lorentz-covariant four-vector first-derivative operators ∂_μ and ∂^μ are given by,

$$\partial_\mu = (c^{-1}\partial/\partial t, \nabla_\mathbf{r}), \tag{1d}$$

and,

$$\partial^{\mu} = (c^{-1}\partial/\partial t, -\nabla_{\mathbf{r}}), \tag{1e}$$

which imply that the Lorentz-scalar contracted second-derivative operator $\partial_{\lambda}\partial^{\lambda}$ comes out to be,

$$\partial_{\lambda}\partial^{\lambda} = c^{-2}\partial^2/\partial t^2 - \nabla^2_{\mathbf{r}}. \tag{1f}$$

Upon contracting both sides of Eq. (1c) with ∂_{μ}, we obtain the *current conservation* source contraint,

$$\partial_{\mu} j^{\mu} = 0. \tag{1g}$$

It turns out that Eq. (1c) *fails to uniquely determine* A^{μ}. It is readily verified that if the four-gradient of an *arbitrary* Lorentz-scalar function $f(\mathbf{r}, t)$ is added to A^{μ}, i.e., if,

$$A^{\mu} \to A^{\mu} + \partial^{\mu} f, \tag{1h}$$

then Eq. (1c) continues to be satisfied. We now take advantage of this *gauge-invariance ambiguity* [2] of Eq. (1c) to formally *simplify* it by imposing on it the Lorentz-invariant *Lorentz condition* [3],

$$\partial_{\nu} A^{\nu} = 0, \tag{2a}$$

which results in its becoming just,

$$\partial_{\lambda}\partial^{\lambda} A^{\mu} = j^{\mu}/c. \tag{2b}$$

The Lorentz condition has *not*, however, *fully removed* the gauge-invariance ambiguity, since we readily see that Eq. (2b) will still be satisfied after the gauge transformation of Eq. (1h) *provided* that the scalar gauge function $f(\mathbf{r}, t)$ satisfies,

$$\partial_{\lambda}\partial^{\lambda} f = 0, \tag{2c}$$

which is *restricted* gauge-invariance ambiguity. We note that the current conservation source constraint of Eq. (1g) follows from Eq. (2b) upon taking account of the Lorentz condition of Eq. (2a).

Experience with electrostatics suggests that the *pure potential* effects [4] which arise from $A^{\mu} = (\phi, \mathbf{A})$ are primarily associated with ϕ, whereas it is almost universally agreed that the *dynamical, radiative* effects that arise from A^{μ} are associated with the *transverse part* of $\mathbf{A}$ [5]. But such distinctions amongst the components of A^{μ} are *not at all* formally apparent at this stage; in fact they are *missing altogether* from Eq. (2b). It turns out, however, that the *Lorentz condition* of Eq. (2a) is able to nail down a *unique longitudinal part* $\mathbf{A}_L$ of $\mathbf{A}$ with the property that $\mathbf{A}_L$ is a *pure homogeneous functional of the time derivative of* ϕ, and, as a wonderful bonus, that the four-vector $(\phi, \mathbf{A}_L)$ *also* satisfies the Lorentz condition!

Upon writing out the Lorentz condition of Eq. (2a) as,

$$\nabla \cdot \mathbf{A} = -\dot{\phi}/c, \tag{3a}$$

we realize its extremely close formal similarity to the Coulomb law $\nabla \cdot \mathbf{E} = \rho$ of Maxwell's equations [1]. The Coulomb law of course uniquely determines the longitudinal part of $\mathbf{E}$ in terms of the charge density ρ. Here we proceed to determine the longitudinal part of $\mathbf{A}$ in terms of the time derivative of ϕ in exactly the same way, and obtain,

$$\mathbf{A} = \mathbf{A}_L + \mathbf{A}_T, \tag{3b}$$

where,

$$\mathbf{A}_L(\mathbf{r}, t) = c^{-1}\nabla((4\pi)^{-1}\textstyle\int d^3\mathbf{r}'\, \dot{\phi}(\mathbf{r}', t)/|\mathbf{r} - \mathbf{r}'|), \tag{3c}$$

and $\mathbf{A}_T$ *must* be transverse, i.e.,

$$\nabla \cdot \mathbf{A}_T = 0. \tag{3d}$$

That Eqs. (3b) through (3d) satisfy Eq. (3a) is readily verified by use of the Coulomb potential Green's function identity,

$$\nabla^2_{\mathbf{r}}(1/|\mathbf{r} - \mathbf{r}'|) = -4\pi\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{3e}$$

