Equivalence of Maxwell's source-free equations to the time-dependent Schrödinger equation for a solitary particle with two polarizations and Hamiltonian $|c\widehat{\mathbf{p}}|$

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Abstract

It was pointed out in a previous paper that although neither the Klein-Gordon equation nor the Dirac Hamiltonian produces sound solitary free-particle relativistic quantum mechanics, the natural square-root relativistic Hamiltonian for a nonzero-mass free particle does achieve this. Failures of the Klein-Gordon and Dirac theories are reviewed: the solitary Dirac free particle has, *inter alia*, an invariant speed well in excess of c and staggering spontaneous Compton acceleration, but no pathologies whatsoever arise from the square-root relativistic Hamiltonian. Dirac's key misapprehension of the underlying four-vector character of the time-dependent, configuration-representation Schrödinger equation for a solitary particle is laid bare, as is the invalidity of the standard "proof" that the nonrelativistic limit of the Dirac equation is the Pauli equation. Lorentz boosts from the particle rest frame point uniquely to the square-root Hamiltonian, but these don't exist for a massless particle. Instead, Maxwell's equations are dissected in spatial Fourier transform to separate nondynamical longitudinal from dynamical transverse field degrees of freedom. Upon their decoupling in the absence of sources, the transverse field components are seen to obey two identical time-dependent Schrödinger equations (owing to two linear polarizations), which have the massless free-particle diagonalized square-root Hamiltonian. Those fields are readily *modified* to conform to the attributes of solitary-photon *wave functions*. The wave functions' relations to the *potentials* in radiation gauge are also worked out. The exercise is then repeated *without* the considerable benefit of the spatial Fourier transform.

Introduction

It was pointed out in a previous paper [1] that a *solitary* free relativistic nonzero-mass particle is described *without any pathology whatsoever* by the *natural* time-dependent Schrödinger equation,

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$$i\hbar\partial(|\psi(t)\rangle)/\partial t = \sqrt{m^2c^4 + |c\widehat{\mathbf{p}}|^2}\,|\psi(t)\rangle, \tag{1}$$

whereas the widely used relativistic free-particle Klein-Gordon and Dirac equations are generally acknowledged not to be up to this simple task [2]. For example, negative energy solutions of the Klein-Gordon equation *fail* to be orthogonal to their *positive* energy counterparts that have the *same momentum*. This contradicts a fundamental property of quantum theory that makes its probability interpretation possible; unsurprisingly it is well-known that Klein-Gordon theory can yield *negative* probabilities [2]. This particular pathology of the second-order in time Klein-Gordon equation cannot arise if the solitary free particle is described by a standard first-order in time Schrödinger equation with a Hermitian Hamiltonian operator, such as that of Eq. (1). The *particular* Hamiltonian operator of Eq. (1), namely $\sqrt{m^2c^4 + |c\widehat{\mathbf{p}}|^2}$, has the *additional* virtue of *fully adhering* to the classical Correspondence Principle, being that it is the *direct* quantization of the *correct classical Hamiltonian* for a solitary relativistic free particle of mass m. It is to be noted in particular that solitary free *relativistic* particles *cannot* have negative energies if solitary free *nonrelativistic* particles are to be restricted to having only *nonnegative kinetic energies*!

In light of the problems the second-order in time Klein-Gordon equation has in describing the solitary relativistic free particle, Dirac appreciated the need for elementary *relativistic* quantum mechanics to *return* to the standard first-order in time Schrödinger equation format with Hermitian Hamiltonian operator that serves elementary *nonrelativistic* solitary-particle quantum mechanics so admirably. Lamentably, however, Dirac was less responsive to the exacting requirements of the classical Correspondence Principle than he was, like Klein, Gordon and Schrödinger, misdirectedly concerned about the fact that the solitary free particle Hamiltonian operator $\sqrt{m^2c^4 + |c\widehat{\mathbf{p}}|^2}$ turns out to be a *nonlocal* entity in configuration representation: it doesn't seem to have occurred to these pioneers that this fact in *no* way stymies the fruitful application of *perturbation approximations*—the relativistic *corrections* to the atomic physics in which they were interested are obviously *very well-suited* to this approach, being *compatibly small*. Dirac unfortunately *rejected* the Correspondence Principle appropriate square-root Hamiltonian operator of Eq. (1) in favor of a *misconceived linearization* of it in terms of the components of the momentum operator $\widehat{\mathbf{p}}$ and the mass m, for which he argued on the basis of a *fundamental misapprehension* of the manner in which the *solitary-particle* time-dependent Schrödinger equation in configuration representation,

$$i\hbar\partial(\langle\mathbf{r}|\psi(t)\rangle)/\partial t = \langle\mathbf{r}|\widehat{H}|\psi(t)\rangle,$$

is related to the *covariance requirements of special relativity* [3, 4, 2]. It is clear that the operator $\partial/\partial t$ on the *left-hand side* of this equation is the *time component* of the *four-vector operator* $c\partial/\partial x_\mu$, and the operator $\widehat{H}$ on the *right-hand side* of this equation is the *time component* of the *four-vector operator* $c\widehat{p^\mu}$, where $\widehat{p^\mu} \stackrel{\text{def}}{=} (\widehat{H}/c, \widehat{\mathbf{p}})$. Moreover, it was *postulated* by Schrödinger, and is a *basic consequence* of Dirac's *own* canonical commutation rule, that,

$$-i\hbar\nabla_{\mathbf{r}}(\langle\mathbf{r}|\psi(t)\rangle) = \langle\mathbf{r}|\widehat{\mathbf{p}}|\psi(t)\rangle,$$

so that the *full four-vector equation*,

$$i\hbar\partial(\langle\mathbf{r}|\psi(t)\rangle)/\partial x_\mu = \langle\mathbf{r}|\widehat{p^\mu}|\psi(t)\rangle,$$

is *guaranteed to hold* in solitary-particle quantum mechanics! Since the operator $\partial/\partial x_\mu$ is *patently* a *Lorentz covariant* four-vector, the covariance requirements of special relativity are met in solitary-particle quantum mechanics by simply requiring that the Hamiltonian operator $\widehat{H}$ be selected so as to ensure that the four-vector operator $\widehat{p^\mu} = (\widehat{H}/c, \widehat{\mathbf{p}})$ *also* transforms between inertial frames as a *Lorentz covariant* four-vector. This requirement is *automatically fulfilled* by *scrupulous adherence to the strongest possible form of the classical Correspondence Principle*, i.e., that $\widehat{H}$ be the *quantization* of precisely that *classical Hamiltonian* H which has been carefully checked to be appropriate to fully relativistic solitary-particle *classical* mechanics! For the *free* solitary particle of nonzero mass m, this physically *methodical* and *highly conservative* approach leaves us with *no option* but to *accept* Eq. (1) as its *correct* time-dependent Schrödinger equation description! This *even* extends to *free* spin $\frac{1}{2}$ particles of nonzero mass: notwithstanding that spin $\frac{1}{2}$ itself is a nonclassical attribute, the nonrelativistic Pauli Hamiltonian for such a particle *automatically reduces* to the *usual* nonrelativistic *purely kinetic-energy Hamiltonian* in the *free-particle* limit, and one can *always* find an inertial frame of reference in which a *free particle of nonzero mass* is *completely nonrelativistic*!

Dirac, however, was much too focused on trying to cobble up a relativistic solitary-particle Hamiltonian operator which is *local in configuration representation* to be in any frame of mind to appreciate this *profound link* between the strongest form of the classical Correspondence Principle and the *requirement of Lorentz*

covariance in solitary-particle *quantum mechanics.* Instead of pondering the *details* of *how* the requirement of Lorentz covariance actually impacts the time-dependent solitary-particle Schrödinger equation in configuration representation, Dirac was content to assume that relativistic covariance merely requires that there be essentially complete symmetry in the formal treatment of space and time coordinates [3, 2, 4]. As a result, he *completely missed the point* that the time-dependent Schrödinger equation relates the *time derivative* operator to an *energy* operator, *neither of which* are Lorentz *scalars,* but *each of which* is the *time component* of a Lorentz *four-vector.* Not having *assimilated* these basics, he conjured from whole cloth a *nonexistent Lorentz scalar* which he perceived this equation to *split into two nonscalar fragments for the express purpose of displaying the fragment which is proportional to the time derivative on the left-hand side of the equality sign!* Thus primed, Dirac "concluded" that his phantom scalar's "completion fragment", which is everything on the *right-hand side* of the equality sign, "must" therefore be *linear in the space gradient,* which suited his purpose *perfectly,* as it results in a *local* "Hamiltonian operator" in configuration representation! Following this "eureka moment", which was the *fruit* of his *mistakenly identifying as a scalar the time component of a four-vector,* Dirac *failed to reflect* on whether a Hamiltonian operator that is *linear* in the space gradient, and thus in the momentum, could *really* be relativistically correct in light of the firmly established understanding that a solitary free particle's Hamiltonian is *ineluctably* the *time-component* of a *Lorentz-covariant four-vector* whose *remaining three components* are c times that free particle's three-momentum! This understanding, *conjoined with the Lorentz transformation itself,* in fact *determines* that the *square-root* Hamiltonian which occurs in Eq. (1) is the *only correct one* for the nonzero-mass free particle!

