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CORRECTIONS TO THE WU-SPRUNG POTENTIAL FOR THE RIEMANN ZEROS AND A NEW HAMILTONIAN WHOSE ENERGIES ARE THE PRIME NUMBERS

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- ABSTRACT: We review the Wu-Sprung potential adding a correction involving a fractional derivative of Riemann Zeta function, we study a global semiclassical analysis in order to fit a Hamiltonian H=T+V fitting to the Riemann zeros and another new Hamiltonian whose energy levels are precisely the prime numbers, through these paper we use the notation $\log_e(x)=\ln(x)=\log(x)$ for the logarithm , also unles we specify $\sum_\gamma h(\gamma)$ means that we sum over ALL the imaginary parts of the nontrivial zero on both the upper and lower complex plane.

1.WU-SPRUNG POTENTIAL WITH OSCILLATING TERM

From the point of view of Physics an 'easy' proof to prove the celebrate Riemann Hypothesis would be to find a self-adjoint operator L so $\zeta\left(\frac{1}{2}+i\hat{L}\right)|\Phi\rangle=0$, this would mean that all the zeros of the $\zeta-$function would have real part ½ , this is called the Hilbert-Polya [4] approach , in order to get this Linear operator Wu and Sprung [9] conjectured that if this operator were a Hamiltonian of the form $H=p^2+V(x)$, then the 'smooth' part of this potential would be

$$\pi y_{SM}(V)=V^{-1}(y)=\sqrt{V-V_0}\log\left(\frac{V_0}{2\pi e^2}\right)+\sqrt{V}\log\left(\frac{\sqrt{V-V_0}+\sqrt{V}}{\sqrt{V}-\sqrt{V-V_0}}\right) \qquad (1.1)$$

With $V_0 \approx 9.74123$, the method by Wu and Sprung is based on semiclassical analysis in physics or WKB [7] approach, the idea is that in the semiclassical approximation

$$\frac{dN}{dE} = \sum_n \delta(E - E_n) \approx \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \delta\left(E - p^2 - V(x)\right) = \int_0^V \frac{dx/dV}{\sqrt{E - V}} \quad (1.2)$$

In this WKB approach ,we replace the sum over Energies (imaginary part of the Riemann zeros) by a sum over the phase space (p,x) , also we have used the known properties of Dirac delta function $\delta(f(x)) = \sum_{f(u)=0} \frac{\delta(x-u)}{|f'(u)|}$ and $\int_{-\infty}^{\infty} dx f(x) \delta(x-a) = f(a)$ to integrate over the variable ‘p’ (momentum) . Equation (1.2) is just a type of Abel integral equation (see [11])

$$f(x) = \int_0^x \frac{y(t)dt}{\sqrt{x-t}} \qquad y(x) = \frac{1}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)dt}{\sqrt{x-t}} \qquad (1.3)$$

If we approximate the smooth density of states (Number of imaginary parts of the Riemann zeros , who are less than a given quantity E) by $N(E) = \frac{E}{2\pi} \log \frac{E}{2\pi} - \frac{E}{2\pi} + \frac{7}{8}$, then $f(E) = \frac{dN}{dE}$ solving the last integral on (1.3) we obtain the smooth part of the Wu-Sprung potential defined implicitly in (1.1)

- *Fractional derivative correction to Wu-Sprung potential:*

Although Wu and Sprung considered only the smooth part of the density of zeros N(E) there is an extra term proportional to $\frac{1}{\pi} \arg\left\{\zeta\left(\frac{1}{2} + is\right)\right\}$ if we insert this , inside the Abel integral equation (1.3)., then the oscillating term contribution to the Wu-Sprung potential can be defined in terms of the Riemann-Liouville differintegral [10]

$$\pi y_{SM}(V) = B \Re e \left\{ \frac{d^{-1/2}}{dV^{-1/2}} \frac{\zeta'}{\zeta}\left(\frac{1}{2} + iV\right) \right\} \qquad B \in R \qquad (1.4)$$