What is thus in fact verified is that,

$$\nabla \cdot \mathbf{A}_L = -\dot{\phi}/c, \tag{3f}$$

which implies that the four-vector field,

$$A^\mu_{(0L)} \stackrel{\text{def}}{=} (\phi, \mathbf{A}_L), \tag{3g}$$

also satisfies the Lorentz condition,

$$\partial_\mu A^\mu_{(0L)} = 0, \tag{3h}$$

and this, together with the fact that A^μ satisfies the the Lorentz condition (by Eq. (2a)), implies that,

$$A^\mu_{(T)} \stackrel{\text{def}}{=} A^\mu - A^\mu_{(0L)}, \tag{3i}$$

as well satisfies the Lorentz condition,

$$\partial_\mu A^\mu_{(T)} = 0. \tag{3j}$$

From Eqs. (3g), (3i) and (3b), we see that,

$$A^\mu_{(T)} = (0, \mathbf{A}_T), \tag{3k}$$

which shows that $A^\mu_{(T)}$ is *completely transverse*. We now further note that any four-vector field which satisfies the Lorentz condition is *necessarily* Lorentz-covariant. That is because such a four-vector field contracted with the manifestly Lorentz-covariant four-vector differential operator ∂_μ yields *zero identically*, which is a *manifest Lorentz scalar*. Therefore $A^\mu_{(0L)}$ and $A^\mu_{(T)}$ are Lorentz-covariant four-vector fields.

The Lorentz condition has thus enabled us to covariantly separate A^μ into $A^\mu_{(T)}$, which is *purely transverse* and $A^\mu_{(0L)}$, which is *purely timelike and longitudinal*, with its time component being ϕ *itself*, while its longitudinal part is a pure homogeneous functional of $\dot{\phi}$. Now the current density four-vector $j^\mu = (c\rho, \mathbf{j})$ satisfies the current conservation source constraint given by Eq. (1g), which can be reexpressed as,

$$\nabla \cdot \mathbf{j} = -\dot{\rho}, \tag{4a}$$

in extremely close analogy with Eq. (3a) for $A^\mu = (\phi, \mathbf{A})$. Therefore we have for the current density four-vector j^μ extremely close analogs of *all* the results given by Eqs. (3) for the four-vector gauge field A^μ. We therefore simply list the most important of these results with a minimum of comment,

$$\mathbf{j} = \mathbf{j}_L + \mathbf{j}_T, \tag{4b}$$

where,

$$\mathbf{j}_L(\mathbf{r}, t) = \nabla((4\pi)^{-1} \textstyle\int d^3\mathbf{r}'\, \dot{\rho}(\mathbf{r}', t)/|\mathbf{r} - \mathbf{r}'|), \tag{4c}$$

and $\mathbf{j}_T$ *must* be transverse, i.e.,

$$\nabla \cdot \mathbf{j}_T = 0. \tag{4d}$$

In fact,

$$\nabla \cdot \mathbf{j}_L = -\dot{\rho}, \tag{4e}$$

which implies that the four-vector field,

$$j^\mu_{(0L)} \stackrel{\text{def}}{=} (c\rho, \mathbf{j}_L), \tag{4f}$$

also satisfies the current conservation constraint,

$$\partial_\mu j^\mu_{(0L)} = 0. \tag{4g}$$

We of course have that,

$$j^\mu_{(T)} \stackrel{\text{def}}{=} j^\mu - j^\mu_{(0L)}, \tag{4h}$$

as well satisfies the current conservation constraint,

$$\partial_\mu j^\mu_{(T)} = 0. \tag{4i}$$

We note that,

$$j^\mu_{(T)} = (0, \mathbf{j}_T), \tag{4j}$$

which shows that $j^\mu_{(T)}$ is *completely transverse.* Since $j^\mu_{(0L)}$ and $j^\mu_{(T)}$ both satisfy the current conservation constraint, they are therefore both Lorentz-covariant four-vector fields.