Dirac *also* paid no heed to the fact that a solitary free-particle Hamiltonian operator which is *linear* in the space gradient, and therefore in the momentum operator, has, in light of Heisenberg's equation of motion, the *unavoidable consequence* that the free particle's *velocity* is *completely independent* of its *momentum,* which is an *astounding contradiction of nonrelativistic free-particle physics,* quantum or classical! Dirac determined the *coefficients* of his misconceived linearized Hamiltonian operator by requiring that its *square* be equal to the *square* of the square-root Hamiltonian operator of Eq. (1), which is a perilously weakened *surrogate* for the strong classical Correspondence Principle that *produces* the square-root Hamiltonian operator of Eq. (1) in the first place! It results in these coefficients satisfying the well-known Dirac-matrix *anticommutation relations* [2]. The free-particle *velocity operator,* which involves *only* these coefficients, is thereupon determined to equal the speed of light c times a three-vector of the Dirac matrices, which each square to unity. Therefore the *speed* of *any* free Dirac particle turns out to have the universal *superluminal* value $\sqrt{3}\,c$, *irrespective of its momentum!* The free-particle Dirac equation in fact yields *more* such inordinately *unphysical* results. Upon using the misconceived linearized Dirac Hamiltonian operator in conjunction with Heisenberg's equation of motion to calculate the free particle's spontaneous *acceleration,* one finds that its *magnitude* has a *minimum* value of order of the "Compton acceleration" $mc^3/\hbar$, which, for the electron, is about $10^{28}g$, an *absolutely staggering* violation of Newton's First Law of Motion for a free particle! The natural square-root Hamiltonian operator of Eq. (1) gives *nil* spontaneous acceleration, a result that is, of course, in *complete agreement* with Newton's First Law of Motion for a free particle. It as well gives the *correct expression* for the relativistic free particle's *velocity* in terms of its momentum. The *extreme disparity* of the results of the natural square-root Hamiltonian operator versus those of the *misbegotten linearized* Dirac Hamiltonian operator is an overwhelming object lesson on the dangers inherent in *any weakening* of the strongest sensible version of the classical Correspondence Principle.

Although it is routinely claimed that the Dirac equation reduces to the nonrelativistic Pauli equation for a spin $\frac{1}{2}$ particle when the particle's momentum magnitude is much less than mc [2], the "proof" of this assertion is definitely *invalid,* being *unsalvageably dependent* on the lapse of *forgetting* that at $\mathbf{p} = \mathbf{0}$ the two *lower components* of the Dirac spinor in the *standard representation* have the time-dependence phase factor $e^{+(imc^2/\hbar)t}$, which is *totally different* from the analogous factor $e^{-(imc^2/\hbar)t}$ that occurs in this spinor's two *upper components!* Nor does this assertion remotely accord with some of the most elementary "physics" consequences of the *free* Dirac and Pauli theories at *vanishing momentum.* The latter's Hamiltonian operator is just the nonrelativistic kinetic energy operator $|\widehat{\mathbf{p}}|^2/(2m)$, and its particle speed operator is, of course, $|\widehat{\mathbf{p}}|/m$. Thus a *free* Pauli particle eigenstate of vanishing momentum has vanishing speed. For the *free* Dirac theory, we have *already seen* that the particle speed operator is *even simpler,* namely the *identity operator* times the *universal superluminal speed* $\sqrt{3}\,c$! So a *free* Dirac particle eigenstate of vanishing momentum *still* has this problematic extreme speed!

Furthermore, *notwithstanding its spin $\frac{1}{2}$ attribute,* the *free* Pauli particle's *orbital angular momentum is exactly conserved,* i.e., the rate of change of its orbital angular momentum *vanishes identically.* The *free* Dirac particle's orbital angular momentum, however, is coupled with *astonishing strength* to its spin: as the free Dirac particle's *momentum magnitude* tends toward *zero,* the *dimensionless ratio* of the *magnitude* of the rate of change of its orbital angular momentum to its kinetic energy *increases monotonically without bound,*

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beginning from the *asymptotic* ultrarelativistic *dimensionless ratio value* $\sqrt{2}$! In other words, *far* from having the *exactly conserved orbital angular momentum of the free Pauli particle*, the free Dirac particle's spin-orbit *torque magnitude* always *well exceeds* that particle's *own kinetic energy*, and the dimensionless ratio of these two quantities in fact becomes *arbitrarily* large at *low enough* particle momentum!

All of these *stunningly unphysical* properties of the Klein-Gordon and Dirac equations with regard to the description of a solitary relativistic free particle *notwithstanding*, and *even in spite* of the fact that the simple square-root Hamiltonian operator of Eq. (1)—which is the *unique consequence* of the classical Correspondence Principle for a solitary relativistic free particle—doesn't partake of *any* such pathologies, it *still* has always been the Dirac and Klein-Gordon equations, *rather* than Eq. (1), that are inducted into relativistic quantum field theory. The reason for this, of course, is that *antiparticles* were first observed sometime *after* Dirac began to speculate about mechanisms which could serve to *hide* the physically problematic *unbounded-below* negative energy spectra that are a feature of his and the Klein-Gordon equations, but which simply *do not occur* for Eq. (1). Had Dirac *not* been so speculating, the existence of antiparticles would have been regarded as an *energy degeneracy* of nature's full field theoretic Hamiltonian, and an explanation for that *degeneracy* would have been sought, following the grand tradition established by Wigner, Weyl and others, entirely in terms of the effect of a *symmetry* possessed by that full Hamiltonian. A particle and its antiparticle are *distinguishable*, and each can have only *positive* energy, so it is *entirely natural* that they should be described by *two entirely independent quantum fields, with each having purely positive energy.* In other words, had the Klein-Gordon and Dirac equations, with their problematic unbounded-below negative energy spectra *never been concocted*, it would have been perfectly straightforward to accommodate the discovery of antiparticles in a simple, logical framework that is very strongly grounded in physical precedent. The straightforward use of the *purely positive energy* Eq. (1) in conjunction with *symmetry* postulates to accommodate *antiparticles* has the *theoretical advantage* that it *as well* automatically accommodates a sensible theory of a solitary free relativistic particle, which the Dirac and Klein-Gordon equations are *utterly unsuited* to do. There is no physical reason whatsoever that *nonrelativistic* solitary particle theory should not link to relativistic particle physics in a completely *smooth fashion*, which is what Eq. (1) transparently enables. Furthermore, both the Klein-Gordon and Dirac equations historically arose as *eccentric offshoots* of Eq. (1), motivated *not* by legitimate physics concerns, but by an *irrational distaste* for the *nonlocal* character of Eq. (1) in *configuration representation*. This means that the Klein-Gordon and Dirac equations were *not designed ab initio* to accommodate *both* a particle and its antiparticle: this is a role into which historical happenstance has *pushed* them—by their *actual patrimony* they were *designed* to accommodate *only a single type of particle*! Nowadays, it is known that particle-antiparticle symmetry can be slightly *broken*, as CP noninvariance experiments have shown (given the dominance of particles over antiparticles in our immediate surroundings, it would be *astonishing* if particle-antiparticle symmetry were *not* in fact broken). But the Dirac and Klein-Gordon fields, *not* having been *designed to accommodate two particles*, are highly stressed to accommodate two slightly *nondegenerate* particles, which is what corresponds to the existent symmetry breaking. It is obvious that the model with two *independent* positive-energy fields for particle and antiparticle offers *vastly more flexibility to accommodate symmetry breaking* than do the claustrophobic Dirac and Klein-Gordon models, which shoehorn two particles into a field structure that was *designed to accommodate just one*. As one example, two independent fields easily accommodate two slightly different masses: there is simply no way to have a single Dirac or Klein-Gordon field with more than one mass.

For a nonzero-mass solitary free particle, the relativistic square-root Hamiltonian operator of Eq. (1) is *completely determined* by the Lorentz transformation. This is because one can *always* find an inertial frame in which a *solitary* free particle of mass m is at rest, i.e., has four-momentum $(mc, \mathbf{0})$. The *Lorentz transformation* to the inertial frame in which this particle has velocity $\mathbf{v}$, where $|\mathbf{v}| < c$, then takes the particle's four-momentum to,

$$(mc(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}, \, m\mathbf{v}(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}) = (E(\mathbf{v})/c, \, \mathbf{p}(\mathbf{v})),$$

which, together with the *identity*,

$$mc^2(1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}} = \sqrt{m^2c^4 + |cm\mathbf{v}|^2(1 - |\mathbf{v}|^2/c^2)^{-1}},$$

implies that,

$$E(\mathbf{v}) = \sqrt{m^2c^4 + |c\mathbf{p}(\mathbf{v})|^2}.$$

Since the classical precursor of the square-root Hamiltonian operator for the solitary free-particle of mass m that occurs in Eq. (1) is thus *mandated by the very nature of the Lorentz transformation*, it is little wonder that Dirac's misconceived effort to *linearize* the square-root character of this Hamiltonian operator has consequences

which *terribly violate* well-known relativistic properties of a free particle: we have seen that these consequences include the blatantly unphysical universal *superluminal* free particle speed $\sqrt{3}\,c$ *irrespective* of the particle momentum, a *minimum* spontaneous free-particle acceleration magnitude of order of the Compton acceleration $mc^3/\hbar$, namely about $10^{28}g$ for the electron, which staggeringly violates Newton's First Law of Motion for a free particle, and the gross failure to *conserve* free-particle orbital angular momentum, which the nonrelativistic spin $\frac{1}{2}$ Pauli theory free particle definitely *does*.

For a *zero-mass free particle*, however, there is *no* inertial frame in which that particle is at rest, so we cannot readily derive its Hamiltonian from the Lorentz transformation, as we have done for the nonzero-mass free particle. Confirmation that the Hamiltonian operator given by Eq. (1) continues to be correct for a *massless* solitary free particle must be sought elsewhere. We therefore turn to the study of electromagnetic radiation, which is supposed to consist of *massless photons*. Surprisingly, we shall see that Maxwell's *classical* equations for *pure* electromagnetic radiation can be recast into a form that is in essence that of the time-dependent Schrödinger equation of Eq. (1) with $m = 0$. Because of the particle's vanishing mass, Planck's constant $\hbar$ can be factored out of *both sides* of Eq. (1), since $\widehat{\mathbf{p}} = -i\hbar\nabla_{\mathbf{r}}$ in configuration representation and $\widehat{\mathbf{p}} = \hbar\mathbf{k}$ in Fourier vector variable $\mathbf{k}$-representation. That Planck's constant *drops out of the relativistic solitary free-particle time-dependent Schrödinger equation in the $m = 0$ case* is a key factor in allowing that equation to be related to the putatively "classical" Maxwell theory.