This expression (1.4) are the common definitions for a fractional derivative/integral

$$\frac{d^q f(x)}{dx^q} = \frac{1}{\Gamma(q-m)} \frac{d^m}{dx^m} \int_0^x \frac{dt f(t)}{(x-t)^{q-m+1}} \qquad \frac{d^{-q} f(x)}{dx^q} = \frac{1}{\Gamma(q)} \int_0^x dt f(t)(x-t)^{q-1} \qquad (1.5)$$

(derivative) (integral)

Expressions (1.5) are also useful because they satisfy a semigroup composition property for the fractional derivatives/integrals ,namely $D^a.D^b = D^{a+b}$ this allows to write

$D^1.D^{-1/2}=D^{1/2}$ or $D^{-1}.D^{1/2}=D^{-1/2}$, however the addition of potential (1.4) makes the implicit form of the potential inside (1.1) harder to solve.

2. A TRACE AND ANOTHER INTEGRAL EQUATION FOR THE RIEMANN HYPOTHESIS:

In a previous paper [6] we obtained a Trace formula for Unitary operator $U=e^{iu\hat{H}}$ defined for u >0 .

$$e^{u/2}-e^{-u/2}\frac{d\Psi_0(e^u)}{du}-\frac{e^{u/2}}{e^{3u}-e^u}=\sum_{n=-\infty}^{\infty}e^{iuE_n}=Tr\left\{\hat{U}=e^{iu\hat{H}}\right\}=2\sum_{n=0}^{\infty}\cos(uE_n) \quad u>0 \quad (2.1)$$

And 0 for u <0 .This trace (2.1) follows inmediatly from differentiation with respect to 'x' and setting $x=e^u$ inside the explicit formula for the Chebyshev function

$$\Psi_0(x)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}=\begin{cases} x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{\zeta'(0)}{\zeta(0)}-\frac{1}{2}\log(1-x^{-2}) & x>1 \\ 0 & x<1\end{cases} \quad (2.2)$$

Using the semiclassical approach inside (2.2) for u >0 and the Euler formula for the complex exponential ,we can get the integral equation with $A_{wkb}\in R$ a constant to be fixed by empirical or numerical observations

$$\left(e^{u/2}-e^{-u/2}\frac{d\Psi_0(e^u)}{du}-\frac{e^{-u/2}}{e^{2u}-1}\right)\sqrt{\frac{u}{\pi}}=A_{wkb}\int_0^{\infty}dx\cos\left(uV(x)+\frac{\pi}{4}\right) \quad u>0 \ (2.3)$$

Here, the derivative of the Chebyshev function can be described as an infinite sum over primes p and prime powers $\frac{d\Psi_0(x)}{dx}\frac{1}{\log(x)}=\sum_{\nu=1}^{\infty}\sum_{p}\delta(x-p^{\nu})$, this integral equation can be make 'smoother' by defining a pair of functions g(x) and h(x) with the following properties [6]

- Both g(x)=g(-x) and h(x)=h(-x) are even functions
- $\lim_{x\to 0}\frac{g(x)}{x}$ exists and it is finite
- The functions h(x) and g(x) are related by a Fourier Cosine transorm $\frac{1}{2\pi}\int_{-\infty}^{\infty}dx h(x)e^{i\alpha x}=g(\alpha)$
- The function h(x) tends to 0 as $x\to\infty$ faster than $e^{x/2}$, so the integral $\int_{-\infty}^{\infty}dx e^{iux}g(x)e^{x/2}$ exists as a function

