Analytical proof of the Taylor equation including Taylor's constant S_γ which previously required numerical integration, with applications

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ABSTRACT

British mathematician Sir Geoffrey I. Taylor in secret work for British civil defence in 1941 (declassified in 1950 and published in the *Proceedings of the Royal Society,* vol. 201A, pp. 159-186), derived the strong shock solution equation, namely distance, $R = S_\gamma t^{2/5}E^{1/5}\rho_0^{-1/5}$, where ρ_0 is the ambient (pre-shock) atmospheric density, t is time after explosion, E is the energy released and S_γ is Taylor's calculated function of γ, requiring a complex step-wise numerical integration. We present a proof of the equation $R = \{[75E(\gamma - 1)t^2]/(8\pi\rho_0)\}^{1/5}$, implying that Taylor's so-called constant $S_\gamma = [75(\gamma - 1)/(8\pi)]^{1/5}$, not requiring any complex integration. This is useful for close-in shock waves from nuclear explosions and supernovae explosions. We further obtain the general arrival time of the shock wave $t = R/\{c_0 + [75E(\gamma - 1)/(8\pi\rho_0 R^3)]^{1/2} + Rc_0[4\pi P_0/(3E(\gamma - 1))]^{1/5}\}$, by noting two asymptotic solutions; namely, at very great distances, the blast decays into a sound wave so the arrival time t approaches the ratio of distance to sound velocity (R/c_0), while at very close-in distances the strong shock equation previously derived becomes accurate, and there is also an easily included effect at intermediate distances due to the expansion of the hot air in reducing shock front arrival times. The errors of method made by Taylor for nuclear test explosions in air were also made by Russian mathematician Leonid I. Sedov who applied similar cumbersome numerical integrations in a 1946 paper (published in the *Journal of Applied Mathematics and Mechanics,* vol. 10, pp. 241-50).

INTRODUCTION

Sir Geoffrey Taylor in June 1941 produced a secret mathematical paper entitled "The formation of a blast wave by a very intense explosion" for the British Civil Defence Research Committee of the Ministry of Home Security. It was declassified and published, together with an additional part comparing the prediction to the film-measured effects of the 1945 Trinity nuclear test, in March 1950 in the *Proceedings of the Royal Society* (vol. 201A, pp. 159-186).

Taylor there stated: "The present writer had been told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission… In the then common explosive bomb mechanical effects were produced by the sudden generation of a large amount of gas at a high temperature in a confined space… Would similar effects be produced if energy could be released in a highly concentrated form unaccompanied by the generation of a gas?… The motion and pressure of the surrounding air is calculated. It is found that a spherical shock wave is propagated outwards whose radius R is related to the time t since the explosion started by the equation $R = S_\gamma t^{2/5}E^{1/5}\rho_0^{-1/5}$, where ρ_0 is the atmospheric density, E is the energy released and S_γ is a calculated function of γ, the ratio of the specific heats of air."

The analysis done by Taylor uses analytic boundary conditions for the physical situation for the spherical divergence of energy in the shock wave, but is supplemented by step-by-step numerical integration of the differential equation of motion of air for the a situation where air pressure varies with distance, the equation of continuity (the mass of air being engulfed by the expanding shock wave is added to the total mass within the shock wave), and the equation of state of a gas, to find the value of S_γ, a constant. Taylor was unable to find a complete solution without resorting to numerical methods, although he did find the correct numerical variation of the initial expansion of the 1945 Trinity test fireball, as he states in the second part of his paper: "The relationship predicted in part I, namely, that $R^{5/2}$ would be proportional to t, is surprisingly accurately verified over a range from R = 20 to 185 m. The value of $R^{5/2}t^{-1}$ so found was used… to estimate the energy E which was generated in the explosion."

In the fireball, the *increase* of γ due to radiation-induced ionisation and the thermal dissociation of diatomic air molecules of nitrogen and oxygen into atoms, cancels out the *decrease* in γ due to the absorption of energy in vibrations (which increases C_v, the specific heat capacity for constant volume or constant density). The result of these two factors is that Taylor found that γ remains approximately constant with fireball temperature, at $\gamma = 1.4$. Taylor calculated that the Trinity test fireball energy assuming $\rho_0 = 1.25$ kg/m^3 (which is too high for the Alamogordo test site!) and that 1 kt of TNT releases 4.18×10^{12} J, giving the result of 16,800 long tons, which is equivalent to 17.1 metric kt. Taylor notes that this is "that part of the energy which was not radiated outside the ball of fire" within 62 milliseconds of detonation. Correcting the Trinity air density to 1.004 kg/m^3 gives the true blast energy to be (1.004/1.25) x 17.1 = 13.7 metric kt.

METHOD

In the process of developing this book, an original mathematical proof for the mechanism of the blast wave was produced: the first part of our analysis proves that in Taylor's equation $R = S_\gamma t^{2/5} E^{1/5} \rho_0^{-1/5}$ the constant $S_\gamma = [75(\gamma - 1)/(8\pi)]^{1/5}$, so in fact it does not need to be calculated by Taylor's lengthy numerical integration! Later we use this result in constructing the formula for blast wave arrival time.