To make further progress, we must *dissect* Maxwell's four equations themselves: these are a mixed bag of dynamical field equations of motion and nondynamical constraint conditions. Since the time-dependent Schrödinger equation is *purely dynamical* in character, it will be necessary to properly resolve the nondynamical constraint conditions, a task which we now undertake.

The electromagnetic field as a constrained dynamical system

Since any time-dependent Schrödinger equation is linear and *homogeneous*, only the *source-free* (i.e., pure radiation) version of Maxwell's equations could possibly correspond to such an equation. But the resolution of the two *nondynamical constraints* amongst the the four Maxwell equations can be carried out even in the *presence* of the source terms, so we shall initially retain those terms. The four Maxwell equations for the electromagnetic field $(\mathbf{E}, \mathbf{B})$ with four-current source $(\rho, \mathbf{j}/c)$ are comprised of Coulomb's law,

$$\nabla \cdot \mathbf{E} = \rho, \tag{2a}$$

Faraday's law,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c, \tag{2b}$$

Gauss' law,

$$\nabla \cdot \mathbf{B} = 0, \tag{2c}$$

and Maxwell's law,

$$\nabla \times \mathbf{B} = (\mathbf{j} + \dot{\mathbf{E}})/c, \tag{2d}$$

which, together with Coulomb's law, implies the current conservation condition,

$$\nabla \cdot \mathbf{j} + \dot{\rho} = 0. \tag{2e}$$

Coulomb's and Gauss' laws both involve *no* time derivatives of the electromagnetic field, so they are in the nature of *nondynamical constraints* on that field, whereas Faraday's and Maxwell's law's, which both *do* involve first time derivatives of the electromagnetic field, have the character of dynamical equations of motion of that field. If one is presented with a set of N variables which are subject to both nondynamical equations of constraint and dynamical equations of motion, it is standard practice to search for N *functions* of those N variables with the property that all the dynamical equations of motion involve *only* a subset of $N - k$ of these functions, while all the nondynamical equations of constraint involve *only* the *remaining* subset of k functions. The *first set of $N - k$ functions* is not subject to *any* nondynamical equations of constraint (these apply *exclusively* to the *second* set of k functions), and are regarded as a set of purely *dynamical variables* for

the system. The *second* set of k functions, to which *no* dynamical equations of motion apply, may analogously be regarded as a set of purely *nondynamical variables* for the system. The equations of motion satisfied by the $N - k$ *unconstrained* dynamical variables are then typically summarized by means of a Lagrangian or Hamiltonian from which they follow, respectively, via the Euler-Lagrange or classical Hamiltonian equations equations of motion. Hamiltonization of such a maximal set of unconstrained dynamical variables opens the way to the system's *quantization* via either the Hamiltonian phase-space path integral [5], or, equivalently, the slightly strengthened self-consistent extension of Dirac's canonical commutation rule [6].

For the electromagnetic *field*, N is formally infinite, but we can still usefully discuss the number of *field* degrees of freedom; e.g., the electromagnetic field $(\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$ has six *field* degrees of freedom. *Both* the equations of motion and those of constraint are *linear* for the electromagnetic field, so one can expect the extraction of a maximal subset of unconstrained dynamical variables (actually unconstrained dynamical *fields*) to involve appropriate *linear transformations* of components of the electromagnetic field $(\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$. Furthermore, consideration of the Coulomb and Gauss equations of constraint quickly makes it clear that $\nabla \cdot \mathbf{E}(\mathbf{r}, t)$ and $\nabla \cdot \mathbf{B}(\mathbf{r}, t)$ (which vanishes identically!) are each purely nondynamical *single* field degrees of freedom, and that *no additional* purely *nondynamical* field degrees of freedom are available to be extracted from the six field degrees of freedom of the electromagnetic field system $(\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$. Therefore the electromagnetic field must have *four* unconstrained, purely *dynamical* field degrees of freedom. To cleanly separate the nondynamical $\nabla \cdot \mathbf{E}(\mathbf{r}, t)$ and $\nabla \cdot \mathbf{B}(\mathbf{r}, t)$ from the purely dynamical part of $(\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$, a *hypothetical coordinate system* in which one of the three components of the electric field $\mathbf{E}(\mathbf{r}, t)$ *is* just $\nabla \cdot \mathbf{E}(\mathbf{r}, t)$ and also in which one of the three components of the magnetic field $\mathbf{B}(\mathbf{r}, t)$ *is* just $\nabla \cdot \mathbf{B}(\mathbf{r}, t)$ would be very convenient. In such a hypothetical coordinate system, the set of the remaining two components of $\mathbf{E}(\mathbf{r}, t)$, together with the remaining two components of $\mathbf{B}(\mathbf{r}, t)$, would comprise the four unconstrained, purely *dynamical* electromagnetic field degrees of freedom. It turns out to be technically most straightforward to precede attempted *implementation* of this sort of idea by *spatial Fourier transformation* of the electromagnetic field $(\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$ and its four-current source $(\rho(\mathbf{r}, t), \mathbf{j}(\mathbf{r}, t)/c)$. We define,

$$(\mathbf{E}(\mathbf{k}, t), \mathbf{B}(\mathbf{k}, t)) \stackrel{\text{def}}{=} (2\pi)^{-\frac{3}{2}} \int d^3\mathbf{r} \, e^{-i\mathbf{k}\cdot\mathbf{r}} (\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)), \tag{3a}$$

which is the "unitary" Fourier transform. Also,

$$(\rho(\mathbf{k}, t), \mathbf{j}(\mathbf{k}, t)/c) \stackrel{\text{def}}{=} (2\pi)^{-\frac{3}{2}} \int d^3\mathbf{r} \, e^{-i\mathbf{k}\cdot\mathbf{r}} (\rho(\mathbf{r}, t), \mathbf{j}(\mathbf{r}, t)/c). \tag{3b}$$

In the penultimate section of this paper, we shall essay the trickier task of attempting to reveal the dynamical time-dependent Schrödinger equation character of Maxwell's source-free equations directly in configuration representation, without resort to this spatial Fourier transformation. It is worth remarking at this stage that since the charge density $\rho(\mathbf{r}, t)$ is a real-valued function, and the *same* is true of all the Cartesian components of $(\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t))$ and $\mathbf{j}(\mathbf{r}, t)$, the corresponding spatial Fourier transforms of all these entities have the property that their complex conjugation is equivalent to *reversing the sign* of their *Fourier vector argument* $\mathbf{k}$. Some key manipulations that are carried out further on rely heavily on this technical point. The spatial Fourier transformation of $\nabla \cdot \mathbf{E}(\mathbf{r}, t)$ comes out be $i\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, t)$, which, in a *coordinate system* that has $\mathbf{u}_L(\mathbf{k}) \stackrel{\text{def}}{=} \mathbf{k}/|\mathbf{k}|$ as *one* of its three orthogonal unit vectors, is equal to $i|\mathbf{k}|$ times the $\mathbf{u}_L(\mathbf{k})$-component of $\mathbf{E}(\mathbf{k}, t)$, which we denote as $E_L(\mathbf{k}, t)$. Coulomb's law thus obviously implies the nondynamical equation,

$$E_L(\mathbf{k}, t) = -i\rho(\mathbf{k}, t)/|\mathbf{k}|, \tag{4a}$$

and, analogously, Gauss' law implies the nondynamical equation

$$B_L(\mathbf{k}, t) = 0. \tag{4b}$$

We can therefore be quite confident that $E_L(\mathbf{k}, t)$ and $B_L(\mathbf{k}, t)$ exhaust the nondynamical components of $\mathbf{E}(\mathbf{k}, t)$ and $\mathbf{B}(\mathbf{k}, t)$ respectively, and that the *remaining* two components of each of these two fields will be purely dynamical, i.e., free of any nondynamical constraint. But to demonstrate this in detail, we must explicitly display the *remaining two* mutually orthogonal unit vectors, which are *each* as well orthogonal to $\mathbf{u}_L(\mathbf{k})$, and then work out the consequences of the Maxwell equations for the remaining two components of both $\mathbf{E}(\mathbf{k}, t)$ and $\mathbf{B}(\mathbf{k}, t)$) in *that* coordinate system, in order to verify that *purely dynamical equations of motion* which involve *only* these four components result. In the particular case that the four-current source $(\rho(\mathbf{k}, t), \mathbf{j}(\mathbf{k}, t)/c)$

vanishes, we *also* need to demonstrate that the now *homogeneous* equations of motion obtained for these four unconstrained *dynamical components* of $(\mathbf{E}(\mathbf{k}, t), \mathbf{B}(\mathbf{k}, t))$ are *equivalent* to the schematic Schrödinger Eq. (1) with $m = 0$—note as well that in this source-free case the two *nondynamical components* $E_L(\mathbf{k}, t)$ and $B_L(\mathbf{k}, t)$ of $(\mathbf{E}(\mathbf{k}, t), \mathbf{B}(\mathbf{k}, t))$ *vanish identically*, as is seen from Eqs. (4).

