Funcoids and Reloids*

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Abstract

It is a part of my Algebraic General Topology research.

In this article I introduce the concepts of *funcoids* which generalize proximity spaces and *reloids* which generalize uniform spaces. The concept of funcoid is generalized concept of proximity space, the concept of reloid is cleared from superfluous details (generalized) concept of uniform space. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of analysis and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula are generalized continuity, proximity continuity, and uniform continuity.

Keywords: algebraic general topology, quasi-uniform spaces, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

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1 Common

1.1 Draft status

This article is a draft.

This text refers to a preprint edition of [5]. Theorem number clashes may appear due editing both of these manuscripts.

1.2 Used concepts, notation and statements

Set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will denote $\langle f\rangle X = \{f\alpha \mid \alpha \in X\}$.

For simplicity I will assume that all sets in consideration are subsets of universal set $\mho$.

1.2.1 Filters

In this work the word *filter* will refer to a filter on a set $\mho$ (in contrast to [5] where are considered filters on arbitrary posets).

I will call the set of filters ordered reverse to set-theoretic inclusion of filters *the set of filter objects* $\mathfrak{F}$ and its element *filter objects* (f.o. for short). I will denote $\operatorname{up}\mathcal{F}$ the filter corresponding to a filter object $\mathcal{F}$. So we have $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \operatorname{up}\mathcal{A} \supseteq \operatorname{up}\mathcal{B}$ for every filter objects $\mathcal{A}$ and $\mathcal{B}$. We also will equate filter objects corresponding to principal filters with corresponding sets. (Thus we have $\mathscr{P}\mho \subseteq \mathfrak{F}$.) See [5] for formal definition of filter objects in the framework of ZF. Filters (and filter objects) are studied in the work [5].

Filter objects corresponding to ultrafilters are atoms of the lattice $\mathfrak{F}$ and will be called *atomic filter objects*.

Also we will need to introduce the concept of *generalized filter base*.

Definition 1. *Generalized filter base* is a set $S \in \mathscr{P}\mathfrak{F} \setminus \{\emptyset\}$ such that

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S: \mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Proposition 2. Let S is a generalized filter base. If $A_1, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists \mathcal{C} \in S: \mathcal{C} \subseteq A_1 \cap \ldots \cap A_n.$$

Proof. Can be easily proved by induction. □

Theorem 3. If S is a generalized filter base, then $\mathrm{up} \bigcap^{\mathfrak{F}} S = \bigcup \langle \mathrm{up} \rangle S$.

Proof. Obviously $\mathrm{up} \bigcap^{\mathfrak{F}} S \supseteq \bigcup \langle \mathrm{up} \rangle S$. Reversely, let $K \in \bigcap^{\mathfrak{F}} S$; then $K = A_1 \cap \ldots \cap A_n$ where $A_i \in \mathrm{up} \mathcal{A}_i \in \mathcal{S}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n \subseteq A_1 \cap \ldots \cap A_n = K$, $K \in \mathrm{up}\, \mathcal{C}$, $K \in \bigcup \langle \mathrm{up} \rangle S$. □

Corollary 4. If S is a generalized filter base, then $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in S$.

Proof. $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in \mathrm{up} \bigcap^{\mathfrak{F}} S \Leftrightarrow \emptyset \in \bigcup \langle \mathrm{up} \rangle S \Leftrightarrow \exists \mathcal{X} \in S: \emptyset \in \mathrm{up} \mathcal{X} \Leftrightarrow \emptyset \in S$. □

Definition 5. I will call a *partially ordered (pre)category* a (pre)category together with partial order on each of its Hom-sets.

1.3 Earlier works

Some mathematician were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Some references to predecessors:

- In [1] and [2] are studied semi-uniformities and proximities.
- [3] and [4] contains recent progress in quasi-uniform spaces.

2 Partially ordered dagger categories

2.1 Partially ordered categories

Definition 6. I will call a *partially ordered (pre)category* a (pre)category together with partial order $\subseteq$ on each of its Hom-sets with the additional requirement that

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for any morphisms f_1, g_1, f_2, g_2 such that $\mathrm{Src}\, f_1 = \mathrm{Src}\, f_2 \wedge \mathrm{Dst}\, f_1 = \mathrm{Dst}\, f_2 = \mathrm{Src}\, g_1 = \mathrm{Src}\, g_2 \wedge \mathrm{Dst}\, g_1 = \mathrm{Dst}\, g_2$.

2.2 Dagger categories

Definition 7. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a *dagger precategory* is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1. $f^{\dagger\dagger} = f$;

2. $(g\circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 8. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x\mapsto x^\dagger$.

In other words, a *dagger category* is a category equipped with a function $x\mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1. $f^{\dagger\dagger} = f$;
2. $(g\circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Theorem 9. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A. \qquad \square$$

For a partially ordered dagger (pre)category I will additionally require (for any morphisms f and g)

$$f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 10. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

Definition 11. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.

Definition 12. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f\circ f$.

Theorem 13. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^\dagger \circ f$.

Proof.

(1)⇒(2). If f is symmetric and transitive then $f^\dagger\circ f = f\circ f = f$.

(2)⇒(1). $f^\dagger = (f^\dagger\circ f)^\dagger = f^\dagger\circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger\circ f = f\circ f$, so f is transitive. $\square$

2.2.1 Monovalued and entirely defined morphisms

Definition 14. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f\circ f^\dagger \subseteq 1_{\mathrm{Dst}\, f}$.

Definition 15. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \supseteq 1_{\mathrm{Src}\, f}$.

Remark 16. Easy to show that this is a generalization of monovalued and entirely defined binary relations as morphisms of the category **Rel**.

Definition 17. For a given partially ordered dagger category C the *category of monovalued (entirely defined) morphisms* of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined) morphisms is monovalued (entirely defined) and that identity morphisms are monovalued and entirely defined.

Proof.

Monovalued. Let f and g are monovalued morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \subseteq g \circ 1_{\mathrm{Dst}\, f} \circ g^\dagger = g \circ 1_{\mathrm{Src}\, g} \circ g^\dagger = g \circ g^\dagger \subseteq 1_{\mathrm{Dst}\, g} = 1_{\mathrm{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\mathrm{Dst}\, 1_A} \subseteq 1_{\mathrm{Dst}\, 1_A}$.

Entirely defined. Let f and g are entirely defined morphisms, $\mathrm{Dst}\, f = \mathrm{Src}\, g$. $(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \supseteq f^\dagger \circ 1_{\mathrm{Src}\, g} \circ f = f^\dagger \circ 1_{\mathrm{Dst}\, f} \circ f = f^\dagger \circ f \supseteq 1_{\mathrm{Src}\, f} = 1_{\mathrm{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\mathrm{Src}\, 1_A} \subseteq 1_{\mathrm{Src}\, 1_A}$. □

3 Funcoids

3.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of "spaces" (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement "f is a continuous function from a space μ to a space ν" can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see my yet unpublished article "Generalized continuity" for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A\, \delta\, B$ is defined for any sets A and B. We will extend it from sets to filter objects by the formula:

$$\mathcal{A}\, \delta'\, \mathcal{B} \Leftrightarrow \forall A \in \mathrm{up}\, \mathcal{A}, B \in \mathrm{up}\, \mathcal{B}: A\, \delta\, B.$$

Then (as will be proved below) exist two functions $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A}\, \delta'\, \mathcal{B} \Leftrightarrow \mathcal{B} \cap \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap \beta \mathcal{B} \neq \emptyset.$$

The pair $(\alpha; \beta)$ is called *funcoid* when $\mathcal{B} \cap \alpha \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap \beta \mathcal{B} \neq \emptyset$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of discrete funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by $\langle f \rangle|_{\mathscr{P}\mho}$ that is by values of function $\langle f \rangle$ on sets.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^{\mho}$. Then we define $\alpha' X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\alpha X \mid x \in X\}$ for any set X. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathscr{P}\mho}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p exists unique funcoid such that $\langle f \rangle|_{\mathscr{P}\mho} = p$; in this case $\langle f \rangle|_{\mathscr{P}\mho} \in \mathscr{P}\mho^{\mathscr{P}\mho}$.