Applying the Fourier integral transform $h(u)=\int_{-\infty}^{\infty} dx e^{iux} g(x)$ (2.3) becomes for our test functions g and h

$$\begin{aligned}&\sqrt{iD}h\left(r+\frac{i}{2}\right)+\sqrt{-iD}h\left(r-\frac{i}{2}\right)-\sum_{n=2}^{\infty}\sqrt{\frac{\log n}{n}}\Lambda(n)g(\log n)\left(\frac{1}{n^{ir}}+\frac{1}{n^{-ir}}\right)\\&-2\int_{0}^{\infty}\frac{due^{-u/2}\sqrt{u}g(u)\cos(ur)}{e^{2u}-1}=A_{wkb}\int_{-\infty}^{\infty}dx\left(\sqrt{i\pi}h\left(V(x)+r\right)+\sqrt{-i\pi}h\left(V(x)-r\right)\right)\end{aligned} \qquad (2.4)$$

With $\Lambda(n)=\Psi(n)-\Psi(n-1)$ being the Mangoldt function and $\sqrt{D}=\frac{d^{1/2}}{dx^{1/2}}$ the half-derivative, this fractional derivative appears from the property of the Fourier transform $\int_{-\infty}^{\infty} dx f(x) x^{u} e^{i\alpha x}=\left(-i\frac{d}{d\alpha}\right)^{u}\int_{-\infty}^{\infty} dx f(x) e^{i\alpha x}$. (2.4) is now the most general non-linear integral equation that can be obtained using the WKB approach for the potential V(x) and that is compatible with the Tracial condition (2.1) many authors forget this fact and do not use (2.1) to construct a more general potential than Wu-Sprung one (in fact this Wu-sprung potential is incomplete since it does not take into account the oscillating part of the Riemann zeros , which is very important in the search of an operator realization for the Riemann Hypothesis) , in order to convert (2.4) into a linear equation it is enough define the change of variable $x=V^{-1}(y)$. A similar formla to (2.4) can be used to compute sums over the imaginary part of the zeros γ (trace formula)

$$\begin{aligned}&h\left(r+\frac{i}{2}\right)+h\left(r-\frac{i}{2}\right)-\sum_{n=1}^{\infty}\Lambda(n)\left(\frac{1}{n^{1/2+ir}}+\frac{1}{n^{1/2-ir}}\right)g(\log n)\\&-2\int_{0}^{\infty}\frac{e^{-u/2}\cos(ru)g(u)}{e^{2u}-1}du=A_{wkb}\left(\sum_{\gamma}h(\gamma-r)+\sum_{\gamma}h(\gamma+r)\right)\end{aligned} \qquad (2.5)$$

- *Trace formula for Riemann Zeros and Wu-Sprung potential:*

A more general formula than (2.5) valid only in the distributional sense expressing the sum over Riemann zeros without involving the prime numbers

$$\sum_{\gamma}A_{wkb}\pi\delta(s-\gamma)+\sum_{\gamma}\pi A_{wkb}\delta(s+\gamma)=\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)+\frac{\zeta'}{\zeta}\left(\frac{1}{2}-is\right)-\frac{4}{1+4s^{2}}-\sum_{n=0}^{\infty}\frac{(4n-1)}{\left((2n-1/2)^{2}+s^{2}\right)} \qquad (2.6)$$

Integration from 0 to E in the variable ‘s’ will give the oscillating term $\frac{1}{\pi}\arg\left\{\zeta\left(\frac{1}{2}+is\right)\right\}$, the factor $\left(1+4s^{2}\right)^{-1}$ is due to the pole of the Zeta function at s=1 and the last sum comes from the non-trivial zeros

Proof: if we set $h(x)=\delta(x)$ inside (2.5) and use the analytic continuation [5] , [6] of the divergent series (is a regularization more than a 'sum' definition)

$$2\sum_{n=1}^{\infty}\frac{\Lambda(n)\cos(s\log n)}{\sqrt{n}}=\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)+\frac{\zeta'}{\zeta}\left(\frac{1}{2}-is\right) \qquad \frac{1}{1-e^{as}}=\sum_{n=0}^{\infty}e^{ans} \qquad (2.7)$$