In order to obtain two mutually orthogonal unit vectors which are both *also* orthogonal to the unit vector $\mathbf{u}_L(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|$, we display $\mathbf{u}_L(\mathbf{k})$ in *Cartesian coordinates*: it is simply the well-known unit vector in the *radial* direction that the Fourier vector argument $\mathbf{k}$ points toward, expressed in terms of that vector's spherical polar angles $\phi_{\mathbf{k}}$ and $\theta_{\mathbf{k}}$,

$$\mathbf{u}_L(\mathbf{k}) = (\cos \phi_{\mathbf{k}} \sin \theta_{\mathbf{k}}, \sin \phi_{\mathbf{k}} \sin \theta_{\mathbf{k}}, \cos \theta_{\mathbf{k}}). \tag{5a}$$

Now because $\mathbf{u}_L(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|$, $\mathbf{u}_L(-\mathbf{k}) = -\mathbf{u}_L(\mathbf{k})$, i.e., $\mathbf{u}_L(\mathbf{k})$ has the same *odd parity* that $\mathbf{k}$ has. Therefore the parity flip mapping $\mathbf{k} \to -\mathbf{k}$ corresponds to the polar angular mapping $\theta_{\mathbf{k}} \to \theta_{\mathbf{k}} + \pi$, because this sends $\sin \theta_{\mathbf{k}} \to -\sin \theta_{\mathbf{k}}$ and $\cos \theta_{\mathbf{k}} \to -\cos \theta_{\mathbf{k}}$, thus sending, from Eq. (5a), $\mathbf{u}_L(\mathbf{k}) \to -\mathbf{u}_L(\mathbf{k})$. However, if we instead choose to carry out the polar angular mapping $\theta_{\mathbf{k}} \to \theta_{\mathbf{k}} + \pi/2$, then $\sin \theta_{\mathbf{k}} \to \cos \theta_{\mathbf{k}}$, $\cos \theta_{\mathbf{k}} \to -\sin \theta_{\mathbf{k}}$, and $\mathbf{u}_L(\mathbf{k}) \to \mathbf{u}_1(\mathbf{k})$, where,

$$\mathbf{u}_1(\mathbf{k}) \stackrel{\text{def}}{=} (\cos \phi_{\mathbf{k}} \cos \theta_{\mathbf{k}}, \sin \phi_{\mathbf{k}} \cos \theta_{\mathbf{k}}, -\sin \theta_{\mathbf{k}}), \tag{5b}$$

is readily checked to be a *unit vector* that is *orthogonal* to $\mathbf{u}_L(\mathbf{k})$. The parity flip angular mapping $\theta_{\mathbf{k}} \to \theta_{\mathbf{k}} + \pi$ reveals that $\mathbf{u}_1(\mathbf{k})$ is *also* of *odd* parity. With the mutually orthogonal unit vectors $\mathbf{u}_L(\mathbf{k})$ and $\mathbf{u}_1(\mathbf{k})$ in hand, we can now readily construct a *third* unit vector $\mathbf{u}_2(\mathbf{k})$ which is orthogonal to *both* of these,

$$\mathbf{u}_2(\mathbf{k}) \stackrel{\text{def}}{=} \mathbf{u}_L(\mathbf{k}) \times \mathbf{u}_1(\mathbf{k}) = (-\sin \phi_{\mathbf{k}}, \cos \phi_{\mathbf{k}}, 0). \tag{5c}$$

It is immediately seen that $\mathbf{u}_2(\mathbf{k})$ is of *even* parity. By using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, or, alternatively, the spherical polar angular representations given by Eqs. (5a)–(5c), it is readily checked that $\mathbf{u}_1(\mathbf{k})$, $\mathbf{u}_2(\mathbf{k})$ and $\mathbf{u}_L(\mathbf{k})$ comprise a "right-handed" orthonormal local vector triad, i.e.,

$$\mathbf{u}_1(\mathbf{k}) \times \mathbf{u}_2(\mathbf{k}) = \mathbf{u}_L(\mathbf{k}), \ \ \mathbf{u}_2(\mathbf{k}) \times \mathbf{u}_L(\mathbf{k}) = \mathbf{u}_1(\mathbf{k}) \text{ and } \mathbf{u}_L(\mathbf{k}) \times \mathbf{u}_1(\mathbf{k}) = \mathbf{u}_2(\mathbf{k}). \tag{5d}$$

Turning now to the implications of Maxwell's equations in this coordinate system, we have already noted that the Coulomb and Gauss laws imply the two nondynamical Eqs. (4a) and (4b). Upon spatial Fourier transformation, Faraday's law, Eq. (2b), becomes,

$$i\mathbf{k} \times \mathbf{E}(\mathbf{k}, t) = -\dot{\mathbf{B}}(\mathbf{k}, t)/c. \tag{6a}$$

Noting that $\mathbf{k} = |\mathbf{k}|\mathbf{u}_L(\mathbf{k})$, and that,

$$\mathbf{E}(\mathbf{k}, t) = E_1(\mathbf{k}, t)\mathbf{u}_1(\mathbf{k}) + E_2(\mathbf{k}, t)\mathbf{u}_2(\mathbf{k}) + E_L(\mathbf{k}, t)\mathbf{u}_L(\mathbf{k}),$$

where,

$$E_1(\mathbf{k}, t) \stackrel{\text{def}}{=} \mathbf{u}_1(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}, t), \ \ E_2(\mathbf{k}, t) \stackrel{\text{def}}{=} \mathbf{u}_2(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}, t) \text{ and } E_L(\mathbf{k}, t) \stackrel{\text{def}}{=} \mathbf{u}_L(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}, t),$$

and analogously for $\dot{\mathbf{B}}(\mathbf{k}, t)$, for which the Gauss law result embodied by Eq. (4b) *already* permits us to conclude that $\dot{B}_L(\mathbf{k}, t) = 0$, we apply Eq. (5d) to the left-hand side of Eq. (6a), and thereby obtain the two *additional* equations,

$$i\dot{B}_1(\mathbf{k}, t) = -|c\mathbf{k}|E_2(\mathbf{k}, t), \tag{6b}$$

and,

$$i\dot{B}_2(\mathbf{k}, t) = |c\mathbf{k}|E_1(\mathbf{k}, t). \tag{6c}$$

Before we turn to Maxwell's law, Eq. (2d), it is convenient to treat the current conservation condition, Eq. (2e), which is a *constraint on the four-current source* that follows from Maxwell's and Coulomb's laws. Upon spatial Fourier transformation, Eq. (2e) becomes,

$$i\mathbf{k} \cdot \mathbf{j}(\mathbf{k}, t) = -\dot{\rho}(\mathbf{k}, t), \tag{6d}$$

which immediately yields the longitudinal source current component in terms of the rate of change of the charge density,

$$j_L(\mathbf{k}, t) = i\dot{\rho}(\mathbf{k}, t)/|\mathbf{k}|. \tag{6e}$$

Upon spatial Fourier transformation, Maxwell's law, Eq. (2d), becomes,

$$i\mathbf{k} \times \mathbf{B}(\mathbf{k}, t) = (\mathbf{j}(\mathbf{k}, t) + \dot{\mathbf{E}}(\mathbf{k}, t))/c. \tag{6f}$$

The left-hand side of Eq. (6f) has a vanishing component in the $\mathbf{u}_L(\mathbf{k})$-direction, and the joint consequence of that and the Coulomb law result embodied by Eq. (4a) for its right-hand side is simply the constraint on the longitudinal source current component that is embodied by eq. (6e). More interesting are the two equations that follow from the components of Eq. (6f) in the $\mathbf{u}_1(\mathbf{k})$ and $\mathbf{u}_2(\mathbf{k})$ directions—these bear a strong resemblance to the Eqs. (6b) and (6c) which follow from Faraday's law,

$$i\dot{E}_1(\mathbf{k}, t) = |c\mathbf{k}|B_2(\mathbf{k}, t) - ij_1(\mathbf{k}, t), \tag{6g}$$

and,

$$i\dot{E}_2(\mathbf{k}, t) = -|c\mathbf{k}|B_1(\mathbf{k}, t) - ij_2(\mathbf{k}, t). \tag{6h}$$

Aside from the purely *source* constraint requirement of Eq. (6e) and the reconfirmation that $\dot{B}_L(\mathbf{k}, t)$ must vanish, which is already a consequence of Eq. (4b) (which is itself the result of the Gauss law), the Faraday and Maxwell laws have yielded four *dynamical* equations of motion, namely Eqs. (6b), (6c), (6g) and (6h), which involve *only* the four *transverse* field components $E_1(\mathbf{k}, t)$, $B_2(\mathbf{k}, t)$, $E_2(\mathbf{k}, t)$ and $B_1(\mathbf{k}, t)$. Absolutely *no* nondynamical equations of constraint for *any* of these four transverse field components have eventuated from any of the Maxwell equations. It is therefore clear that the six field degrees of freedom of $(\mathbf{E}(\mathbf{k}, t), \mathbf{B}(\mathbf{k}, t))$ have now been *successfully partitioned* into *four* unconstrained, purely dynamical *transverse* field degrees of freedom and *two* purely nondynamical *longitudinal* field degrees of freedom $E_L(\mathbf{k}, t)$ and $B_L(\mathbf{k}, t)$, whose values are actually *given* by the simple nondynamical constraints of Eqs. (4a) and (4b). In *addition*, it has, of course, transpired that the four-current source $(\rho(\mathbf{k}, t), \mathbf{j}(\mathbf{k}, t)/c)$ *cannot* be chosen *arbitrarily*, but is subject to the *source constraint* given by Eq. (6e).