It is interesting to use Eq. (2b) and Eqs. (3) to determine the equations that are satisfied $A^\mu_{(0L)}$ and $A^\mu_{(T)}$. From the time component of Eq. (2b), we, of course obtain that,

$$\partial_\lambda \partial^\lambda \phi = \rho, \tag{5a}$$

or,

$$\ddot{\phi}/c^2 - \nabla^2 \phi = \rho. \tag{5b}$$

Thus we see that the operators which comprise $\partial_\lambda \partial^\lambda$ are $c^{-2}\partial^2/\partial t^2$ and $-\nabla^2$. Now from Eqs. (3c), (3e) and (4c) we readily work out that we can write $\mathbf{A}_L$ and $\mathbf{j}_L$ in the very convenient schematic operator forms,

$$\mathbf{A}_L = c^{-1}\nabla(-\nabla^2)^{-1}\dot{\phi}, \tag{5c}$$

and,

$$\mathbf{j}_L = \nabla(-\nabla^2)^{-1}\dot{\rho}. \tag{5d}$$

We clearly see that the two operators which comprise the operator $\partial_\lambda \partial^\lambda$ *both commute* with *all* of the three operators that appear in front of ϕ in Eq. (5c) for $\mathbf{A}_L$. Now the action of $\partial_\lambda \partial^\lambda$ on ϕ is given by Eq. (5a). With that and the form of Eq. (5d), we conclude that,

$$\partial_\lambda \partial^\lambda \mathbf{A}_L = \mathbf{j}_L/c. \tag{5e}$$

We can now combine our result of Eq. (5e) with the forms for $A^\mu_{(0L)}$ and $j^\mu_{(0L)}$ given by Eqs. (3g) and (4f) respectively, plus Eq. (5a), to obtain that,

$$\partial_\lambda \partial^\lambda A^\mu_{(0L)} = j^\mu_{(0L)}/c. \tag{5f}$$

If we combine the definitions of $A^{\mu}_{(T)}$ and $j^{\mu}_{(T)}$ that are given in Eqs. (3i) and (4h) with the results of Eqs. (5f) and (2b), we also obtain,

$$\partial_\lambda \partial^\lambda A^{\mu}_{(T)} = j^{\mu}_{(T)}/c. \tag{5g}$$

The four-vector potential $A^{\mu}_{(0L)}$ is completely determined by ϕ because, as we see from Eqs. (3g) and (5c), its time component is ϕ itself and its remaining longitudinal part is a purely homogeneous functional of $\dot{\phi}$. The relation of ϕ to ρ, however, is given by Eq. (5a), which leaves open the possibility that the relation of ϕ to ρ is *inhomogeneous* and *not fully determined by just ρ itself.* What we face here is simply an aspect of the *restricted* gauge-invariance ambiguity in the face of the imposition of *only* the Lorentz condition, as was pointed out in the discussion centered on Eq. (2c). Indeed it is entirely clear from Eq. (5a) that any contribution to ϕ which is *inhomogeneous* in ρ must satisfy *precisely* the relation pointed out in Eq. (2c). To jettison this annoying remnant of gauge-invariance ambiguity, we need to *supplement* the Lorentz condition with a *restriction on the solution space of Eq. (5a) for ϕ* that *rejects* any elements which are *inhomogeneous* in ρ. The intuitively/physically *most appealing* way to achieve this is to simply assert that the relation of ϕ to ρ is a *totally causal* one in *both* space and time. This *uniquely* pins down the following extremely well-known and justly celebrated homogeneous causal *retarded Coulomb transformation* of ρ as the desired solution of Eq. (5a) for ϕ, i.e.,

$$\phi(\mathbf{r}, t) = (4\pi)^{-1} \int d^3\mathbf{r}' \, \rho(\mathbf{r}', t - c^{-1}|\mathbf{r} - \mathbf{r}'|)/|\mathbf{r} - \mathbf{r}'|. \tag{6}$$