Together with the result $\int_{0}^{\infty}dxe^{-(s-n)x}\cos(ax)=\frac{s-n}{(s-n)^2+a^2}$,and the zeta regularization for Dirichlet series [6] and [5] so we can regularize the divergent series by $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}e^{is\log n}=-\frac{\zeta'}{\zeta}\left(\frac{1}{2}-is\right)$, then we can give a 'formal' proof to (2.6) . A direct application of this formula can be to evaluate sums of the form $\sum_{\gamma}h(\gamma)$ and to obtain an expansion for the smooth part of Wu-Sprung potential

$$Cx_{SM}(V)=\left(\frac{1}{i\left(s+i/2\right)^{3/2}}-\frac{1}{i\left(s-i/2\right)^{3/2}}+i\sum_{n=0}^{\infty}\frac{2n+1/2}{(s+ia_n)^{3/2}}-i\sum_{n=0}^{\infty}\frac{2n+1/2}{(s-ia_n)^{3/2}}\right) \qquad (2.8)$$

With $\lim_{\lambda\to -1}\left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\right)=\frac{1}{C}$ and $a_n=2n+1/2$ this definition is obtained by just applying the half-derivative operator $\frac{d^{1/2}}{dx^{1/2}}$ and the expansion inside the equation (2.6)

3. A POTENTIAL FUNCTION FOR THE PRIMES

A final question in our paper would be , could we obtain via semiclassical approximations a Hamiltonian $H=p^2+Q(x)$, so we could obtain the inverse of potential Q(x) using the WKB approximations for the density of states or the complex exponential sum of density of states

$$\sum_{n}\delta(E-E_n)\approx\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\delta\left(E-p^2-Q(x)\right) \qquad \sum_{n}e^{-uE_n}\approx\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dpe^{-u(p^2+Q(x))} \qquad (3.1)$$

So $E_n=p_n$ is the n-th prime number and u >0 is a Real number , for the second case we will need 2 fundamental properties of Laplace transform [1]

$$L\left[\int_{0}^{t}f(u)g(t-u)du\right] \qquad L\left[f(t)\right]=L\left[g(t)\right]\to f=g \qquad L\left[f(t)\right]=\int_{0}^{\infty}dtf(t)e^{-st} \qquad (3.2)$$

The first formula for the potential Q(x) is inmediate, the 'density' of primes is given by the derivative of the Prime Number counting function [2] $\frac{d\pi}{dx}=\sum_{p}\delta(x-p)$ suing

(3.1) (1.2) and the solution for the Abel integral equation (1.3) together with the defintion of the fractional derivative and integral we find

$$Q^{-1}(x) \approx A_p \frac{d^{1/2}\pi(x)}{dx^{1/2}} = A_p \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{\pi(t)dt}{\sqrt{x-t}} \qquad (3.3)$$

The constant A_p can be determined by empirical or numerical observations and must be independent of the choice of potential Q(x) and its inverse.

The second method based on the Laplace direct and inverse transform is the following , using again the WKB approach to replace the sum by an integral

$$\sum_n e^{-uE_n} \approx \int_0^\infty dx \int_0^\infty dp e^{-u(p^2+Q(x))} = \sqrt{\frac{\pi}{u}}\int_0^\infty dx e^{-uQ(x)} = \sqrt{\frac{\pi}{u}}\int_0^\infty dx e^{-ux}\frac{dQ^{-1}}{dx} \quad (3.4)$$

On the other hand there is an exact expression for the density of primes/energies defined by $\sum_n e^{-uE_n} = s\int_0^\infty dx\pi(x)e^{-ux}$, so equating (3.4) and this expression and using the unicity property for the Laplace transform (if two functions have the same Laplace transform , then these functions are equal f=g) then (3.4) becomes

$$\sum_n e^{-uE_n} = s\int_0^\infty dx\pi(x)e^{-ux} = s^{-1/2}\sqrt{\pi}\int_0^\infty dx e^{-ux}\frac{dQ^{-1}}{dx} = s^{1/2}\int_0^\infty dx e^{-ux}Q^{-1}(x) \quad (3.5)$$