Linear algebraic decoupling of the four transverse dynamical fields

Eq. (6g) for the dynamical transverse fields $E_1(\mathbf{k}, t)$ and $B_2(\mathbf{k}, t)$ is clearly coupled to Eq. (6c), and likewise Eq. (6h) for the dynamical transverse fields $E_2(\mathbf{k}, t)$ and $B_1(\mathbf{k}, t)$ is clearly coupled to Eq. (6b). Some investigators may be tempted to decouple these equations by taking *second* time derivatives, but such an approach is *entirely unnecessary* and involves a risk of *introducing* extraneous solutions that *don't* actually apply to these equations—indeed taking an unwarranted second time derivative is precisely how the unphysical, unbounded-below negative energy spectrum was inadvertently forced into the relativistic Klein-Gordon equation for a nonzero-mass free particle. Eqs. (6g) and (6c) are easily decoupled by the straightforward expedient of taking their sum and difference, and the same applies to Eqs. (6h) and (6b). Adding Eq. (6c) to Eq. (6g) yields,

$$i\partial(E_1(\mathbf{k}, t) + B_2(\mathbf{k}, t))/\partial t = |c\mathbf{k}|(E_1(\mathbf{k}, t) + B_2(\mathbf{k}, t)) - ij_1(\mathbf{k}, t), \tag{7a}$$

while subtracting Eq. (6b) from Eq. (6h) yields,

$$i\partial(E_2(\mathbf{k}, t) - B_1(\mathbf{k}, t))/\partial t = |c\mathbf{k}|(E_2(\mathbf{k}, t) - B_1(\mathbf{k}, t)) - ij_2(\mathbf{k}, t). \tag{7b}$$

One can also subtract Eq. (6c) from Eq. (6g) to obtain,

$$i\partial(E_1(\mathbf{k}, t) - B_2(\mathbf{k}, t))/\partial t = -|c\mathbf{k}|(E_1(\mathbf{k}, t) - B_2(\mathbf{k}, t)) - ij_1(\mathbf{k}, t), \tag{7c}$$

and add Eq. (6b) to Eq. (6h) to obtain,

$$i\partial(E_2(\mathbf{k},t) + B_1(\mathbf{k},t))/\partial t = -|c\mathbf{k}|(E_2(\mathbf{k},t) + B_1(\mathbf{k},t)) - ij_2(\mathbf{k},t). \tag{7d}$$

Now it turns out that Eq. (7c) is *not independent* of Eq. (7a); in fact, Eq. (7c) is actually *equivalent* to Eq. (7a)! The reason for this is somewhat involved; it is related to the previously mentioned fact that for any Cartesian component of $\mathbf{E}(\mathbf{k},t)$, $\mathbf{B}(\mathbf{k},t)$, or $\mathbf{j}(\mathbf{k},t)$, complex conjugation is equivalent to changing the *sign* of the Fourier vector argument $\mathbf{k}$. Making matters a bit more complicated is the fact that $E_1(\mathbf{k},t)$ and $E_2(\mathbf{k},t)$ are *not Cartesian components* of $\mathbf{E}(\mathbf{k},t)$ because $E_1(\mathbf{k},t) = \mathbf{E}(\mathbf{k},t) \cdot \mathbf{u}_1(\mathbf{k})$ and $E_2(\mathbf{k},t) = \mathbf{E}(\mathbf{k},t) \cdot \mathbf{u}_2(\mathbf{k})$. Because $\mathbf{u}_1(\mathbf{k})$ is of *odd parity* in its argument $\mathbf{k}$, complex conjugation of $E_1(\mathbf{k},t)$ is equivalent to changing *both* the sign of its *argument* $\mathbf{k}$ *and* its *overall* sign! However, because $\mathbf{u}_2(\mathbf{k})$ is of *even parity* in its argument $\mathbf{k}$, complex conjugation of $E_2(\mathbf{k},t)$ is equivalent to merely changing the sign of its *argument* $\mathbf{k}$. Exactly the same distinction with regard to complex conjugation holds between $B_1(\mathbf{k},t)$ and $B_2(\mathbf{k},t)$, and as well between $j_1(\mathbf{k},t)$ and $j_2(\mathbf{k},t)$. Now if we take the complex conjugate of both sides of Eq. (7c) and apply what we have just learned, the result is,

$$-i\partial(-E_1(-\mathbf{k},t) - B_2(-\mathbf{k},t))/\partial t = -|c\mathbf{k}|(-E_1(-\mathbf{k},t) - B_2(-\mathbf{k},t)) - ij_1(-\mathbf{k},t). \tag{7e}$$

Upon combining signs in Eq. (7e), we find that it resembles Eq. (7a) in every respect, except for the fact that all occurrences of the Fourier vector *argument* $\mathbf{k}$ have effectively had their sign reversed. However, because Eq. (7e) is supposed to hold irrespective of *what* value is assumed by $\mathbf{k}$, we are free to make the simple one-to-one formal transformation $\mathbf{k} \to -\mathbf{k}$ which turns Eq. (7e) into Eq. (7a). Furthermore, if we take the complex conjugate of both sides of Eq. (7d) and apply to it what we have learned above, the result is,

$$-i\partial(E_2(-\mathbf{k},t) - B_1(-\mathbf{k},t))/\partial t = -|c\mathbf{k}|(E_2(-\mathbf{k},t) - B_1(-\mathbf{k},t)) + ij_2(-\mathbf{k},t). \tag{7f}$$

Upon reversing the sign of both sides of Eq. (7f), we find that it resembles Eq. (7b) in every respect, except for the fact that all occurrences of the Fourier vector *argument* $\mathbf{k}$ have effectively had their sign reversed. But we are, of course, again justified in making the simple one-to-one formal transformation $\mathbf{k} \to -\mathbf{k}$ which turns Eq. (7f) into Eq. (7b).

We have thus succeeded in replacing the *four coupled* equations of motion for the dynamical transverse fields by *two* nontrivially complex-valued and *fully decoupled* such equations, namely Eqs. (7a) and (7b). If we multiply both of these equations through by $\hbar$, and then set both of the transverse source currents $j_1(\mathbf{k},t)$ and $j_2(\mathbf{k},t)$ to zero, Eqs. (7a) and (7b) assume precisely the schematic form of Eq. (1) with $m = 0$, i.e., they are of the form of time-dependent Schrödinger equations for a solitary relativistic massless free particle. The fact that there are *two* such equations suggests, in light of the detailed electromagnetic field composition of each of their two apparent "wave functions", that they describe the amplitudes for two *linear polarization* states of the solitary massless particle. We shall now further investigate this interesting source-free limit of Maxwell's equations.

The Schrödinger character of the source-free Maxwell equations

When the four-current source $(\rho(\mathbf{k},t), \mathbf{j}(\mathbf{k},t)/c)$ vanishes altogether, Eqs. (4) show that the two nondynamical longitudinal electromagnetic field elements $E_L(\mathbf{k},t)$ and $B_L(\mathbf{k},t)$ vanish identically as well. The *only* physics that remains is *purely dynamical and transverse*, and is fully described by the two relativistic, massless, solitary free-particle Schrödinger-style equations,

$$i\hbar\partial(E_1(\mathbf{k},t) + B_2(\mathbf{k},t))/\partial t = |c\hbar\mathbf{k}|(E_1(\mathbf{k},t) + B_2(\mathbf{k},t)), \tag{8a}$$

and,

$$i\hbar\partial(E_2(\mathbf{k},t) - B_1(\mathbf{k},t))/\partial t = |c\hbar\mathbf{k}|(E_2(\mathbf{k},t) - B_1(\mathbf{k},t)), \tag{8b}$$

which follow from Eqs. (7a) and (7b) in the source-free situation. The detailed structure of the two putative "wave functions" in terms of the transverse electromagnetic field components strongly suggests that they represent the amplitudes for the two possible transverse linear polarization states of the solitary, massless, free electromagnetic field particle. There is a technical snag, however, which bars such an interpretation from being immediately made: the "wave functions" that appear in the Schrödinger equations of Eqs. (8) are sums

and differences of *transverse electromagnetic field components*, which have the character of *energy-density amplitudes*, whereas *true solitary-particle wave functions* have the character of *probability-density amplitudes*. To get a feeling for just *what* energy is represented by the two complex-valued "wave functions" of Eqs. (8), we wish to integrate the sum of their absolute squares over all of the Fourier vector-variable **k**-space. We begin by integrating over just the absolute square of the "wave function" of Eq. (8a),

$$\int d^3\mathbf{k}(|E_1 + B_2|^2 = \int d^3\mathbf{k}(|E_1|^2 + |B_2|^2) + \int d^3\mathbf{k}(E_1^* B_2 + B_2^* E_1), \tag{9a}$$

where we have temporarily suppressed writing out the *arguments* of the transverse field components to save space. However, bearing in mind the discussion between Eqs. (7d) and (7e), we have that, $(E_1(\mathbf{k}, t))^* = -E_1(-\mathbf{k}, t)$ and $B_2(\mathbf{k}, t) = (B_2(-\mathbf{k}, t))^*$, from which we readily deduce that $\int d^3\mathbf{k}\, E_1^* B_2 = -\int d^3\mathbf{k}\, B_2^* E_1$, and therefore that the second integral on the right-hand side of Eq. (9a) vanishes. Analogous arguments show that when one integrates over the absolute square of the "wave function" of Eq. (8b), the integration over the corresponding two cross terms vanishes as well. Therefore, the result of integrating over the *sum* of the absolute squares of these two "wave functions" is,

$$\int d^3\mathbf{k}(|E_1 + B_2|^2 + |E_2 - B_1|^2) = \int d^3\mathbf{k}(|E_1|^2 + |E_2|^2 + |B_1|^2 + |B_2|^2). \tag{9b}$$

Now the integral on the right-hand side of Eq. (9b) is equal to twice the total energy present in the transverse electromagnetic field components, which is, of course twice the total electromagnetic field energy, since the longitudinal components of the electromagnetic field vanish identically in the source-free case. Therefore the energy represented by the two complex-valued "wave functions" of Eqs. (8) is twice the total electromagnetic field energy.