With the use of Eq. (6), a *particular gauge* has at long last been precisely determined. This gauge is as *close to the Coulomb gauge* as it is *possible* to get *without* clashing with the tenets of special relativity. Recall that the Coulomb-gauge version of ϕ simply *omits* the time retardation of the functional on the right-hand side of Eq. (6), and thereby manifests an *intantaneous response* of ϕ throughout the *whole of space* to *any change* in ρ, which is relativistically problematic. The two gauges agree *perfectly*, however, when ρ is time-independent (static). In that case the ϕ of Eq. (6) is *also* time-independent, and therefore, from Eq. (5c), the *longitudinal part $\mathbf{A}_L$ of $\mathbf{A}$ vanishes identically*, which is what the Coulomb gauge *mandates under all circumstances* by relativistically dubious *fiat.*

In the gauge determined by the Lorentz condition and Eq. (6), which we dub the causal Lorentz gauge, *there is nothing remotely dynamical about the timelike/longitudinal four-potential* $A^{\mu}_{(0L)} = (\phi, \mathbf{A}_L)$, because Eq. (6) *completely ties* ϕ to ρ without *the least trace* of dynamical freedom, and, of course, Eq. (5c) *just as completely ties* $\mathbf{A}_L$ to $\dot{\phi}$. Therefore, in causal Lorentz gauge, the timelike/longitudinal four-potential $A^{\mu}_{(0L)}$ *isolates the nondynamical sector of the Abelian gauge theory*, and it does so in *relativistically compliant fashion*, satisfying its own covariant Lorentz condition, Eq. (3h), and its own covariant "equation of motion", Eq. (5f) (to which *only the completely causal solution* that is set out in Eqs. (6) and (5c) is selected), a very far cry indeed from the *utter disregard for relativity* inherent in the Coulomb gauge.

Thus *devoid* in causal Lorentz gauge of *any dynamical content*, the timelike/longitudinal four-potential $A^{\mu}_{(0L)}$ *cannot be independently quantized*; in causal Lorentz gauge the *independently quantizable part of the gauge theory* resides *entirely* in the relativistically compliant *transverse, dynamical four-potential* $A^{\mu}_{(T)}$, which satisfies its own covariant Lorentz condition, Eq. (3j), and its own covariant transverse equation of motion, Eq. (5g). With the dynamical transverse and the nondynamical timelike/longitudinal sectors of the gauge theory thus cleanly and covariantly *separated* in causal Lorentz gauge, the possibility of *discarding* one of these sectors may be entertained. Whether that could be called for is an empirical issue: some parton-style analyses of empirical *hadronic* data have suggested that *quarks alone* are inadequate to account for that data, that strong participation by *gluons* (independently-quantized, transverse-spin dynamical gauge particles) is in fact required.

Feeding back the nondynamical four-vector potential in causal Lorentz gauge

A fascinating aspect of the Lorentz condition is that it *precisely parallels* the four-current conservation constraint. Thus a gauge field which adheres to the Lorentz condition could conceivably be made to contribute to its *own input four-current*, but it would need to be multiplied by a factor which has dimensions of inverse length squared in order to be four-current compatible. For the transverse *dynamical* four-potential in causal Lorentz gauge, such a maneuver would, at least naively, appear to endow the *independently-quantized* transverse gauge *particle* with *mass.* That seems uncomfortable from a physics standpoint, however, since the inherently *transverse* quantized gauge particle only has *two* spin degrees of freedom, not the *three* that a spin 1 particle with *mass* evidently *requires*.

In causal Lorentz gauge, the dynamical transverse and nondynamical timelike/longitudinal four-potentials, however, each *individually* adheres to the Lorentz condition, and each *also* has its *own* particular *individually conserved* input four-current, so it is possible for the *nondynamical* timelike/longitudinal four-potential in causal Lorentz gauge to be made to contribute to *itself only*, and to therefore *not* give rise to problematic gauge-particle *mass*, but to nevertheless *transform* its time component ϕ into an entity with *Yukawa-like* properties. Such a *fed-back* ϕ in causal Lorentz gauge would very likely be endowed with an exponentially *growing* Yukawa-type component in *addition* to a *traditional* exponentially *decaying* Yukawa component—it obviously requires a near-miracle for an exponentially *growing* Yukawa component to *not* develop in consequence of feedback. It is naturally very tempting to speculate that such an exponentially *growing* fed-back ϕ might be the mechanism of permanent quark-gluon confinement.