Using the convolution and the unicity properties of the Laplace transform then we reach to the same conclussion as in (3.3) $Q^{-1}(x) \approx A_p \frac{d^{1/2}\pi(x)}{dx^{1/2}}$, the problem here is that still we do not know the value of the Prime counting function $\pi(x)$, one of the most common approaches to this function are given by the Prime Number theorem , or the Ramanujan approximation [3] , then our potential now becomes

$$Q^{-1}(x) \approx A_p \frac{d^{1/2}Li(x)}{dx^{1/2}} \quad \text{or} \qquad Q^{-1}(x) \approx A_p \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{x^{1/n}}{x\lg x}\right) \qquad (3.6)$$

With $Li(x) = \int_2^x \frac{dt}{\ln t}$ being the logarithmic integral , and $\mu(n)$ the Möbius function. For a better definition and a good introduction to Möbius and other Numer theoretic functions Apostol's book is the best reference [2]

CONCLUSIONS AND FINAL REMARKS:

We have investigated the Wu-Sprung potential and its generalizations (2.4) , we have shown related to this problem , how if Trace formula for the Hamiltonian H that reproduces the imaginary part of the non-trivial zeros is true (see (2.1-2-5)) one can get a semiclassical WKB approach to obtain the inverse of the potential V(x) , this improves the approximate result by Wu and Sprung, since they did not take care of this necessary condition (2.1) in order to obtain a Hamiltonian operator realization of Riemann Hypothesis, also in a similar manner we managed to use fractional Calculus to obtain the oscillatory part of the Wu-sprung potential and applied the smae in (3.6) to get the inverse of the potential inside a Hamiltonian whose energy levels are precisely the prime numbers $E_n = p_n$. In the semiclassical WKB approach , for 1-D dynamical systems we have the asymptotic formula for the inverse of the potential $V^{-1}(x) \approx A \frac{d^{1/2} N(x)}{dx^{1/2}}$ for some Real constant 'A' , and N(E) being the density of states , for the case of the potential for prime numbers p_n the density of states is just $\pi(x)$, for the case of the Riemann Hypothesis, we assume that the Riemann Zeta function is the dynamical zeta function of a certain dynamical system defined by a Hamiltonian function $H(q,p) = p^2 + V(x)$, in this case we also have the relationship

$$\rho(E) = \rho_{smooth}(E) + \frac{1}{\pi} \Im m \left(\frac{\partial \log Z\left(\frac{1}{2} + iE\right)}{\partial E} \right) \qquad \rho(E) = \frac{dN}{dE} \qquad (3.7)$$

In our case $Z(s) = \zeta(s)$, so taking the half-integral of $\rho(E) = \frac{dN}{dE}$, the oscillating part of the potential must be proportional to the Real part of $\frac{d^{-1/2}}{dx^{-1/2}} \left(\sqrt{-i}\, \frac{\zeta'}{\zeta}\left(\frac{1}{2} + ix\right) \right)$, in Appendix A we will show (A.5) how this series is related via Zeta regularization to an infinite (divergent) sum over primes and prime powers. A similar formula involving the half-integral of the logarithmic derivative for the Zeta function had been obtained in our previous paper [6] using the Zeta regularization algorithm.