Now let us suppose that the *sole source* of the electromagnetic field energy which is present is a *solitary* photon. That photon's energy density in **k**-space (which is effectively momentum-space, since $\hbar\mathbf{k}$ *is* the photon's momentum) is then equal to *half* of the sum of the absolute squares of the two complex-valued "wave functions" of Eq. (8), as we have learned from Eq. (9b). Now insofar as the solitary photon *has* its Fourier vector variable equal to **k**, i.e., insofar as it *has* momentum $\hbar\mathbf{k}$, it clearly *also has energy* $|c\hbar\mathbf{k}|$. Therefore, we can *convert* our photon's *energy density* in **k**-space—which is *half* of the sum of the squares of the two complex-valued "wave functions" of Eqs. (8)—to its *probability density* in **k**-space by simply dividing that energy density by $|c\hbar\mathbf{k}|$. This implies that we can *convert* each of the two transverse electromagnetic field "wave functions" of Eqs. (8) to a proper *solitary photon* wave function (whose absolute square yields a *probability density*) by dividing it by $(2|c\hbar\mathbf{k}|)^{\frac{1}{2}}$. It is clear that both of these proper *solitary photon wave functions* will satisfy the *very same Schrödinger equation* that the two transverse electromagnetic field component "wave functions" of Eqs. (8) satisfy: the factor of $(2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}$ *doesn't interfere* with the validity of that time-dependent Schrödinger equation. Therefore, when only a solitary photon is present, its two linear polarization wave function components (complex-valued probability amplitudes) are given in terms of the corresponding transverse electromagnetic field components by,

$$\langle \mathbf{k}|\psi_1(t)\rangle = (2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(E_1(\mathbf{k}, t) + B_2(\mathbf{k}, t)), \tag{10a}$$

and,

$$\langle \mathbf{k}|\psi_2(t)\rangle = (2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(E_2(\mathbf{k}, t) - B_1(\mathbf{k}, t)). \tag{10b}$$

It is convenient to as well explicitly write down the parity-reversed complex conjugates of these solitary-photon linear polarization wave function components,

$$\langle \psi_1(t)| - \mathbf{k}\rangle = -(2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(E_1(\mathbf{k}, t) - B_2(\mathbf{k}, t)), \tag{10c}$$

and,

$$\langle \psi_2(t)| - \mathbf{k}\rangle = (2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(E_2(\mathbf{k}, t) + B_1(\mathbf{k}, t)), \tag{10d}$$

because, with these in hand, the relationships of the solitary-photon linear polarization wave function components to the transverse electromagnetic field components can be *inverted*,

$$E_1(\mathbf{k}, t) = (|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_1(t)\rangle - \langle\psi_1(t)| - \mathbf{k}\rangle), \tag{10e}$$

$$E_2(\mathbf{k}, t) = (|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_2(t)\rangle + \langle\psi_2(t)| - \mathbf{k}\rangle), \tag{10f}$$

$$B_1(\mathbf{k}, t) = -(|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_2(t)\rangle - \langle\psi_2(t)| - \mathbf{k}\rangle), \tag{10g}$$

$$B_2(\mathbf{k}, t) = (|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_1(t)\rangle + \langle\psi_1(t)| - \mathbf{k}\rangle). \tag{10h}$$

It is worth explicitly reiterating that the two complex-valued linear polarization components of the solitary-photon wave function satisfy the massless case of the relativistic free-particle time-dependent Schrödinger equation that is given by Eq. (1),

$$i\hbar\partial(\langle\mathbf{k}|\psi_1(t)\rangle)/\partial t = |c\hbar\mathbf{k}|\langle\mathbf{k}|\psi_1(t)\rangle, \tag{11a}$$

$$i\hbar\partial(\langle\mathbf{k}|\psi_2(t)\rangle)/\partial t = |c\hbar\mathbf{k}|\langle\mathbf{k}|\psi_2(t)\rangle. \tag{11b}$$

Finally, it is worthwhile to relate the solitary free-photon's complex-valued wave function to the components of the electromagnetic *four-vector potential* to which it corresponds. The electromagnetic four-vector potential does have a gauge ambiguity issue which unfortunately is not fully resolved by the relativistically invariant Lorentz condition—suppression of the ensuing timelike and longitudinal "ghost radiation" [7, 8] requires a further stipulation: probably the most intuitively appealing is to require the scalar potential to be *uniquely determined*, in strictly *homogeneous* and *causal* fashion, by the charge density, which is, after all, its notional *source* after imposition of the Lorentz condition. This produces results that are no less definite than those of the Coulomb gauge—in fact these two gauges produce *identical* results for all *static* charge densities—but without the Coulomb gauge's disconcerting instantaneous scalar potential response at arbitrarily large distances to charge density *change*. In the present source-free case, both gauges are, in fact, identical to the *radiation gauge* [9], $\nabla \cdot \mathbf{A}(\mathbf{r}, t) = \phi(\mathbf{r}, t) = 0$, which causes the four-vector potential to have *only four transverse dynamical field degrees of freedom*, in complete *agreement* with the situation discussed above for the *electromagnetic field* in this source-free case. The relation of the electromagnetic field to the four-vector potential is, of course, given by,

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t), \tag{12a}$$

and,

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) - \dot{\mathbf{A}}(\mathbf{r}, t)/c, \tag{12b}$$

which, in spatial Fourier transform become,

$$\mathbf{B}(\mathbf{k}, t) = i\mathbf{k} \times \mathbf{A}(\mathbf{k}, t), \tag{12c}$$

and,

$$\mathbf{E}(\mathbf{k}, t) = -i\mathbf{k}\phi(\mathbf{k}, t) - \dot{\mathbf{A}}(\mathbf{k}, t)/c. \tag{12d}$$

Upon applying to it Eq. (5d), Eq. (12c) readily yields the two transverse components of $\mathbf{A}(\mathbf{k}, t)$ in terms of those of $\mathbf{B}(\mathbf{k}, t)$,

$$A_1(\mathbf{k}, t) = -iB_2(\mathbf{k}, t)/|\mathbf{k}|, \tag{13a}$$

$$A_2(\mathbf{k}, t) = iB_1(\mathbf{k}, t)/|\mathbf{k}|, \tag{13b}$$

and Eq. (12d) *immediately* yields the two transverse components of $\dot{\mathbf{A}}(\mathbf{k}, t)$ in terms of those of $\mathbf{E}(\mathbf{k}, t)$,

$$\dot{A}_1(\mathbf{k}, t) = -cE_1(\mathbf{k}, t), \tag{13c}$$

$$\dot{A}_2(\mathbf{k}, t) = -cE_2(\mathbf{k}, t), \tag{13d}$$

Upon putting Eqs. (10e) through (10h) into Eqs. (13a) through (13d) above, we obtain,

$$A_1(\mathbf{k}, t) = -i(|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_1(t)\rangle + \langle\psi_1(t)| - \mathbf{k}\rangle)/|\mathbf{k}|, \tag{14a}$$

$$A_2(\mathbf{k}, t) = -i(|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_2(t)\rangle - \langle\psi_2(t)| - \mathbf{k}\rangle)/|\mathbf{k}|, \tag{14b}$$

$$\dot{A}_1(\mathbf{k}, t) = -c(|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_1(t)\rangle - \langle\psi_1(t)| - \mathbf{k}\rangle), \tag{14c}$$

$$\dot{A}_2(\mathbf{k}, t) = -c(|c\hbar\mathbf{k}|/2)^{+\frac{1}{2}}(\langle\mathbf{k}|\psi_2(t)\rangle + \langle\psi_2(t)| - \mathbf{k}\rangle). \tag{14d}$$

Eqs. (14a) through (14d) can now be inverted,

$$\langle\mathbf{k}|\psi_1(t)\rangle = (2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(i|\mathbf{k}|A_1(\mathbf{k}, t) - \dot{A}_1(\mathbf{k}, t)/c), \tag{15a}$$

$$\langle\mathbf{k}|\psi_2(t)\rangle = (2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(i|\mathbf{k}|A_2(\mathbf{k}, t) - \dot{A}_2(\mathbf{k}, t)/c), \tag{15b}$$

$$\langle\psi_1(t)| - \mathbf{k}\rangle = (2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(i|\mathbf{k}|A_1(\mathbf{k}, t) + \dot{A}_1(\mathbf{k}, t)/c), \tag{15c}$$

$$\langle\psi_2(t)| - \mathbf{k}\rangle = -(2|c\hbar\mathbf{k}|)^{-\frac{1}{2}}(i|\mathbf{k}|A_2(\mathbf{k}, t) + \dot{A}_2(\mathbf{k}, t)/c). \tag{15d}$$

From Eqs. (15a) and (15b) it is apparent that the correct Schrödinger equation quantization of the solitary free photon requires *not only* the two transverse components of $\mathbf{A}(\mathbf{k}, t)$, but *as well* the two transverse components of $\dot{\mathbf{A}}(\mathbf{k}, t)$. The two linear polarization state wave function components, $\langle\mathbf{k}|\psi_1(t)\rangle$ and $\langle\mathbf{k}|\psi_2(t)\rangle$, are *each* ineluctably complex-valued objects in a way that is thoroughly *nonsuperficial*: it requires *two* "classical" field degrees of freedom, such as *both* $E_1(\mathbf{k}, t)$ and $B_2(\mathbf{k}, t)$, or *both* $A_1(\mathbf{k}, t)$ and $\dot{A}_1(\mathbf{k}, t)$, to comprise *one* such deeply complex-valued quantum wave function component. Of course this bodes well for the *next* level of quantization, wherein our Schrödinger equation wave function components are themselves promoted to become operators which have prescribed commutation relations with their own Hermitian conjugates: this reflects their complex-valued makeup from *independent fields* which are interpreted as being *mutually canonically conjugate*, a status for which the pair $A_1(\mathbf{k}, t)$ and $\dot{A}_1(\mathbf{k}, t)$ are, of course, prime candidates. We see that the automatic solitary photon "first quantization" that is simply part and parcel of the very *nature* of

Maxwell's supposedly "classical" equations also *automatically* has properties which *anticipate* and *facilitate* "second quantization". Once *transverse source currents* are present, the solitary photon Schrödinger equation becomes *inhomogeneous*, i.e., it no longer *is* a Schrödinger equation, as we clearly see from Eqs. (7a) and (7b). The *inhomogeneity* of what, in the source-free case, had been the solitary photon Schrödinger equation, of course bespeaks *the creation and destruction* of such photons. It is quite remarkable, however, just how well-organized the solitary photon wave function is *ab initio* for rising to the challenges of the eventually necessary "second quantization".