A technical/mathematical difficulty with exponentially *growing* Yukawa-type potentials is that since they *cannot* be spatially Fourier transformed, *neither* can they ever be the *result* of *any* approach which *entails spatial Fourier analysis* for obtaining the feedback potential response occasioned by an external charge density. Convenient handling of such potentials might conceivably entail unusual techniques such as use of the Laplace transformation. In what follows, we cope with this issue by first writing down a standard *inverse* space-time Fourier-transformation expression which applies to the causal retarded exponentially-*decaying* Yukawa feedback homogeneous potential response to a given external charge density, on which we carry out *closed-form evaluation* of the *spatial part* of this inverse Fourier transformation in order to obtain the explicit decaying Yukawa potential response *directly in configuration space*, with *only its time* still Fourier-transformed, and then *demonstrate* that this result can be straightforwardly *extended* to a sizable *class* of causal retarded exponentially-*growing* Yukawa-type feedback homogeneous potential responses to the *very same* external charge density. Some physically-based criterion for *choosing* amongst the *many different types* of causal retarded exponentially-growing Yukawa feedback homogeneous potential responses to an external charge density will eventually need to be developed: perhaps in the more realistic non-Abelian color-gauge context, that choice can somehow be tied to *color* in such a way that color *singlet* states uniquely encounter the exponentially-*decaying* Yukawa feedback potential, whilst any *nonsinglet* color configuration encounters an exponentially-*growing* version of the Yukawa feedback retarded homogeneous potential which serves to effectively *confine* that nonsinglet color configuration.

Without feedback, we recall that in causal Lorentz gauge the nondynamical timelike/longitudinal four-potential consists of $(\phi, \mathbf{A}_L)$, where the longitudinal three-potential $\mathbf{A}_L$ is, in fact, the homogeneous functional of $\dot{\phi}$ that is given by Eq. (5c) in such a way that the Lorentz condition $\nabla \cdot \mathbf{A}_L + \dot{\phi}/c = 0$ is *identically satisfied*. Similarly, in causal Lorentz gauge, this four-potential's conserved four-current source $(c\rho, \mathbf{j}_L)$ has its longitudinal part $\mathbf{j}_L$ given as a homogeneous functional of $\dot{\rho}$ by Eq. (5d) in such a way that the current conservation constraint $\nabla \cdot \mathbf{j}_L + \dot{\rho}$ is *identically satisfied*. The equation satisfied by $(\phi, \mathbf{A}_L)$ in terms of its source four-current $(c\rho, \mathbf{j}_L)$ is,

$$\partial_\lambda \partial^\lambda (\phi, \mathbf{A}_L) = (c\rho, \mathbf{j}_L)/c.$$

Now given a *nonnegative constant* κ whose dimension is inverse length, we proceed to make the nondynamical timelike/longitudinal $(\phi, \mathbf{A}_L)$ four-potential *contribute to its own source* by *adding* to its above *purely external* causal Lorentz gauge source current the conserved "nondynamical timelike/logitudinal four-potential self-current" $(-c\kappa^2\phi, -c\kappa^2\mathbf{A}_L)$ to produce the new *net* four-current source, $(c(\rho - \kappa^2\phi), \mathbf{j}_L - c\kappa^2\mathbf{A}_L)$, which, in view of the Lorentz condition, clearly *also identically satisfies* the current conservation constraint. If we now *replace* the original current source $(c\rho, \mathbf{j}_L)$ by this new *fed-back* net current source, the equation satisfied by $(\phi, \mathbf{A}_L)$ becomes,

$$\partial_\lambda \partial^\lambda (\phi, \mathbf{A}_L) = (\rho - \kappa^2\phi, \mathbf{j}_L/c - \kappa^2\mathbf{A}_L),$$

or,

$$(\partial_\lambda \partial^\lambda + \kappa^2)(\phi, \mathbf{A}_L) = (\rho, \mathbf{j}_L/c).$$