If we define the Xi function by $\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$, the condition for the inverse of potential and the quantization (spectral determinant) become

$$V^{-1}(x) \approx A \Re e \left(\frac{d^{-1/2}}{dx^{-1/2}} \left(\sqrt{-i}\, \frac{\xi'}{\xi}\left(\frac{1}{2} + ix\right) \right) \right) \qquad \xi\left(\frac{1}{2} + iE\right) = g(E) \det\left(E - H\right) = 0 \quad (3.8)$$

Here g(E) is a function with no zeros (real or complex) , the idea is that the Xi function defined on the critical line is proportional to the spectral determinant $\det\left(E-H\right) = 0$, this last equation inside (3.8) is the secular equation for the Eignevalues inside the Hamiltonian $H = p^2 + V(x)$, using the half-integral of the logarithmic derivative for the Riemann Xi-function we could get an asymptotic for the inverse of the potential

APPENDIX A: A CONSTRUCTION OF THE INVERSE OF THE POTENTIAL V(x) FROM THE SUM $\sum_{n=-\infty}^{\infty} e^{iuE_n}$

We have devised a Trace formula (2.3) using the semiclassical WKB plus the relation of the derivative of the Chebyshev function , the idea is can we solve (2.3) to get a real valued quantity ? , first of all we will not use our Trace formula, but a similar trace obtained by Riemann and Weyl [1]

$$\sum_{\gamma>0} h(\gamma) = 2h(i/2) - g(0)\log\pi - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n) + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)dr \quad \text{(A.1)}$$

Taking the inverse function for the potential $x = x(V)$, and using the approximation for the sum over the energies plus integration by parts

$$\sum_{n=-\infty}^{\infty} e^{iuE_n} \approx \sqrt{\frac{\pi}{u}}\int_{-\infty}^{\infty} dx e^{iuV(x)} \qquad \int_{-\infty}^{\infty} dV\frac{dx}{dV}e^{iuV} = -iu\int_{-\infty}^{\infty} dV x(V) e^{iuV} \qquad \text{(A.2)}$$

So taking the Fourier inverse transform , and taking into account that the trace is only nonzerop for u >0 , in order to get x(V) we should evaluate the integral

$$2\int_{0}^{\infty} du \sum_{\gamma}\frac{e^{-iu\gamma - iuV + \pi/4}}{|\pi u|^{1/2}} = \sum_{\gamma>0}\frac{1}{|x-\gamma|^{1/2}} + \sum_{\gamma>0}\frac{1}{|x+\gamma|^{1/2}} = x(V) \qquad \text{(A.3)}$$

In order to get rid off the sum over the Non-trivial zeros, we could use (A.1) with $g(x) = \frac{1}{|x|^{1/2}}$ to get the most general potential compatible with (2.3) and (A.1)

$$x(V) \approx \frac{A}{|2V+i|^{1/2}} + BCos(cV + \pi/4) + D + E\sum_{n=1}^{\infty}\frac{\Lambda(n)\cos(V\log n + \pi/4)}{|n\log n|^{1/2}} + Fp.v\left(\int_{-\infty}^{\infty} dr \frac{\Gamma}{\Gamma}\left(\frac{1}{4}+\frac{ir}{2}\right)\left(\frac{1}{|V+r|^{1/2}} + \frac{1}{|V-r|^{1/2}}\right)\right) \qquad \text{(A.4)}$$

Here, A,B,C ,D ,E and F are REAL numbers that describe the inverse of the potential V(x), since the Wu-Aprung potential and ours have been obtained by using WKB methods comparing the two potentials (at least the smooth parts) we could get the value of these constants , the sum involving $\Lambda(n)$ is divergent , and needs to be regularized to extract some meaningful finite information, from the Dirichlet generating function, and integrating over ‘s’ we get via Zeta regulariztion

$$2\sum_{n=1}^{\infty}\frac{\Lambda(n)\cos(V\log n + \pi/4)}{|n\log n|^{1/2}} = -\sqrt{-i}\,\frac{d^{-1/2}}{dV^{-1/2}}\left(\frac{\zeta'(1/2+iV)}{\zeta(1/2+iV)}\right) - \sqrt{i}\,\frac{d^{-1/2}}{dV^{-1/2}}\left(\frac{\zeta'(1/2-iV)}{\zeta(1/2-iV)}\right) \qquad \text{(A.5)}$$

(A.5) is get by a simple half-integration with respect to ‘s’ insde the zeta regularized identity $\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2+is}}=-\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)$. In general this formula A.5 will be divergent, so for Numerical purposes we should truncate it up to some finite N to extract a finite contribution for the oscillating part of the potential.