For any binary relation p exists unique funcoid f such that $\forall X \in \mathscr{P}\mho$: $\langle f \rangle X = \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *discrete*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of discrete funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \left\langle (g \circ f)^{-1} \right\rangle = \left\langle f^{-1} \right\rangle \circ \left\langle g^{-1} \right\rangle.$$

By the same formulas we can define composition of any two funcoids.

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below will be defined domain and image of a funcoid (the domain and the image of a funcoid are filter objects).

3.2 Basic definitions

Definition 18. Let's call *a funcoid* a pair $(\alpha; \beta)$ where $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{Y} \cap^{\mathfrak{F}} \alpha \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta \mathcal{Y} \neq \emptyset).$$

Definition 19. $\langle (\alpha; \beta) \rangle \overset{\text{def}}{=} \alpha$ for a funcoid $(\alpha; \beta)$.

Definition 20. $(\alpha; \beta)^{-1} = (\beta; \alpha)$ for a funcoid $(\alpha; \beta)$.

Proposition 21. If f is a funcoid then f^{-1} is also a funcoid.

Proof. Follows from symmetry in the definition of funcoid. □

Obvious 22. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 23. The relation $[f] \in \mathscr{P}\mathfrak{F}^2$ is defined by the formula (for any filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f)

$$\mathcal{X} [f] \mathcal{Y} \overset{\text{def}}{=} \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset.$$

Obvious 24. $\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle \mathcal{Y}$ for any filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f.

Obvious 25. $\left[f^{-1} \right] = [f]^{-1}$ for a funcoid f.

Theorem 26.

1. For given value of $\langle f \rangle$ exists no more than one funcoid f.
2. For given value of $[f]$ exists no more than one funcoid f.

Proof. Let f and g are funcoids.

Obviously $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\left\langle f^{-1} \right\rangle = \left\langle g^{-1} \right\rangle \Rightarrow [f] = [g]$. So enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} [f] \mathcal{Y} \Leftrightarrow \mathcal{X} [g] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset$ and consequently $\langle f \rangle \mathcal{X} = \langle g \rangle \mathcal{X}$ for any f.o. $\mathcal{X}$ and $\mathcal{Y}$ because the set of filter objects is separable [5], thus $\langle f \rangle = \langle g \rangle$. □

Proposition 27. $\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})=\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}$ for any funcoid f and $\mathcal{I},\mathcal{J}\in\mathfrak{F}$.

Proof.

$$\begin{aligned}
\star\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J}) &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\right\} &= \text{(by corollary 10 in [5])}\\
\left\{\mathcal{Y}\in\mathfrak{F}\mid(\mathcal{I}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y})\cup^{\mathfrak{F}}(\mathcal{J}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y})\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{I}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\vee\mathcal{J}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{I}\neq\emptyset\vee\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{J}\neq\emptyset\right\} &= \\
\left\{\mathcal{Y}\in\mathfrak{F}\mid(\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{I})\cup^{\mathfrak{F}}(\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{J})\neq\emptyset\right\} &= \text{(by corollary 10 in [5])}\\
\left\{\mathcal{Y}\in\mathfrak{F}\mid\mathcal{Y}\cap^{\mathfrak{F}}(\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J})\neq\emptyset\right\} &= \\
\star(\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}).&
\end{aligned}$$

Thus $\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})=\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}$ because $\mathfrak{F}$ is separable. □

3.2.1 Composition of funcoids

Definition 28. *Composition* of funcoids is defined by the formula

$$(\alpha_2;\beta_2)\circ(\alpha_1;\beta_1)=(\alpha_2\circ\alpha_1;\beta_1\circ\beta_2).$$

Proposition 29. If f, g are funcoids then $g\circ f$ is funcoid.

Proof. Let $f=(\alpha_1;\beta_1)$, $g=(\alpha_2;\beta_2)$.

$\mathcal{Y}\cap^{\mathfrak{F}}(\alpha_2\circ\alpha_1)\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\alpha_2\alpha_1\mathcal{X}\neq\emptyset\Leftrightarrow\alpha_1\mathcal{X}\cap^{\mathfrak{F}}\beta_2\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\beta_1\beta_2\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}(\beta_1\circ\beta_2)\mathcal{Y}\neq\emptyset$.

So $(\alpha_2\circ\alpha_1;\beta_1\circ\beta_2)$ is a funcoid. □

Obvious 30. $\langle g\circ f\rangle=\langle g\rangle\circ\langle f\rangle$ for any funcoids f and g.

Proposition 31. $(h\circ g)\circ f=h\circ(g\circ f)$ for any funcoids f, g, h.

Proof.
$\langle(h\circ g)\circ f\rangle=\langle h\circ g\rangle\circ\langle f\rangle=(\langle h\rangle\circ\langle g\rangle)\circ\langle f\rangle=\langle h\rangle\circ(\langle g\rangle\circ\langle f\rangle)=\langle h\rangle\circ\langle g\circ f\rangle=\langle h\circ(g\circ f)\rangle$. □

Theorem 32. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for any funcoids f and g.

Proof. $\left\langle(g\circ f)^{-1}\right\rangle=\left\langle f^{-1}\right\rangle\circ\left\langle g^{-1}\right\rangle=\left\langle f^{-1}\circ g^{-1}\right\rangle$. □

3.3 Funcoid as continuation

Theorem 33. For any funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\operatorname{up}\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X},Y\in\operatorname{up}\mathcal{Y}:X[f]Y$.

Proof. 2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\operatorname{up}\mathcal{Y}:Y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\operatorname{up}\mathcal{Y}:\mathcal{X}[f]Y$.
Analogously $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}:X[f]\mathcal{Y}$. Combining these two equalities we get

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X},Y\in\operatorname{up}\mathcal{Y}:X[f]Y.$$

1. $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset$
Let's denote $W=\left\{\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\mid X\in\operatorname{up}\mathcal{X}\right\}$. We will prove that W is a generalized filter base. To prove this enough to show that $V=\{\langle f\rangle X\mid X\in\operatorname{up}\mathcal{X}\}$ is a generalized filter base.

Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle A$, $\mathcal{Q}=\langle f\rangle B$ where $A,B\in\operatorname{up}\mathcal{X}$; $A\cap B\in\operatorname{up}\mathcal{X}$ and $\mathcal{R}\subseteq\mathcal{P}\cap^{\mathfrak{F}}\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle(A\cap B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$\emptyset \notin W \Leftrightarrow \bigcap^{\mathfrak{F}} W \not\ni \emptyset$ by the corollary 4 of the theorem 3. That is

$$\forall X \in \mathrm{up}\,\mathcal{X} \colon \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle X \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \mathrm{up}\,\mathcal{X} \neq \emptyset.$$

Comparing with the above, $\mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \mathrm{up}\,\mathcal{X} \neq \emptyset$. So $\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \langle f \rangle \rangle \mathrm{up}\,\mathcal{X}$. □

Theorem 34.