We in fact *only* need solve the *time-component* feedback equation,

$$(\partial_\lambda \partial^\lambda + \kappa^2)\phi = \rho, \tag{7a}$$

for ϕ, because, in view of the Lorentz condition, $\mathbf{A}_L$ is *still* the homogeneous functional of $\dot{\phi}$ which is given by Eq. (5c). If we restrict ourselves to merely solving for a *Green's function* $G(\mathbf{r}, t; \kappa)$ which satisfies,

$$(\partial_\lambda \partial^\lambda + \kappa^2) G(\mathbf{r}, t; \kappa) = \delta(t)\delta^{(3)}(\mathbf{r}), \tag{7b}$$

then a solution of Eq. (7a) which is *homogeneous in the charge density* ρ, a *key property that we insist on*, will be given by,

$$\phi(\mathbf{r}, t) = \int dt' d^3\mathbf{r}'\, \rho(\mathbf{r}', t') G(\mathbf{r} - \mathbf{r}', t - t', \kappa). \tag{7c}$$

Now a causal *retarded* Green's function that satisfies,

$$G(\mathbf{r}, t; \kappa) = 0 \text{ when } t < 0, \tag{7d}$$

and whose static reduction is a *purely exponentially decaying* Yukawa potential, is given by the *inverse space-time Fourier transformation* expression [2],

$$G_{-1}(\mathbf{r}, t; \kappa) = (2\pi)^{-4} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} \int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}} [-((\omega/c) + i\epsilon)^2 + |\mathbf{k}|^2 + \kappa^2]^{-1}. \tag{7e}$$

Because,

$$(\partial_\lambda \partial^\lambda + \kappa^2) = c^{-2}\partial^2/\partial t^2 - \nabla^2 + \kappa^2, \tag{7f}$$

it is easily seen $G_{-1}(\mathbf{r}, t; \kappa)$ satisfies the basic Green's function requirement of Eq. (7b). Since both poles in the ω-dependence of the integrand of G_{-1} occur in the lower-half ω-plane, and because, for $t < 0$, the ω contour must be closed in the upper-half ω-plane, we see that for $t < 0$, G_{-1} vanishes, which makes it a causal *retarded* Green's function. With the aid of a contour integration, the *inverse spatial* $d^3\mathbf{k}$-integration of Eq. (7e) for G_{-1} can be analytically carried out in *closed form*. We have proceeded to straightforwardly *extend* the *particular* expression which thereby results for G_{-1} to a *class* of objects whose *members* we denote as $G_\eta(\mathbf{r}, t; \kappa)$. We will now demonstrate that *each* such object G_η satisfies the basic Green's function requirement of Eq. (7b) *irrespective* of the value of η, while *still retaining* the causal *retarded* nature of G_{-1}. The expression for G_η is in the form of an *inverse time Fourier transformation* times the factor (2π),

$$G_\eta(\mathbf{r}, t; \kappa) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} [\theta(1 - (\omega/(c\kappa))^2) g_\eta(r, \kappa(1 - (\omega/(c\kappa))^2)^{\frac{1}{2}}) + \theta(1 - (c\kappa/\omega)^2) h(r, \omega(1 - (c\kappa/\omega)^2)^{\frac{1}{2}})], \tag{8a}$$

where $r \stackrel{\text{def}}{=} |\mathbf{r}|$, θ is the standard Heaviside unit step function, which equals zero for negative argument and unity for positive argument, and

$$g_\eta(r, \kappa') \stackrel{\text{def}}{=} (8\pi^2(r + \epsilon))^{-1} [\cosh(\kappa' r) + \eta \sinh(\kappa' r)], \tag{8b}$$

$$h(r, \omega') \stackrel{\text{def}}{=} (8\pi^2(r + \epsilon))^{-1} e^{i\omega' r/c}, \tag{8c}$$