Another formula erquivalent to (A.4) can be given, if we consider in distributional sense $\frac{d^{1/2}H(x)}{dx^{1/2}}=\frac{H(x)}{\sqrt{\pi x}}$ and $f_{+}(x-a)=f(x-a)H(x-a)$, then the sum (up to a constant) $\sum_{\gamma>0}\frac{1}{|x-\gamma|^{1/2}}+\sum_{\gamma>0}\frac{1}{|x+\gamma|^{1/2}}$ could be viewed as the half-derivative of the Density of zeros inside the Critical line so in the WKB approximation the inverse for the potential is $A\frac{d^{1/2}N_{smooth}(x)}{dx^{1/2}}+A\frac{d^{1/2}N_{osc}(x)}{dx^{1/2}}\approx V^{-1}(x)$,with N(T)

$$\frac{1}{\pi}Arg\left(\zeta\left(\frac{1}{2}+iT\right)\right)\quad \frac{1}{\pi}Arg\left(\Gamma\left(\frac{1}{4}+i\frac{T}{2}\right).\left(\frac{1}{8}+\frac{T^{2}}{2}\right)\pi^{-\frac{1}{4}-i\frac{T}{2}}\right)\approx\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}+\frac{7}{8} \tag{A.6}$$

Then using the property $D^{1/2}.D^{-1}=D^{-1}.D^{1/2}$, $\pi x(V)$ can be rewritten (at first approximation ignoring possible terms proportional to O(1/V)) as

$$-A\Re\mathrm{e}\left(\sqrt{-i}\,\frac{d^{-1/2}}{dV^{-1/2}}\left(\frac{\zeta'(1/2+iV)}{\zeta(1/2+iV)}\right)\right)+\sqrt{\pi}A\frac{d^{1/2}}{dV^{1/2}}\left(\frac{V}{2}\log\left(\frac{V}{2\pi}\right)\right)+\frac{7A}{8}\sqrt{\frac{\pi^{2}}{V}}-\frac{A}{2}\sqrt{V} \tag{A.7}$$

Then, the Wu-Sprung potential is equivalent to our Trace formula (2.2) and (2.3) , however Wu and Sprung avoided the ‘oscillating’ term coming from the half-integral of the expression (in the sense of Zeta regularization) $-\frac{\zeta'}{\zeta}\left(\frac{1}{2}+is\right)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1/2+is}}$, this term is obtained from the derivative of the Chebyshev step function $\frac{d\Psi_{0}}{dx}$ controlling how prime and prime powers are distributed, Also the Trace (2.3) for $Tr\left\{e^{iu\hat{H}}\right\}$ is more general since it can be used to give meaning to any sum $\sum_{\gamma}h(\gamma)$ not only for $\sum_{\gamma}\delta(x-\gamma)=\frac{dN}{dx}$ as we have proved in formula (2.6), in (A.7) ‘A’ is a real constant.

The result $A\frac{d^{1/2}N(x)}{dx^{1/2}}\approx V^{-1}(x)$, is valid for any 1-D Hamiltonian with potential V(x) and it is based on the semiclassical WKB approximation, for example for the Harmonic oscillator $N(E)\approx\frac{E}{\omega}$, then taking the half-derivative we find $A\omega^{-1}\sqrt{x}\approx V^{-1}(x)$, the inverse of this expression yields $V(x)\approx A^{-2}\omega^{2}x^{2}$, which is almost the exact value of the potential inside the Harmonic oscillator, note that in this case this WKB approximation is EXACT

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