1. A function $\alpha \in \mathfrak{F}^{\mathscr{P}\mho}$ conforming to the formulas (for any $I, J \in \mathscr{P}\mho$)

$$\alpha\emptyset = \emptyset, \quad \alpha(I \cup J) = \alpha I \cup \alpha J$$

can be continued to the function $\langle f \rangle$ for a unique funcoid f;

$$\langle f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha \rangle \mathrm{up}\,\mathcal{X} \tag{1}$$

for any filter object $\mathcal{X}$.

2. A relation $\delta \in \mathscr{P}(\mathscr{P}\mho)^2$ conforming to the formulas (for any $I, J, K \in \mathscr{P}\mho$)

$$\begin{array}{l} \neg(\emptyset\,\delta\,I), \;\; I \cup J\,\delta\,K \Leftrightarrow I\,\delta\,K \vee J\,\delta\,K, \\ \neg(I\,\delta\,\emptyset), \;\; K\,\delta\,I \cup J \Leftrightarrow K\,\delta\,I \vee K\,\delta\,J \end{array} \tag{2}$$

can be continued to the relation $[f]$ for unique funcoid f;

$$\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y} \colon X\,\delta\,Y \tag{3}$$

for any filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (1) and (3) follow from the previous theorem.

2. Let define $\alpha \in \mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\alpha X) = \{Y \in \mathscr{P}\mho \mid X\,\delta\,Y\}$ for any $X \in \mathscr{P}\mho$. Analogously can be defined $\beta \in \mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\beta X) = \{X \in \mathscr{P}\mho \mid X\,\delta\,Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}^{\mathfrak{F}}$ and $\beta' \in \mathfrak{F}^{\mathfrak{F}}$ by the formulas

$$\alpha'\mathcal{X} = \bigcap^{\mathfrak{F}} \langle \alpha \rangle \mathrm{up}\,\mathcal{X} \quad \text{and} \quad \beta'\mathcal{X} = \bigcap^{\mathfrak{F}} \langle \beta \rangle \mathrm{up}\,\mathcal{X}.$$

and δ to $\delta' \in \mathscr{P}\mathfrak{F}^2$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y} \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y} \colon X\,\delta\,Y.$$

$\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle \alpha \rangle \mathrm{up}\,\mathcal{X} \neq \emptyset \Leftrightarrow \bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle \alpha \rangle \mathrm{up}\,\mathcal{X} \neq \emptyset$. Let's prove that

$$W = \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle \alpha \rangle \mathrm{up}\,\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle \alpha \rangle \mathrm{up}\,\mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle \alpha \rangle \mathrm{up}\,\mathcal{X}$ then exist $X_1, X_2 \in \mathrm{up}\,\mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{A} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle \alpha \rangle \mathrm{up}\,\mathcal{X}$. So $\langle \alpha \rangle \mathrm{up}\,\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

Accordingly the corollary 4 of the theorem 3, $\bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle \alpha \rangle \mathrm{up}\,\mathcal{X} \neq \emptyset$ is equivalent to

$$\forall X \in \mathrm{up}\,\mathcal{X} \colon \mathcal{Y} \cap^{\mathfrak{F}} \alpha X \neq \emptyset,$$

what is equivalent to $\forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y} \colon Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y} \colon Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \mathrm{up}\,\mathcal{X}, Y \in \mathrm{up}\,\mathcal{Y} \colon X\,\delta\,Y$. Combining the equivalencies we get $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$. Analogously $\mathcal{X} \cap^{\mathfrak{F}} \beta'\mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$. So $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta'\mathcal{Y} \neq \emptyset$, that is $(\alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}\,\delta'\,\mathcal{Y}$ follows that $[(\alpha'; \beta')]$ is a continuation of δ.

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}\mho)^2$ by the formula $X\,\delta\,Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$. Then the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some funcoid f.

$\forall X, Y \in \mathscr{P}\mho \colon X[f]Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$, consequently $\forall X \in \mathscr{P}\mho \colon \alpha X = \langle f \rangle X$. So $\langle f \rangle$ is a continuation of α. □

Note that by the last theorem to every proximity δ corresponds exactly one funcoid. So funcoids are a generalization of proximity structures.

Definition 35. Any (multivalued) function f will be considered as a funcoid, where by definition $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\mathcal{X}$ for any $\mathcal{X}\in\mathfrak{F}$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff.

Definition 36. Funcoids corresponding to binary relation are called *discrete funcoids*.

We may equate discrete funcoids with corresponding binary relations by the method of appendix B in [5]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below). For simplicity I will not dive here into formal definition of equating discrete funcoids with binary relations (by the method shown in appendix B in [5]) but we simply will (informally) assume that discrete funcoids can be equated with binary relations.

I will denote FCD the set of funcoids or the category of funcoids (see below) dependently on context.

3.4 Lattice of funcoids

Definition 37. $f\subseteq g\overset{\mathrm{def}}{=}[f]\subseteq[g]$ for $f,g\in\mathsf{FCD}$.

Thus FCD is a poset.

Conjecture 38. The filtrator of funcoids is:

1. with separable core;
2. with co-separable core.

Definition 39. I will call the *filtrator of funcoids* (see [5] for the definition of filtrators) the filtrator $(\mathsf{FCD};\mathscr{P}\mho^2)$.

Theorem 40. The set of funcoids is a complete lattice. For any $R\in\mathscr{P}\mathsf{FCD}$ and $X,Y\in\mathscr{P}\mho$

1. $X\left[\bigcup^{\mathsf{FCD}}R\right]Y\Leftrightarrow\exists f\in R: X[f]Y$;
2. $\left\langle\bigcup^{\mathsf{FCD}}R\right\rangle X=\bigcup^{\mathfrak{F}}\{\langle f\rangle X\mid f\in R\}$.

Proof.

2. $\langle h\rangle X\overset{\mathrm{def}}{=}\bigcup^{\mathfrak{F}}\{\langle f\rangle X\mid f\in R\}$. $\langle h\rangle\emptyset=\emptyset$;

$$\begin{aligned}\langle h\rangle(I\cup J) &= \bigcup^{\mathfrak{F}}\{\langle f\rangle(I\cup J)\mid f\in R\}\\ &= \bigcup^{\mathfrak{F}}\{\langle f\rangle(I\cup^{\mathfrak{F}}J)\mid f\in R\}\\ &= \bigcup^{\mathfrak{F}}\{\langle f\rangle I\cup^{\mathfrak{F}}\langle f\rangle J\mid f\in R\}\\ &= \bigcup^{\mathfrak{F}}\{\langle f\rangle I\mid f\in R\}\cup^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\{\langle f\rangle J\mid f\in R\}\\ &= \langle h\rangle I\cup^{\mathfrak{F}}\langle h\rangle J.\end{aligned}$$

So $\langle h\rangle$ can be continued to a funcoid. Obviously

$$\forall f\in R: h\supseteq f. \tag{4}$$

And h is the least funcoid for which holds the condition (4). So $h=\bigcup^{\mathsf{FCD}}R$.

1. $X\left[\bigcup^{\mathsf{FCD}}R\right]Y\Leftrightarrow Y\cap^{\mathfrak{F}}\left\langle\bigcup^{\mathsf{FCD}}R\right\rangle X\neq\emptyset\Leftrightarrow Y\cap^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\{\langle f\rangle X\mid f\in R\}\neq\emptyset\Leftrightarrow\exists f\in R$: $Y\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset\Leftrightarrow\exists f\in R: X[f]Y$. □

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 41. For any funcoids f and g and a filter object $\mathcal{X}$

1. $\langle f \cup^{\mathsf{FCD}} g \rangle \mathcal{X} = \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$;
2. $[f \cup^{\mathfrak{F}} g] = [f] \cup [g]$.