Configuration space approach to the Schrödinger character of Maxwell's source-free equations

Having gained insight from dissection of the Maxwell equations in spatially Fourier-transformed formulation, we now try our hand at teasing out the Schrödinger character of their transverse dynamical segment directly in configuration representation. We have learned that "dynamical" and "transverse" are effectively synonyms for Maxwell's equations, so we simply focus on separating the electromagnetic fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, and also the current $\mathbf{j}(\mathbf{r}, t)$, into their physically natural transverse and longitudinal parts. This, of course, requires no action whatsoever for $\mathbf{B}(\mathbf{r}, t)$, which Gauss' law, Eq. (2c), marks a purely transverse. The longitudinal part of $\mathbf{E}(\mathbf{r}, t)$ is the negative of the gradient of the very same scalar potential $\phi(\mathbf{r}, t)$ which describes $\mathbf{E}(\mathbf{r}, t)$ fully in the electrostatic limit, i.e.,

$$\mathbf{E}_L(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t), \tag{16a}$$

where, of course,

$$\phi(\mathbf{r}, t) \stackrel{\text{def}}{=} (4\pi)^{-1} \int d^3\mathbf{r}' \, \rho(\mathbf{r}', t)/|\mathbf{r} - \mathbf{r}'|. \tag{16b}$$

Because of the Green's function identity,

$$\nabla_{\mathbf{r}}^2(1/|\mathbf{r} - \mathbf{r}'|) = -4\pi \, \delta^{(3)} \, (\mathbf{r} - \mathbf{r}'),$$

we have that,

$$\nabla^2\phi(\mathbf{r}, t) = -\rho(\mathbf{r}, t), \tag{16c}$$

and,

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = \nabla \cdot \mathbf{E}_L(\mathbf{r}, t) = \rho(\mathbf{r}, t), \tag{16d}$$

as required by the Coulomb law, Eq. (2a). Eq. (16d) follows from Eq. (16a), Eq. (16c) and,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_L(\mathbf{r}, t) + \mathbf{E}_T(\mathbf{r}, t), \tag{16e}$$

the separation of $\mathbf{E}(\mathbf{r}, t)$ into its longitudinal and transverse parts, where, of course, by definition,

$$\nabla \cdot \mathbf{E}_T(\mathbf{r}, t) = 0 \text{ and } \nabla \times \mathbf{E}_L(\mathbf{r}, t) = \mathbf{0}. \tag{16f}$$

Now the current conservation condition given by Eq. (2e) has a formal structure that is very similar to that of Coulomb's law, and therefore permits an analogous determination of $\mathbf{j}_L(\mathbf{r}, t)$, the longitudinal part of the current $\mathbf{j}(\mathbf{r}, t)$,

$$\mathbf{j}_L(\mathbf{r}, t) = \nabla\dot{\phi}(\mathbf{r}, t), \tag{16g}$$

which, together with Eq. (16c) and the relations for $\mathbf{j}(\mathbf{r}, t)$, $\mathbf{j}_L(\mathbf{r}, t)$ and $\mathbf{j}_T(\mathbf{r}, t)$ which are analogous to Eqs. (16e) and (16f), implies that,

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) = \nabla \cdot \mathbf{j}_L(\mathbf{r}, t) = -\dot{\rho}(\mathbf{r}, t), \tag{16h}$$

as required by the current conservation condition of Eq. (2e).

We are now in a position to reexpress the Maxwell and Faraday laws in terms of *only* the *transverse* fields $\mathbf{E}_T(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, and the transverse current $\mathbf{j}_T(\mathbf{r}, t)$. In particular, Eqs. (16g) and (16a) permit us to deduce that,

$$\mathbf{j}(\mathbf{r}, t) + \dot{\mathbf{E}}(\mathbf{r}, t) = \mathbf{j}_T(\mathbf{r}, t) + \dot{\mathbf{E}}_T(\mathbf{r}, t),$$

which permits the Maxwell law of Eq. (2d) to be rewritten,

$$\nabla \times \mathbf{B} = (\mathbf{j}_T + \dot{\mathbf{E}}_T)/c. \tag{17a}$$

From Eqs. (16a) and (16e), or from Eq. (16f), we can deduce that $\nabla \times \mathbf{E}(\mathbf{r}, t) = \nabla \times \mathbf{E}_T(\mathbf{r}, t)$, which permits the Faraday law of Eq. (2b) to be rewritten,

$$\nabla \times \mathbf{E}_T = -\dot{\mathbf{B}}/c. \tag{17b}$$

This completes the divorce of the dynamical Maxwell and Faraday laws from the purely nondynamical variable $\mathbf{E}_L(\mathbf{r}, t)$, whose value is given in detail by Eqs. (16a) and (16b). We can now *combine* the purely dynamical and transverse versions of the Maxwell and Faraday laws which are given by Eqs. (17a) and (17b) into a *single* complex-valued equation by adding Eq. (17b) to Eq. (17a) multiplied through by the imaginary unit i. This is readily seen to produce the result,

$$i\partial(\mathbf{E}_T(\mathbf{r}, t) + i\mathbf{B}(\mathbf{r}, t))/\partial t = c\nabla \times (\mathbf{E}_T(\mathbf{r}, t) + i\mathbf{B}(\mathbf{r}, t)) - i\mathbf{j}_T(\mathbf{r}, t). \tag{17c}$$

If we instead *subtract* Eq. (17b) from Eq. (17a) multiplied through by the imaginary unit i, we obtain,

$$i\partial(\mathbf{E}_T(\mathbf{r}, t) - i\mathbf{B}(\mathbf{r}, t))/\partial t = -c\nabla \times (\mathbf{E}_T(\mathbf{r}, t) - i\mathbf{B}(\mathbf{r}, t)) - i\mathbf{j}_T(\mathbf{r}, t), \tag{17d}$$

The difference between Eqs. (17c) and (17d) above bears a strong similarity to the difference between Eqs. (7a) and (7c) or that between Eqs. (7b) and (7d). As in those instances, it is readily shown that Eqs. (17c) and (17d) above are, in fact, *equivalent*. We kept Eqs. (7a) and (7b) *in preference* to Eqs. (7c) and (7d) because, when the transverse current source terms were dropped, the former two evidenced a manifestly *nonnegative* Hamiltonian operator, which is *physically appropriate* for a solitary free particle. Here, unfortunately, *neither* of the candidate Hamiltonian operators $(c\hbar\nabla\times)$ and $(-c\hbar\nabla\times)$ turns out to be *nonnegative*, and *both* have the *further peculiarity* of being *odd parity operators*, which is *unacceptable* for a solitary free photon Hamiltonian operator. We shall, in fact, need to *meld* a *piece* from *each* of these two Hamiltonian operators to one another in such a way as *produces in configuration representation* the *physically appropriate* Hamiltonian operator for the solitary free photon which Eqs. (7a) and (7b), sans their transverse current source terms, *have already delivered to us in Fourier vector variable representation as* $|c\hbar\mathbf{k}|$. But before we discuss this melding of pieces from the two Hamiltonian operators, we need to look at the details of the passage to the source-free situation, and we *also* need to pass from the complex-valued transverse-vector *electromagnetic field strengths* that are found in Eqs. (17c) and (17d) to complex-valued transverse-vector *probability density amplitudes*, since it is *these* which befit the quantum description of a solitary particle (the transverse-vector character of these probability density amplitudes reflects the the fact that the particle has *only two* polarization degrees of freedom). In the source-free case that $\mathbf{j}(\mathbf{r}, t) = \mathbf{0}$ and $\rho(\mathbf{r}, t) = 0$, it follows that $\phi(\mathbf{r}, t) = 0$, and therefore also $\mathbf{E}_L(\mathbf{r}, t) = \mathbf{0}$, which, in turn, causes $\mathbf{E}_T(\mathbf{r}, t)$ to be equal to $\mathbf{E}(\mathbf{r}, t)$. Thus, after they are multiplied through by $\hbar$, Eqs. (17c) and (17d) become the homogeneous time-dependent Schrödinger-like equations,

$$i\hbar\partial(\mathbf{E}(\mathbf{r}, t) + i\mathbf{B}(\mathbf{r}, t))/\partial t = c\hbar\nabla \times (\mathbf{E}(\mathbf{r}, t) + i\mathbf{B}(\mathbf{r}, t)), \tag{17e}$$

and,

$$i\hbar\partial(\mathbf{E}(\mathbf{r}, t) - i\mathbf{B}(\mathbf{r}, t))/\partial t = -c\hbar\nabla \times (\mathbf{E}(\mathbf{r}, t) - i\mathbf{B}(\mathbf{r}, t)), \tag{17f}$$

whose respective "wave functions" $(\mathbf{E}(\mathbf{r}, t) \pm i\mathbf{B}(\mathbf{r}, t))$ are complex-valued vectors that are *purely transverse*, since, in the source-free case, $\nabla \cdot \mathbf{E} = 0$ from the Coulomb law, as well as $\nabla \cdot \mathbf{B} = 0$ from the Gauss law. Their respective "Hamiltonian operators" $(\pm c\hbar\nabla\times)$ are *Hermitian* on such a "wave function" space, since it is readily shown, using integration by parts that,

$$\int d^3\mathbf{r}\,[(\mathbf{a}(\mathbf{r},t))^* \cdot (\pm\nabla \times \mathbf{b}(\mathbf{r},t))] = \int d^3\mathbf{r}\,[(\pm\nabla \times \mathbf{a}(\mathbf{r},t))^* \cdot \mathbf{b}(\mathbf{r},t)]. \tag{17g}$$