and ϵ is a positive infinitesimal length. The expression in square brackets in the *integrand* of Eq. (8a) equals $(2\pi)^{-1}$ times $G_\eta(\mathbf{r}, \omega; \kappa)$, which is the *time Fourier transformation of* $G_\eta(\mathbf{r}, t; \kappa)$. From that expression and Eq. (8c) we can see that for *asymptotically large values of* $|\omega|$ (i.e., asymptotically high Fourier frequencies), $G_\eta(\mathbf{r}, \omega; \kappa)$ behaves as $(4\pi(r+\epsilon))^{-1} e^{i\omega r/c}$, which is an *outgoing spherical wave* in light of the time-dependence $e^{-i\omega t}$ of the integrand of Eq. (8a). This purely *outgoing* spherical wave high-Fourier-frequency asymptotic behavior of the integrand of the inverse time Fourier transformation of $G_\eta(\mathbf{r}, t; \kappa)$ shows that $G_\eta(\mathbf{r}, t; \kappa)$ is a causal *retarded* Green's function. We *also* note from Eqs. (8a) through (8c) that at the somewhat confusing "critical points", $\omega = \pm c\kappa$, $G_\eta(\mathbf{r}, \omega; \kappa)$ is *continuous* as a function of ω and assumes the value $(4\pi(r + \epsilon))^{-1})$ *irrespective* of the value of η, which ensures, for *any* value of η, that $G_\eta(\mathbf{r}, \omega; \kappa)$ is a sensibly *continuous* well-defined function of ω. It turns out that the $G_\eta(\mathbf{r}, t; \kappa)$ of Eq. (8a) can be expressed in terms of Bessel functions [2], but it doesn't appear to be necessary or worthwhile to write them as such here. The positive infinitesimal length ϵ in Eqs. (8b) and (8c) can usually be dropped, but the key exception is that, if one bears in mind the identity,

$$\nabla^2 f(r) = r^{-1} d^2(rf(r))/dr^2, \tag{8d}$$

then meticulously careful calculation yields,

$$-\nabla^2((8\pi^2(r+\epsilon))^{-1}) = (4\pi^2 r)^{-1}\epsilon/(r+\epsilon)^3 \overset{\epsilon\to 0+}{\rightarrow} (2\pi)^{-1}\delta^{(3)}(\mathbf{r}). \tag{8e}$$

From Eqs. (8b) through (8e), it is readily shown that,

$$-\nabla^2 g_\eta(r,\kappa') = (2\pi)^{-1}\delta^{(3)}(\mathbf{r}) - (\kappa')^2 g_\eta(r,\kappa'), \tag{8f}$$

and,

$$-\nabla^2 h(r,\omega') = (2\pi)^{-1}\delta^{(3)}(\mathbf{r}) + (\omega'/c)^2 h(r,\omega'). \tag{8g}$$

From Eqs. (8f) and (8g), together with Eq. (7f) we obtain,

$$(\partial_\lambda\partial^\lambda + \kappa^2)(e^{-i\omega t} g_\eta(r, \kappa(1-(\omega/(c\kappa))^2)^{\frac{1}{2}})) = (2\pi)^{-1} e^{-i\omega t}\delta^{(3)}(\mathbf{r}), \tag{8h}$$

and,

$$(\partial_\lambda\partial^\lambda + \kappa^2)(e^{-i\omega t} h(r, \omega(1-(c\kappa/\omega)^2)^{\frac{1}{2}})) = (2\pi)^{-1} e^{-i\omega t}\delta^{(3)}(\mathbf{r}), \tag{8i}$$

and therefore,

$$(\partial_\lambda\partial^\lambda + \kappa^2)G_\eta(\mathbf{r},t;\kappa) = (2\pi)^{-1}\delta^{(3)}(\mathbf{r})\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} = \delta(t)\delta^{(3)}(\mathbf{r}), \tag{8j}$$

which shows that *all* $G_\eta(\mathbf{r},t;\kappa)$, *regardless* of the value of η, are Green's functions of $(\partial_\lambda\partial^\lambda + \kappa^2)$. In the limit that we do *not* feed back, i.e., that $\kappa \to 0+$, we obtain,

$$G_\eta(\mathbf{r},t;\kappa=0) = \int_{-\infty}^{+\infty} d\omega\, e^{-i\omega t} h(r,\omega) = (8\pi^2 r)^{-1}\int_{-\infty}^{+\infty} d\omega\, e^{-i\omega(t-r/c)} = (4\pi r)^{-1}\delta(t-r/c), \tag{9}$$

which is the standard causal *retarded* Green's function of the Abelian gauge theory—together with Eq. (7c) it yields the celebrated Eq. (6). Note that in this no-feedback limit where $\kappa \to 0+$, there is *no dependence whatsoever on* η.