Proof.

1. Let $\langle h \rangle \mathcal{X} \overset{\text{def}}{=} \langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}$; $\langle h^{-1} \rangle \mathcal{Y} \overset{\text{def}}{=} \langle f^{-1} \rangle \mathcal{Y} \cup^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y}$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. Then

$$\begin{aligned} \mathcal{Y} \cap^{\mathfrak{F}} \langle h \rangle \mathcal{X} \neq \emptyset &\Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{Y} \neq \emptyset \vee \mathcal{X} \cap^{\mathfrak{F}} \langle g^{-1} \rangle \mathcal{Y} \neq \emptyset \\ &\Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle h^{-1} \rangle \mathcal{Y} \neq \emptyset. \end{aligned}$$

   So h is a funcoid. Consequently $f \cup^{\mathsf{FCD}} g = h$.

2. $\mathcal{X}[f \cup^{\mathsf{FCD}} g]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \cup^{\mathsf{FCD}} g \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} (\langle f \rangle \mathcal{X} \cup^{\mathfrak{F}} \langle g \rangle \mathcal{X}) \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset \vee \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}[f]\mathcal{Y} \vee \mathcal{X}[g]\mathcal{Y}$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. □

3.5 More on composition of funcoids

Proposition 42. $[g \circ f] = [g] \circ \langle f \rangle = \langle g^{-1} \rangle^{-1} \circ [f]$ for $f, g \in \mathsf{FCD}$.

Proof. $\mathcal{X}[g \circ f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \circ f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g \rangle \langle f \rangle \mathcal{X} \neq \emptyset \Leftrightarrow \langle f \rangle \mathcal{X}[g]\mathcal{Y} \Leftrightarrow \mathcal{X}([g] \circ \langle f \rangle)\mathcal{Y}$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$. $[g \circ f] = [(f^{-1} \circ g^{-1})^{-1}] = [f^{-1} \circ g^{-1}]^{-1} = ([f^{-1}] \circ \langle g^{-1} \rangle)^{-1} = \langle g^{-1} \rangle^{-1} \circ [f]$. □

The following theorem is the variant for funcoids of the statement (which defines compositions of relations) that $x(g \circ f)z \Leftrightarrow \exists y(x f y \wedge y g z)$ for any x and z and any binary relations f and g.

Theorem 43. For any $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$ and $f, g \in \mathsf{FCD}$

$$\mathcal{X}[g \circ f]\mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X}[f]y \wedge y[g]\mathcal{Z}).$$

Proof.

$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X}[f]y \wedge y[g]\mathcal{Z}) &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle y \neq \emptyset \wedge y \cap^{\mathfrak{F}} \langle f \rangle \mathcal{X} \neq \emptyset) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle y \neq \emptyset \wedge y \subseteq \langle f \rangle \mathcal{X}) \\ &\Rightarrow \mathcal{Z} \cap^{\mathfrak{F}} \langle g \rangle \langle f \rangle \mathcal{X} \neq \emptyset \\ &\Leftrightarrow \mathcal{X}[g \circ f]\mathcal{Z}. \end{aligned}$$

Reversely, if $\mathcal{X}[g \circ f]\mathcal{Z}$ then $\langle f \rangle \mathcal{X}[g]\mathcal{Z}$, consequently exists $y \in \operatorname{atoms}^{\mathfrak{F}} \langle f \rangle \mathcal{X}$ such that $y[g]\mathcal{Z}$; we have $\mathcal{X}[f]y$. □

Theorem 44. If f, g, h are funcoids then

1. $f \circ (g \cup^{\mathsf{FCD}} h) = f \circ g \cup^{\mathsf{FCD}} f \circ h$;
2. $(g \cup^{\mathsf{FCD}} h) \circ f = g \circ f \cup^{\mathsf{FCD}} h \circ f$.

Proof. I will prove only the first equality because the other is analogous.

For any $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$

$$\begin{aligned} \mathcal{X}[f \circ (g \cup^{\mathsf{FCD}} h)]\mathcal{Z} &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X}[g \cup^{\mathsf{FCD}} h]y \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : ((\mathcal{X}[g]y \vee \mathcal{X}[h]y) \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X}[g]y \wedge y[f]\mathcal{Z} \vee \mathcal{X}[h]y \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X}[g]y \wedge y[f]\mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\mathcal{X}[h]y \wedge y[f]\mathcal{Z}) \\ &\Leftrightarrow \mathcal{X}[f \circ g]\mathcal{Z} \vee \mathcal{X}[f \circ h]\mathcal{Z} \\ &\Leftrightarrow \mathcal{X}[f \circ g \cup^{\mathsf{FCD}} f \circ h]\mathcal{Z}. \end{aligned}$$

□

3.6 Domain and range of a funcoid

Definition 45. Let $\mathcal{A}\in\mathfrak{F}$. The *identity funcoid* $I_{\mathcal{A}}=(\mathcal{A}\cap^{\mathfrak{F}};\mathcal{A}\cap^{\mathfrak{F}})$.

Proposition 46. The identity funcoid is a funcoid.

Proof. We need to prove that $(\mathcal{A}\cap^{\mathfrak{F}}\mathcal{X})\cap^{\mathfrak{F}}\mathcal{Y}\neq\emptyset\Leftrightarrow(\mathcal{A}\cap^{\mathfrak{F}}\mathcal{Y})\cap^{\mathfrak{F}}\mathcal{X}\neq\emptyset$ what is obvious. □

Obvious 47. $(I_{\mathcal{A}})^{-1}=I_{\mathcal{A}}$.

Obvious 48. $\mathcal{X}[I_{\mathcal{A}}]\mathcal{Y}\Leftrightarrow\mathcal{A}\cap^{\mathfrak{F}}\mathcal{X}\cap^{\mathfrak{F}}\mathcal{Y}\neq\emptyset$ for any $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$.

Definition 49. I will define *restricting* of a funcoid f to a filter object $\mathcal{A}$ by the formula $f|_{\mathcal{A}}\overset{\text{def}}{=}f\circ I_{\mathcal{A}}$.

Obviously the last definition does not contradict to the previous.

Definition 50. *Image* of a funcoid f will be defined by the formula $\operatorname{im} f=\langle f\rangle\mho$.
Domain of a funcoid f is defined by the formula $\operatorname{dom} f=\operatorname{im} f^{-1}$.

Proposition 51. $\langle f\rangle\mathcal{X}=\langle f\rangle(\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f)$ for any $f\in\mathsf{FCD}$, $\mathcal{X}\in\mathfrak{F}$.

Proof. For any filter object $\mathcal{Y}$ we have $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle(\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f)\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\operatorname{im} f^{-1}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$. □

Proposition 52. $\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f\neq\emptyset\Leftrightarrow\langle f\rangle\mathcal{X}\neq\emptyset$ for any $f\in\mathsf{FCD}$, $\mathcal{X}\in\mathfrak{F}$.

Proof. $\mathcal{X}\cap^{\mathfrak{F}}\operatorname{dom} f\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mho\neq\emptyset\Leftrightarrow\mho\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\langle f\rangle\mathcal{X}\neq\emptyset$. □

Corollary 53. $\operatorname{dom} f=\bigcup^{\mathfrak{F}}\{a\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mho,\langle f\rangle a\neq\emptyset\}$.