Because the total purely electromagnetic field energy, which is $\frac{1}{2}\int d^3\mathbf{r}\,(|\mathbf{E}(\mathbf{r},t)|^2 + |\mathbf{B}(\mathbf{r},t)|^2)$, is readily shown to be *independent of time* in the source-free case (indeed this follows from the real and imaginary parts of either of Eqs. (17e) or (17f), in conjunction with Eq. (17g)), we can define *two* candidate solitary-particle complex-valued vector wave functions which are *properly normalized to unity*, namely,

$$\boldsymbol{\Psi}(\mathbf{r},t) \stackrel{\text{def}}{=} (\mathbf{E}(\mathbf{r},t) + i\mathbf{B}(\mathbf{r},t))/(\int d^3\mathbf{r}'\,(|\mathbf{E}(\mathbf{r}',t)|^2 + |\mathbf{B}(\mathbf{r}',t)|^2))^{\frac{1}{2}}, \tag{18a}$$

and *also* $(\boldsymbol{\Psi}(\mathbf{r},t))^*$, the *complex conjugate* of this $\boldsymbol{\Psi}(\mathbf{r},t)$. It is clear from Eq. (18a) that,

$$\int d^3\mathbf{r}\,[(\boldsymbol{\Psi}(\mathbf{r},t))^* \cdot \boldsymbol{\Psi}(\mathbf{r},t)] = 1, \tag{18b}$$

a property which $(\boldsymbol{\Psi}(\mathbf{r},t))^*$ obviously shares,

$$\int d^3\mathbf{r}\,[\boldsymbol{\Psi}(\mathbf{r},t) \cdot (\boldsymbol{\Psi}(\mathbf{r},t))^*] = 1. \tag{18c}$$

We further note that, because in the source-free case $\nabla \cdot \mathbf{E}(\mathbf{r},t) = 0$ and $\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0$, both $\boldsymbol{\Psi}(\mathbf{r},t)$ and $(\boldsymbol{\Psi}(\mathbf{r},t))^*$ are *strictly transverse*, i.e.,

$$\nabla \cdot \boldsymbol{\Psi}(\mathbf{r},t) = 0 \text{ and } \nabla \cdot (\boldsymbol{\Psi}(\mathbf{r},t))^* = 0, \tag{18d}$$

but that, as we see from Eqs. (17e) and (17f), the time-dependent Schrödinger equations which $\boldsymbol{\Psi}(\mathbf{r},t)$ and $(\boldsymbol{\Psi}(\mathbf{r},t))^*$ obey have Hamiltonian operators of opposite sign,

$$i\hbar\dot{\boldsymbol{\Psi}}(\mathbf{r},t) = c\hbar\nabla \times \boldsymbol{\Psi}(\mathbf{r},t), \tag{18e}$$

$$i\hbar(\dot{\boldsymbol{\Psi}}(\mathbf{r},t))^* = -c\hbar\nabla \times (\boldsymbol{\Psi}(\mathbf{r},t))^*. \tag{18f}$$

Unfortunately, *neither* of the Hamiltonian operators $(c\hbar\nabla\times)$ and $(-c\hbar\nabla\times)$ in the two Schrödinger equations just given is *nonnegative*, which, in light of the *nonnegativity of the purely electromagnetic field energy* $\frac{1}{2}\int d^3\mathbf{r}\,(|\mathbf{E}(\mathbf{r},t)|^2 + |\mathbf{B}(\mathbf{r},t)|^2)$, is an *absolutely necessary requirement* for the physically proper description of a solitary free photon. In *addition*, both of these Hamiltonian operators are *odd parity operators*, which implies that *reversing the parity* of any of their energy eigenstates produces an energy eigenstate whose eigenenergy has the *opposite sign* to that of the eigenenergy of the *original* energy eigenstate! This *complete energy eigenspectrum sign symmetry* is again incompatible with the need for solitary free photons to be of strictly nonnegative energy. The nonconservation of parity for a free solitary photon which such a Hamiltonian implies is as well *incompatible with the conservation of parity on the part of electromagnetic theory*.

In order to overcome these problems, we must *meld* the *nonnegative part* of the eigenenergy/eigenstate spectrum of $(c\hbar\nabla\times)$ with the *complementary positive part* of the eigenenergy/eigenstate spectrum of $(-c\hbar\nabla\times)$. The resulting *nonnegative Hamiltonian operator* will clearly be the *absolute value* of the operator $(c\hbar\nabla\times)$, i.e., the operator $|c\hbar\nabla \times |$. To gain further insight into the precise nature of this *melded* Hamiltonian operator $|c\hbar\nabla \times |$, let us look at the *purely transverse* eigenstates of $(c\hbar\nabla\times)$. Each of these eigenstates is clearly a complex-valued transverse vector field which consists of a *single Fourier component* whose Fourier vector variable *direction* is *strictly orthogonal* to the *longitudinal unit vector* $\mathbf{u}_L(\mathbf{k})$. Upon operating on such a single Fourier component proportional to $e^{i\mathbf{k}\cdot\mathbf{r}}$, the operator $(\nabla\times)$ clearly becomes $i(\mathbf{k}\times) = i|\mathbf{k}|(\mathbf{u}_L(\mathbf{k})\times)$, whereas the *two-dimensional transverse Fourier vector variable space* is clearly *fully spanned* by the *two transverse unit vectors* $\mathbf{u}_1(\mathbf{k})$ and $\mathbf{u}_2(\mathbf{k})$, or, *much more usefully in this particular instance*, the *two complex-valued transverse unit vectors* $(2)^{-\frac{1}{2}}(\mathbf{u}_1(\mathbf{k}) \pm i\mathbf{u}_2(\mathbf{k}))$. By making use of the cross product identities of Eq. (5d) in conjunction with the facts just discussed, one readily verifies that the *purely transverse* eigenstates of the Hermitian operator $(\nabla\times)$ are of the form,

$$\boldsymbol{\Psi}_{\mathbf{k}}^{(\pm)}(\mathbf{r}) \stackrel{\text{def}}{=} (\mathbf{u}_1(\mathbf{k}) \pm i\mathbf{u}_2(\mathbf{k}))g^{(\pm)}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{19}$$

and have the *corresponding eigenvalues* $\pm|\mathbf{k}|$ for the operator $(\nabla\times)$. Therefore the corresponding *two* eigenvalues of the *melded* Hamiltonian operator $|c\hbar\nabla\times|$ on the above two *purely transverse* eigenstates are *both* simply $|c\hbar\mathbf{k}|$, in *complete agreement* with those of the fully diagonalized Hamiltonian for the *two* time-dependent Schrödinger equations that are given by Eqs. (11a) and (11b). Thus we clearly see that an *equivalent* way to express the *action* of our *melded* configuration-space Hamiltonian operator $|c\hbar\nabla\times|$ on the *purely transverse subspace* is as *simply* $|c\hbar\widehat{\mathbf{k}}|$, which can be loosely styled as $|c(-i\hbar\nabla)|$ in configuration representation. Because of the *purely transverse, two-dimensional nature* of the *subspace* on which it is *obliged to operate*, the non-negative *melded* configuration-space Hamiltonian $|c\hbar\nabla\times|$ *loses all trace of its curl character* and becomes indistinguishable from the loosely styled $|c(-i\hbar\nabla)|$, thus falling into line with the $m=0$ case of the Eq. (1) Hamiltonian, namely $|c\widehat{\mathbf{p}}| = |c\hbar\widehat{\mathbf{k}}| = |c(-i\hbar\nabla)|$.

Summarizing, we have that the solitary free-photon wave function in configuration representation is a complex-valued vector field $\mathbf{\Psi}(\mathbf{r},t)$ which is *strictly transverse*, i.e.,

$$\nabla\cdot\mathbf{\Psi}(\mathbf{r},t) = 0, \tag{20a}$$

generally normalizable to unity (unless idealized),

$$\int d^3\mathbf{r}\,[(\mathbf{\Psi}(\mathbf{r},t))^* \cdot \mathbf{\Psi}(\mathbf{r},t)] = 1, \tag{20b}$$

and satisfies the time-dependent Schrödinger equation,

$$i\hbar\dot{\mathbf{\Psi}}(\mathbf{r},t) = |c(-i\hbar\nabla)|\mathbf{\Psi}(\mathbf{r},t), \tag{20c}$$

in accord with the $m=0$ instance of Eq. (1). The loosely styled configuration-space Hamiltonian operator $|c(-i\hbar\nabla)|$ of the above Schrödinger equation is *actually*, in configuration representation, a symmetric nonlocal *integral operator* (something which Klein, Gordon, Schrödinger and Dirac sought *desperately* to avoid, thereby playing *havoc* with physical cogency), whose kernel is given by,

$$\langle\mathbf{r}|\,|c(-i\hbar\nabla)|\,|\mathbf{r}'\rangle = \langle\mathbf{r}|\,|c\hbar\widehat{\mathbf{k}}|\,|\mathbf{r}'\rangle = -(c\hbar/(2\pi^2 R))d^2(R/(R^2+\epsilon^2))/dR^2, \tag{21}$$

where $R \stackrel{\text{def}}{=} |\mathbf{r}-\mathbf{r}'|$ and ϵ is an infinitesimal length.

Conclusion

Finally, it is to be noted that the very first quantum theorist was James Clerk Maxwell. His celebrated equations faithfully encompassed the correct quantum description of the solitary free photon long before Erwin Schrödinger was to accomplish the same feat for the solitary *nonrelativistic* free particle. And Maxwell's formidable theoretical physics machinery *already* yielded up the first instance of *intrinsic* particle degrees of freedom, with all their subtlety! By the grace of almost cosmic coincidence, Maxwell, unlike any of his quantum theory successors, could accomplish all this with no use whatsoever of Planck's world-changing constant, which, still undiscovered, silently awaited the future—the massless nature of the photon permits Maxwell's magnificent equations to simply slip away from $\hbar$'s grasp.

References

[1] S. K. Kauffmann, arXiv:0909.4025 [physics.gen-ph] (2009).

[2] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[3] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961).

[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A117**, 610 (1928).

[5] S. K. Kauffmann, arXiv:0910.2490 [physics.gen-ph] (2009).

[6] S. K. Kauffmann, arXiv:0908.3755 [quant-ph] (2009).

[7] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950).

[8] K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[9] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).