Finally, in the interesting case that the source charge density $\rho(\mathbf{r},t)$ has *no time dependence*, i.e., is *static*, we readily see from Eq. (7c) that the corresponding *static Green's function*, $G_\eta(\mathbf{r};\kappa)$, is simply given by the *integral over all time of the dynamical Green's function*, $G_\eta(\mathbf{r},t;\kappa)$. Since,

$$\int_{-\infty}^{+\infty} dt\, e^{-i\omega t} = (2\pi)\delta(\omega), \tag{10a}$$

we obtain from this and Eqs. (8a) and (8b) that,

$$G_\eta(\mathbf{r};\kappa) = (2\pi) g_\eta(r,\kappa) = (4\pi r)^{-1}[\cosh(\kappa r) + \eta\sinh(\kappa r)], \tag{10b}$$

which properly reduces to the point-charge static Coulomb potential $(4\pi r)^{-1}$ in the limit of no feedback, i.e., $\kappa \to 0+$. For $\eta = -1$, it is the classic purely exponentially *decaying* point-charge static Yukawa potential, $(4\pi r)^{-1}\exp(-\kappa r)$, which is the static reduction of the classic retarded causal Fourier-transformation Green's function that is given by Eq. (7e). In the case of non-vanishing feedback, we *began* with this particular inverse space-time Fourier-transformation Green's function of Eq. (7e), which we denoted as $G_{-1}(\mathbf{r},t;\kappa)$. We then *extended* it's inverse *purely time* Fourier transformation to a whole additional *class* of *extremely similar* inverse *purely time* Fourier transformations for which, however, the corresponding inverse *spatial* Fourier transformations simply *fail to exist because those Green's functions feature growing exponential components in configuration space.* We have denoted the *members* of this class as $G_\eta(\mathbf{r},t;\kappa)$ for arbitrary values of η, and their static reductions, which *prominently display* those *growing* exponential components (*except* for the case $\eta = -1$), are given by Eq. (10b) above.

In diametrical opposition to the classic purely exponentially *decaying* static Yukawa potential associated with $\eta = -1$, there is the purely exponentially *growing* static "contra-Yukawa" potential associated with $\eta = +1$, namely $(4\pi r)^{-1}\exp(+\kappa r)$, which although patently unavailable via inverse spatial Fourier transformation, is *just as much a legitimate consequence of the potential feedback idea* as is the classic purely exponentially *decaying* Yukawa case. Indeed the *range* of η-values which have *some admixture* of exponentially growing static Yukawa potential component *utterly swamps* the *single* $\eta = -1$ value for which that component is precariously just canceled out. But because of the *immense bias* introduced by the almost *universally employed* inverse spatial Fourier transformation [2], the *overwhelming prevalence* of feedback-Yukawa static potentials which *grow* exponentially *rather* than decay exponentially has been effectively *entirely wiped off the radar screen*!

Conclusion

One can at least entertain the *hope* that the feedback-Yukawa exponentially *growing* static potential components provide a vital clue as to the mechanism of quark-gluon confinement. Of course this idea, here treated only in a bare-bones Abelian gauge theory, needs to be properly implemented in the non-Abelian Yang-Mills color gauge theory, where the whole η-range of mixed-component exponentially growing and exponentially decaying static Yukawa potentials can hopefully be *linked to color* in such a way that *only the traditional* $\eta = -1$ *purely exponentially decaying Yukawa potential* operates for *color singlet* entities. The apparently exponentially *growing* nature of the confinement potential *also* provides some feeling for why a quark-gluon "plasma" that is too "hot" (strongly thermally disorganized) to be able to readily reorganize itself into *unconfined color singlets*—i.e., "hot" enough to be "colorful"—can *never* have characteristics that are *at all akin* to those of a free gas.

It is worthwhile to point out once again that the timelike/longitudinal *potential phenomena* that we have treated here in causal Lorentz gauge are *strictly non-dynamical* in nature, and thus are *absolutely not subject to independent quantization*: the independent dynamical gauge *quanta* are all associated to the *transverse* components of the gauge field, which in causal Lorentz gauge can be cleanly and covariantly *separated* from the timelike/longitudinal nondynamical potential phenomena discussed here.

References

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