Proof. This follows from that $\mathfrak{F}$ is an atomistic lattice. □

3.7 Category of funcoids

I will define the category FCD of funcoids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f;\mathcal{A};\mathcal{B})$ where f is a funcoid such that $\operatorname{dom} f\subseteq\mathcal{A}$, $\operatorname{im} f\subseteq\mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}};\mathcal{A};\mathcal{A})$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 54. For any funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcup^{\mathfrak{F}}\langle\langle f\rangle\rangle\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{X},y\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{Y}:x[f]y$.

Proof. 1.

$$\begin{aligned}\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}: x\cap^{\mathfrak{F}}\langle f^{-1}\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{X}: \mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset.\end{aligned}$$

$\partial\langle f\rangle\mathcal{X} = \bigcup \langle\partial\rangle\langle\langle f\rangle\rangle\text{atoms}^{\mathfrak{F}}\mathcal{X} = \partial\bigcup \langle\langle f\rangle\rangle\text{atoms}^{\mathfrak{F}}\mathcal{X}$.

2. If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$, consequently exists $y\in\text{atoms}^{\mathfrak{F}}\mathcal{Y}$ such that $y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$, $\mathcal{X}[f]y$. Repeating this second time we get that there exist $x\in\text{atoms}^{\mathfrak{F}}\mathcal{X}$ such that $x[f]y$. From this follows

$$\exists x\in\text{atoms}^{\mathfrak{F}}\mathcal{X}, y\in\text{atoms}^{\mathfrak{F}}\mathcal{Y}: x[f]y.$$

The reverse is obvious. □

Theorem 55.

1. A function $\alpha\in\mathfrak{F}^{\text{atoms}^{\mathfrak{F}}\mho}$ such that (for any $a\in\text{atoms}^{\mathfrak{F}}\mho$)

$$\alpha a\supseteq\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\text{atoms}^{\mathfrak{F}}\right\rangle\text{up}\, a \tag{5}$$

can be continued to the function $\langle f\rangle$ for a unique funcoid f;

$$\langle f\rangle\mathcal{X} = \bigcup^{\mathfrak{F}}\langle\alpha\rangle\text{atoms}^{\mathfrak{F}}\mathcal{X} \tag{6}$$

for any filter object $\mathcal{X}$.

2. A relation $\delta\in\mathscr{P}(\text{atoms}^{\mathfrak{F}}\mho)^2$ such that (for any $a,b\in\text{atoms}^{\mathfrak{F}}\mho$)

$$\forall X\in\text{up}\, a, Y\in\text{up}\, b\,\exists x\in\text{atoms}^{\mathfrak{F}}X, y\in\text{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Rightarrow a\,\delta\,b \tag{7}$$

can be continued to the relation $[f]$ for a unique funcoid f;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\text{atoms}^{\mathfrak{F}}\mathcal{X}, y\in\text{atoms}^{\mathfrak{F}}\mathcal{Y}: x\,\delta\,y \tag{8}$$

for any filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the previous theorem.

1. Consider the function $\alpha'\in\mathfrak{F}^{\mho}$ defined by the formula (for any $X\in\mathscr{P}\mho$)

$$\alpha' X = \bigcup^{\mathfrak{F}}\langle\alpha\rangle\text{atoms}^{\mathfrak{F}}X.$$

Obviously $\alpha'\emptyset=\emptyset$. For any $I,J\in\mathscr{P}\mho$

$$\begin{aligned}\alpha'(I\cup J) &= \bigcup^{\mathfrak{F}}\langle\alpha'\rangle\text{atoms}^{\mathfrak{F}}(I\cup J)\\ &= \bigcup^{\mathfrak{F}}\langle\alpha'\rangle(\text{atoms}^{\mathfrak{F}}I\cup\text{atoms}^{\mathfrak{F}}J)\\ &= \bigcup^{\mathfrak{F}}\big(\langle\alpha'\rangle\text{atoms}^{\mathfrak{F}}I\cup\langle\alpha'\rangle\text{atoms}^{\mathfrak{F}}J\big)\\ &= \bigcup^{\mathfrak{F}}\langle\alpha'\rangle\,\text{atoms}^{\mathfrak{F}}I\cup^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\langle\alpha'\rangle\text{atoms}^{\mathfrak{F}}.\\ &= \alpha' I\cup^{\mathfrak{F}}\alpha' J.\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 25): $\langle f\rangle\mathcal{X} = \bigcap^{\mathfrak{F}}\langle\alpha'\rangle\text{up}\,\mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\text{atoms}^{\mathfrak{F}}\right\rangle\text{up}\, a &= \bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right\rangle\left\langle\text{atoms}^{\mathfrak{F}}\right\rangle\text{up}\, a\\ &\supseteq \bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\ &= \bigcap^{\mathfrak{F}}\left\{\left(\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right)\{a\}\right\}\\ &= \bigcap^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\langle\alpha\rangle\{a\}\right\}\\ &= \bigcap^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\{\alpha a\}\right\} = \bigcap^{\mathfrak{F}}\{\alpha a\} = \alpha a.\end{aligned}$$

Finally,

$$\alpha a = \bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\text{atoms}^{\mathfrak{F}}\right\rangle\text{up}\, a = \bigcap^{\mathfrak{F}}\langle\alpha'\rangle\text{up}\, a = \langle f\rangle a,$$

so $\langle f\rangle$ is a continuation of α.

2. Consider the relation $\delta'\in\mathscr{P}(\mathscr{P}\mho)^2$ defined by the formula (for any $X,Y\in\mathscr{P}\mho$)

$$X\,\delta'\,Y\Leftrightarrow\exists x\in\text{atoms}^{\mathfrak{F}}X, y\in\text{atoms}^{\mathfrak{F}}Y: x\,\delta\,y.$$

Obviously $\neg(X\,\delta'\,\emptyset)$ and $\neg(\emptyset\,\delta'\,Y)$.

$$\begin{aligned}(I\cup J)\,\delta'\,Y &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}(I\cup J), y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}I\cup\operatorname{atoms}^{\mathfrak{F}}J, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow \exists x\in\operatorname{atoms}^{\mathfrak{F}}I, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\vee\exists x\in\operatorname{atoms}^{\mathfrak{F}}J, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow I\,\delta'\,Y\vee J\,\delta'\,Y;\end{aligned}$$

analogously $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$. Let's continue δ' till a funcoid f (by the theorem 25):

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}, Y\in\operatorname{up}\mathcal{Y}: X\,\delta'\,Y$$

The reverse of (7) implication is trivial, so

$$\forall X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow\forall X\in\operatorname{up}a, Y\in\operatorname{up}b: X\,\delta'\,Y\Leftrightarrow a[f]b$.

So $a\,\delta\,b\Leftrightarrow a[f]b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is proof of the following theorem:

Theorem 56. If R is a set of funcoids, $x,y\in\operatorname{atoms}^{\mathfrak{F}}\mho$, then

1. $\langle\bigcap^{\mathsf{FCD}}R\rangle x=\bigcap^{\mathfrak{F}}\{\langle f\rangle x\mid f\in R\}$;
2. $x[\bigcap^{\mathsf{FCD}}R]y\Leftrightarrow\forall f\in R: x[f]y$.

Proof. 2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R: x[f]y$.

$$\begin{gathered}\forall X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow\\ \forall f\in R, X\in\operatorname{up}a, Y\in\operatorname{up}b\,\exists x\in\operatorname{atoms}^{\mathfrak{F}}X, y\in\operatorname{atoms}^{\mathfrak{F}}Y: x[f]y\Rightarrow\\ \forall f\in R, X\in\operatorname{up}a, Y\in\operatorname{up}b: X[f]Y\Rightarrow\\ \forall f\in R: a[f]b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 55, δ can be continued till $[p]$ for some funcoid p.