Equations for the expansion rate of air burst fireball and its energy as function of its time and radius

Our formula (see the proof below) for the radius of the fireball during shock wave expansion is:

$$R = \{[75E(\gamma - 1)t^2]/(8\pi\rho_0)\}^{1/5},$$

and rearranging this gives the fireball energy as:

$$E = 8\pi\rho_0 R^5/[75(\gamma - 1)t^2].$$

These results are useful to interpret the filmed test fireball expansion rates in terms of energy, to analyze supernovae explosions, and in constructing the equation for the arrival time of the blast wave at any distance.

The errors of method made by Taylor for nuclear test explosions in air were also made by Russian mathematician Leonid I. Sedov who applied similar cumbersome numerical integrations in a 1946 paper (published in the *Journal of Applied Mathematics and Mechanics,* vol. 10, pp. 241-50).

PROOF

The total mass (M) of ambient air which has been engulfed by the blast wave up to any given moment is equal to the mass of air enclosed with the blast radius, which equals the normal air density (ρ_0) multiplied by the total spherical volume enclosed by the radius (R) of the blast wave (volume of a sphere = $4\pi R^3/3$), hence:

$$M = 4\pi R^3 \rho_0/3. \quad (\textit{Equation 1.})$$

The total fireball energy ($E_{fireball}$) is equivalent to its kinetic energy ($E_{kinetic} = PV = \frac{1}{2} MU^2$, where P is pressure, V is volume, and U is the outward blast velocity), *multiplied* by the ratio $E_{total}/E_{kinetic}$. Since $\gamma = 1 + PV/E_{total}$, we can write the energy ratio as: $E_{total}/E_{kinetic} = E_{total}/(PV) = 1/(\gamma - 1)$. Now we can use the relation $E_{kinetic} = \frac{1}{2} MU^2$ to give:

$$E_{total} = \tfrac{1}{2} MU^2(E_{total}/E_{kinetic}) = \tfrac{1}{2} MU^2/(\gamma - 1). \quad (\textit{Equation 2.})$$

The outward velocity of the shock, U, is defined as the distance travelled divided by the time taken, *but is decelerating at a constant rate owing to the continuously increasing mass engulfed,* M, *throughout which the available fireball energy is distributed as heat and pressure.* [Although the total mass engulfed increases, by equation (2) above, as R^3, it is the *surface area* of the fireball, not its volume, which is engulfing air, so the mass engulfed per unit area of surface is proportional to mass/area which varies as the ratio $R^3/R^2 = R$.] These considerations prove that U is always a constant fraction (say B) of R/t:

$$U = BR/t. \quad (\textit{Equation 3.})$$

Substituting formulae (1) and (3) above into (2) gives:

$$E = (2/3)\, \pi\rho_0 B^2 (R^5/t^2)/(\gamma - 1). \quad (\textit{Equation 4.})$$

This formula shows that the ratio R^5/t^2 is a *constant* for any given bomb energy and environment density, etc. Therefore, R^5 must be directly proportional to t^2, or, rearranging, R is *directly proportional* to $t^{2/5}$. This fact permits us to find the value of the constant B in equations (3) and (4). First, let: $R = At^{2/5}$, where A is a constant. Differentiating with respect to t then gives:

$$dR/dt = (2/5)At^{-3/5}. \quad (\textit{Equation 5.})$$

Now, employing calculus on equation (3) gives:

$$dR/dt = U = BR/t. \quad (\textit{Equation 6.})$$

Substituting (5) into (7) gives:

$$dR/dt = U = ABt^{-3/5}. \quad (\textit{Equation 7.})$$

Setting (8) equal to (6) gives:

$$dR/dt = U = ABAt^{-3/5} = (2/5)At^{-3/5}. \quad (\textit{Equation 8.})$$

which cancels down yielding the constant:

$$B = 2/5. \quad (\textit{Equation 9.})$$

Substituting (9) into equation (4) and rearranging the result gives:

$$R = \{[75E(\gamma - 1)t^2]/(8\pi\rho_0)\}^{1/5}. \quad (\textit{Equation 10.})$$

which is valid before the supersonic blast has decayed into a sound wave.

Asymptotically accurate general formula for the arrival time of the blast wave from an air burst explosion

We can find an analytical solution to the blast wave arrival time at all distances if we consider three factors. First, at very great distances, the blast decays into a sound wave so the arrival time t approaches the ratio R/c_0, where c_0 is sound speed. Second, the formula above for the supersonic phase shows that the arrival time is less for small distances. Third, the energy of the blast wave ends up as residual kinetic energy of air molecules, which is heat, leading to the total expansion of the air around ground zero by a distance equal to the spherical radius R_0 of air at ambient pressure P_0, which can contain energy $E = P_0V/(\gamma - 1)$, where $V = 4\pi R_0^3/3$, so: $R_0 = [3E(\gamma - 1)/(4\pi P_0)]^{1/3}$. This is important at intermediate distances in reducing the arrival time of the shock front from large detonations. A simple analytic formula that gives incorporates all correct asymptotic limits and the expansion of the air is:

$$t = R/\{c_0 + [75E(\gamma - 1)/(8\pi\rho_0 R^3)]^{1/2} + Rc_0[4\pi P_0/(3E(\gamma - 1))]^{1/5}\}. \quad (\textit{Equation 11.})$$

This new formula accurately predicts the arrival time of the blast wave at any distance. Changing the volume formulae to hemispheres converts this formula to a surface burst. (Cf. graph of experimental arrival time versus distance data in *Operation Castle* nuclear weapon test report WT-934, 1959.)