For any funcoid q such that $\forall f\in R: q\subseteq f$ we have $x[q]y\Rightarrow\forall f\in R: x[f]y\Leftrightarrow x\,\delta\,y\Leftrightarrow x[p]y$, so $q\subseteq f$. Consequently $p=\bigcap^{\mathsf{FCD}}R$.

From this $x[\bigcap^{\mathsf{FCD}}R]y\Leftrightarrow\forall f\in R: x[f]y$.

1. From the former $y\cap^{\mathfrak{F}}\langle\bigcap^{\mathsf{FCD}}R\rangle x\neq\emptyset\Leftrightarrow\forall f\in R: y\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset$ for any $y\in\operatorname{atoms}^{\mathfrak{F}}\mho$. From this follows what we need to prove. □

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is direct product of two filter objects as defined in the theory of funcoids:

Definition 57. *Direct product* of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ that

$$\mathcal{X}[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}]\mathcal{Y}\Leftrightarrow\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\wedge\mathcal{Y}\cap^{\mathfrak{F}}\mathcal{B}\neq\emptyset.$$

Proposition 58. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a funcoid and

$$\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if } \mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset;\\ \emptyset & \text{if } \mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}=\emptyset.\end{cases}$$

Proof. Obvious. □

Obvious 59. $A\times B=A\times^{\mathsf{FCD}}B$ for sets A and B.

Proposition 60. $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\Leftrightarrow\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}$ for any $f\in\mathsf{FCD}$ and $\mathcal{A},\mathcal{B}\in\mathfrak{F}$.

Proof. If $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ then $\operatorname{dom}f\subseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\subseteq\mathcal{A}$, $\operatorname{im}f\subseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\subseteq\mathcal{B}$. If $\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}$ then

$$\forall\mathcal{X},\mathcal{Y}\in\mathfrak{F}:(\mathcal{X}[f]\mathcal{Y}\Rightarrow\mathcal{X}\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\wedge\mathcal{Y}\cap^{\mathfrak{F}}\mathcal{B}\neq\emptyset);$$

consequently $f\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 61. $f\cap^{\mathsf{FCD}}\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}=I_{\mathcal{B}}\circ f\circ I_{\mathcal{A}}$ for any $f\in\mathsf{FCD}$ and $\mathcal{A},\mathcal{B}\in\mathfrak{F}$.

Proof. $h\overset{\text{def}}{=}I_{\mathcal{B}}\circ f\circ I_{\mathcal{A}}$. For any $\mathcal{X}\in\mathfrak{F}$

$$\langle h\rangle\mathcal{X}=\langle I_{\mathcal{B}}\rangle\langle f\rangle\langle I_{\mathcal{A}}\rangle\mathcal{X}=\mathcal{B}\cap\langle f\rangle(\mathcal{A}\cap\mathcal{X}).$$

From this, as easy to show, $h\subseteq f$ and $h\subseteq\mathcal{A}\times\mathcal{B}$. If $g\subseteq f\wedge g\subseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ for a funcoid g then $\operatorname{dom}g\subseteq\mathcal{A}$, $\operatorname{im}g\subseteq\mathcal{B}$,

$$\langle g\rangle\mathcal{X}=\mathcal{B}\cap^{\mathfrak{F}}\langle g\rangle(\mathcal{A}\cap^{\mathfrak{F}}\mathcal{X})\subseteq\mathcal{B}\cap^{\mathfrak{F}}\langle f\rangle(\mathcal{A}\cap^{\mathfrak{F}}\mathcal{X})=\langle I_{\mathcal{B}}\rangle\langle f\rangle\langle I_{\mathcal{A}}\rangle\mathcal{X}=\langle h\rangle\mathcal{X},$$

$g\subseteq h$. So $h=f\cap^{\mathsf{FCD}}\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

Corollary 62. $f|_{\mathcal{A}}=f\cap\mathcal{A}\times^{\mathsf{FCD}}\mho$ for any $f\in\mathsf{FCD}$ and $\mathcal{A}\in\mathfrak{F}$.

Proof. $f\cap^{\mathsf{FCD}}\mathcal{A}\times^{\mathsf{FCD}}\mho=I_{\mho}\circ f\circ I_{\mathcal{A}}=f\circ I_{\mathcal{A}}=f|_{\mathcal{A}}$. □

Corollary 63. $f\cap^{\mathsf{FCD}}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\neq\emptyset\Leftrightarrow\mathcal{A}[f]\mathcal{B}$ for any $f\in\mathsf{FCD}$, $\mathcal{A},\mathcal{B}\in\mathfrak{F}$.

Proof. $f\cap^{\mathsf{FCD}}\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\neq\emptyset\Leftrightarrow\langle f\cap^{\mathsf{FCD}}\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle\mho\neq\emptyset\Leftrightarrow\langle I_{\mathcal{B}}\circ f\circ I_{\mathcal{A}}\rangle\mho\neq\emptyset\Leftrightarrow\langle I_{\mathcal{B}}\rangle\langle f\rangle\langle I_{\mathcal{A}}\rangle\mho\neq\emptyset\Leftrightarrow\mathcal{B}\cap^{\mathsf{FCD}}\langle f\rangle(\mathcal{A}\cap^{\mathfrak{F}}\mho)\neq\emptyset\Leftrightarrow\mathcal{B}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{A}\neq\emptyset\Leftrightarrow\mathcal{A}[f]\mathcal{B}$. □

Corollary 64. The filtrator of funcoids is star-separable.

Proof. The set of direct products of sets is a separation subset of the lattice of funcoids. □

Theorem 65. If $S\in\mathscr{P}\mathfrak{F}^2$ then

$$\bigcap^{\mathsf{FCD}}\{\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\}=\bigcap^{\mathfrak{F}}\operatorname{dom}S\times^{\mathsf{FCD}}\bigcap^{\mathfrak{F}}\operatorname{im}S.$$

Proof. If $x\in\operatorname{atoms}^{\mathfrak{F}}\mho$ then by the theorem 56

$$\left\langle\bigcap^{\mathsf{FCD}}\{\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\}\right\rangle x=\bigcap^{\mathfrak{F}}\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x\mid(\mathcal{A};\mathcal{B})\in S\}.$$

If $x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom}S\neq\emptyset$ then

$$\forall(\mathcal{A};\mathcal{B})\in S:(x\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\wedge\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x=\mathcal{B});$$
$$\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x\mid(\mathcal{A};\mathcal{B})\in S\}=\operatorname{im}S;$$

if $x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom}S=\emptyset$ then

$$\exists(\mathcal{A};\mathcal{B})\in S:(x\cap^{\mathfrak{F}}\mathcal{A}=\emptyset\wedge\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x=\emptyset);$$
$$\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x\mid(\mathcal{A};\mathcal{B})\in S\}\ni\emptyset.$$

So

$$\left\langle\bigcap^{\mathsf{FCD}}\{\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\}\right\rangle x=\begin{cases}\bigcap^{\mathfrak{F}}\operatorname{im}S & \text{if } x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom}S\neq\emptyset;\\ \emptyset & \text{if } x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom}S=\emptyset.\end{cases}$$

From this follows the statement of the theorem. □

Corollary 66. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \cap^{\mathsf{FCD}} \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 = (\mathcal{A}_0 \cap^{\mathsf{FCD}} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathsf{FCD}} \mathcal{B}_1)$ for any $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \cap^{\mathsf{FCD}} \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 = \bigcap^{\mathfrak{F}} \left\{ \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \right\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap^{\mathsf{FCD}} \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap^{\mathsf{FCD}} \mathcal{B}_1)$. □

Theorem 67. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice FCD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Proof. Let $S \in \mathscr{P}\mathfrak{F}$, $X \in \mathscr{P}\mho$, $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$.

$$\begin{aligned}
\left\langle \bigcup^{\mathsf{FCD}} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle X &= \bigcup^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle X \mid \mathcal{B} \in S \} \\
&= \begin{cases} \bigcup^{\mathfrak{F}} S & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } X \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\
&= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcup^{\mathfrak{F}} S \right\rangle X; \\
\left\langle \bigcap^{\mathsf{FCD}} \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle S \right\rangle x &= \bigcap^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid \mathcal{B} \in S \} \\
&= \begin{cases} \bigcap^{\mathfrak{F}} S & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \\ \emptyset & \text{if } x \cap^{\mathfrak{F}} \mathcal{A} = \emptyset \end{cases} \\
&= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcap^{\mathfrak{F}} S \right\rangle x.
\end{aligned}$$

If $\mathcal{A} \neq \emptyset$ then obviously the function $\mathcal{A} \times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a direct product of filter objects) funcoid (of atomic width).

Proposition 68. If a is an atomic filter object, $f \in \mathsf{FCD}$ then $f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a$.

Proof. Let $\mathcal{X} \in \mathfrak{F}$.

$$\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \langle f|_a \rangle \mathcal{X} = \emptyset.$$ □

3.10 Atomic funcoids

Theorem 69. A funcoid is an atom of the lattice of funcoids iff it is direct product of two atomic filter objects.

Proof.

Direct implication. Let f is an atomic funcoid. Let's get elements $a \in \operatorname{atoms}^{\mathfrak{F}} \operatorname{dom} f$ and $b \in \operatorname{atoms}^{\mathfrak{F}} \langle f \rangle a$. Then for any $\mathcal{X} \in \mathfrak{F}$

$$\mathcal{X} \cap^{\mathfrak{F}} a = \emptyset \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = \emptyset \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is an atomic funcoid $f = a \times^{\mathsf{FCD}} b$.

Reverse implication. Let a, $b \in \operatorname{atoms}^{\mathfrak{F}} \mho$, $f \in \mathsf{FCD}$. If $b \cap^{\mathfrak{F}} \langle f \rangle a = \emptyset$ then $\neg(a [f] b)$, $f \cap^{\mathfrak{F}} a \times^{\mathsf{FCD}} b = \emptyset$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}$: $(\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \cap^{\mathsf{FCD}} a \times^{\mathsf{FCD}} b = \emptyset \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic filter object. □

Theorem 70. The lattice of funcoids is atomic.

Proof. Let f is a non-empty funcoid. Then $\operatorname{dom} f \neq \emptyset$, thus by the theorem 46 in [5] exists $a \in \operatorname{atoms} \operatorname{dom} f$. So $\langle f \rangle a \neq \emptyset$ thus exists $b \in \operatorname{atoms} \langle f \rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 71. The lattice of funcoids is atomically separable.

Proof. Let $f, g \in \mathsf{FCD}$, $f \subset g$. Then exists $a \in \operatorname{atoms}^{\mathfrak{F}} \mho$ such that $\langle f \rangle a \subset \langle g \rangle a$. So because the lattice $\mathfrak{F}$ is atomically separable then exists $b \in \operatorname{atoms}^{\mathfrak{F}} \mho$ such that $\langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset$ and $b \subseteq \langle g \rangle a$. For any $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$

$$\langle f \rangle a \cap^{\mathfrak{F}} \left\langle a \times^{\mathsf{FCD}} b \right\rangle a = \langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset,$$
$$x \neq a \Rightarrow \langle f \rangle x \cap^{\mathfrak{F}} \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \emptyset = \emptyset$$

Thus $\langle f \rangle x \cap^{\mathfrak{F}} \langle a \times b \rangle x = \emptyset$ and consequently $f \cap^{\mathsf{FCD}} a \times^{\mathsf{FCD}} b = \emptyset$.

$$\left\langle a \times^{\mathsf{FCD}} b \right\rangle a = b \subseteq \langle g \rangle a,$$
$$x \neq a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = \emptyset \subseteq \langle g \rangle a.$$

Thus $\left\langle a \times^{\mathsf{FCD}} b \right\rangle x = b \subseteq \langle g \rangle x$ and consequently $a \times^{\mathsf{FCD}} b \subseteq g$.

So the lattice of funcoids is separable by the theorem 19 in [5]. □

Corollary 72. The lattice of funcoids is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [5]. □

Remark 73. For more ways to characterize (atomic) separability of the lattice of funcoids see [5], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 74. The lattice of funcoids is an atomistic lattice.

Proof. Let f is a funcoid. Suppose contrary to the statement to be proved that $\bigcup^{\mathfrak{F}} \operatorname{atoms}^{\mathsf{FCD}} f \subset f$. Then exists $a \in \operatorname{atoms}^{\mathsf{FCD}} f$ such that $a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \operatorname{atoms}^{\mathsf{FCD}} f = \emptyset$ what is impossible. □

Proposition 75. $\operatorname{atoms}^{\mathsf{FCD}}(f \cup^{\mathfrak{F}} g) = \operatorname{atoms}^{\mathsf{FCD}} f \cup \operatorname{atoms}^{\mathsf{FCD}} g$ for any funcoids f and g.

Proof. $a \times^{\mathsf{FCD}} b \cap^{\mathsf{FCD}} (f \cup^{\mathsf{FCD}} g) \neq \emptyset \Leftrightarrow a\left[f \cup^{\mathsf{FCD}} g\right] b \Leftrightarrow a[f] b \vee a[g] b \Leftrightarrow a \times^{\mathsf{FCD}} b \cap^{\mathsf{FCD}} f \neq \emptyset \vee a \times^{\mathsf{FCD}} b \cap^{\mathsf{FCD}} g \neq \emptyset$ for any atomic filter objects a and b. □

Corollary 76. For any $f, g, h \in \mathsf{FCD}$, $R \in \mathscr{P}\mathsf{FCD}$

1. $f \cap^{\mathsf{FCD}} (g \cup^{\mathsf{FCD}} h) = (f \cap^{\mathsf{FCD}} g) \cup^{\mathsf{FCD}} (f \cap^{\mathsf{FCD}} h)$;
2. $f \cup^{\mathsf{FCD}} \bigcap^{\mathsf{FCD}} R = \bigcap^{\mathsf{FCD}} \left\langle f \cup^{\mathsf{FCD}} \right\rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice. To be concise I will write atoms instead of $\operatorname{atoms}^{\mathsf{FCD}}$ and $\cap$ and $\cup$ instead of $\cap^{\mathsf{FCD}}$ and $\cup^{\mathsf{FCD}}$.

1. $\operatorname{atoms}(f \cap (g \cup h)) = \operatorname{atoms} f \cap \operatorname{atoms}(g \cup h) = \operatorname{atoms} f \cap (\operatorname{atoms} g \cup \operatorname{atoms} h) = (\operatorname{atoms} f \cap \operatorname{atoms} g) \cup (\operatorname{atoms} f \cap \operatorname{atoms} h) = \operatorname{atoms}(f \cap g) \cup \operatorname{atoms}(f \cap h) = \operatorname{atoms}((f \cap g) \cup (f \cap h))$.
2. $\operatorname{atoms}\left(f \cup \bigcap^{\mathsf{FCD}} R\right) = \operatorname{atoms} f \cup \operatorname{atoms} \bigcap^{\mathsf{FCD}} R = \operatorname{atoms} f \cup \bigcap^{\mathsf{FCD}} \langle \operatorname{atoms} \rangle R = \bigcap^{\mathsf{FCD}} \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R = \bigcap^{\mathsf{FCD}} \langle \operatorname{atoms} \rangle \langle f \cup \rangle R = \operatorname{atoms} \bigcap^{\mathsf{FCD}} \langle f \cup \rangle R$. □

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

The next proposition is one more (among the theorem 43) generalization for funcoids of composition of relations.

Corollary 77. The lattice of funcoids is co-brouwerian.

Proposition 78. For any $f, g \in \mathsf{FCD}$

$\operatorname{atoms}^{\mathsf{FCD}}(g \circ f) = \left\{ x \times^{\mathsf{FCD}} z \mid x, z \in \operatorname{atoms}^{\mathfrak{F}} \mho, \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms}^{\mathsf{FCD}} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms}^{\mathsf{FCD}} g) \right\}$.

Proof. $x \times^{\mathsf{FCD}} z \cap^{\mathsf{FCD}} g \circ f \neq \emptyset \Leftrightarrow x [g \circ f] z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x [f] y \wedge y [g] z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho :$ $(x \times^{\mathsf{FCD}} y \cap^{\mathsf{FCD}} f \neq \emptyset \wedge y \times^{\mathsf{FCD}} z \cap^{\mathsf{FCD}} g \neq \emptyset)$ (were used the theorem 43). □

Conjecture 79. The set of discrete funcoids is the center of the lattice of funcoids.

3.11 Complete funcoids

Definition 80. I will call *co-complete* such a funcoid f that $\forall X \in \mathscr{P}\mho : \langle f \rangle X \in \mathscr{P}\mho$.

Remark 81. I will call *generalized closure* such a function $\alpha \in \mathscr{P}\mho^{\mathscr{P}\mho}$ that

1. $\alpha \emptyset = \emptyset$;
2. $\forall I, J \in \mathscr{P}\mho : \alpha (I \cup J) = \alpha I \cup \alpha J$.

Obvious 82. A funcoid f is co-complete iff $\langle f \rangle|_{\mathscr{P}\mho}$ is a generalized closure.

Remark 83. Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 84. I will call a *complete funcoid* a funcoid whose reverse is co-complete.

Theorem 85. The following conditions are equivalent for every funcoid f:

1. funcoid f is complete.
2. $\forall S \in \mathscr{P}\mathfrak{F}, J \in \mathscr{P}\mho : (\bigcup^{\mathfrak{F}} S [f] J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} [f] J)$;
3. $\forall S \in \mathscr{P}\mathscr{P}\mho, J \in \mathscr{P}\mho : (\bigcup S [f] J \Leftrightarrow \exists I \in S : I [f] J)$;
4. $\forall S \in \mathscr{P}\mathfrak{F} : \langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
5. $\forall S \in \mathscr{P}\mathscr{P}\mho : \langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
6. $\forall A \in \mathscr{P}\mho : \langle f \rangle A = \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in A \}$.

Proof.

(3) ⇒ (1). For any $S \in \mathscr{P}\mathscr{P}\mho$, $J \in \mathscr{P}\mho$

$$\bigcup S \cap^{\mathfrak{F}} \langle f^{-1} \rangle J \neq \emptyset \Leftrightarrow \exists I \in S : I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J \neq \emptyset, \tag{9}$$

consequently by the theorem 52 in [5] we have $\langle f^{-1} \rangle J \in \mathscr{P}\mho$.

(1) ⇒ (2). For any $S \in \mathscr{P}\mathfrak{F}$, $J \in \mathscr{P}\mho$ we have $\langle f^{-1} \rangle J \in \mathscr{P}\mho$, consequently the formula (9) is true. From this follows (2).

(6) ⇒ (5). $\langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in \bigcup S \} = \bigcup^{\mathfrak{F}} \{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle a \mid a \in A \} \mid A \in S \} = \bigcup^{\mathfrak{F}} \{ \langle f \rangle A \mid A \in S \} = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$.

(2) ⇒ (3), (4) ⇒ (5), (5) ⇒ (3), (2) ⇒ (4), (5) ⇒ (6). Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pre-topological spaces.

Proposition 86. To specify a complete funcoid f it is enough to specify $\langle f \rangle$ on one-element sets, values of $\langle f \rangle$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f \rangle$ on one-element sets $\langle f \rangle$ can be found on any sets and then its value can be inferred for any filter objects.

Choosing arbitrarily the values of $\langle f \rangle$ on one-element sets we can define a complete funcoid the following way: $\langle f \rangle X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{ \alpha \} \mid \alpha \in X \}$ for any $X \in \mathscr{P}\mho$. Obviously it is really a complete funcoid. □

Theorem 87. A funcoid is discrete iff it is both complete and co-complete.

Proof.

Direct implication. Obvious.

Reverse implication. Let f is both a complete and co-complete funcoid. Consider the relation g defined by that $\langle g\rangle\{\alpha\} = \langle f\rangle\{\alpha\}$ (g is correctly defined because f is a generalized closure). Because f is a complete funcoid $f = g$. □

Theorem 88. If R is a set of (co-)complete funcoids then $\bigcup^{\mathsf{FCD}} R$ is a (co-)complete funcoid.

Proof. It is enough to prove only for co-complete funcoids. Let R is a set of co-complete funcoids. Then for any $X \in \mathscr{P}\mho$

$$\left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X = \bigcup \{\langle f\rangle X \mid f \in R\} \in \mathscr{P}\mho$$

(used the theorem 40). □

Corollary 89. If R is a set of binary relations then $\bigcup^{\mathsf{FCD}} R = \bigcup R$.

Proof. From two last theorems. □

Theorem 90. The filtrator of funcoids is filtered.

Proof. It's enough to prove that every funcoid is representable as (infinite) intersection (on the lattice of funcoids) of some set of discrete funcoids.

Let $f \in \mathsf{FCD}$, $A \in \mathscr{P}\mho$, $B \in \operatorname{up}\langle f\rangle A$, $g(A; B) \overset{\text{def}}{=} A \times B \cup^{\mathsf{FCD}} \overline{A} \times \mho$. For any $X \in \mathscr{P}\mho$

$$\langle g(A;B)\rangle X = \left\langle A \times^{\mathsf{FCD}} B\right\rangle X \cup \left\langle \overline{A} \times^{\mathsf{FCD}} \mho\right\rangle X = \left(\begin{cases} \emptyset & \text{if } X = \emptyset \\ B & \text{if } \emptyset \neq X \subseteq A \\ \mho & \text{if } X \nsubseteq A \end{cases} \right) \supseteq \langle f\rangle X;$$

so $g(A;B) \supseteq f$. For any $A \in \mathscr{P}\mho$

$$\bigcap^{\mathfrak{F}} \{\langle g(A;B)\rangle A \mid B \in \operatorname{up}\langle f\rangle A\} = \bigcap^{\mathfrak{F}} \{B \mid B \in \operatorname{up}\langle f\rangle A\} = \langle f\rangle A;$$

consequently

$$\bigcap^{\mathsf{FCD}} \{g(A;B) \mid A \in \mathscr{P}\mho, B \in \operatorname{up}\langle f\rangle A\} = f.$$ □

In certain cases the theorem 44 can be generalized for infinite unions.

Theorem 91. Let $f \in \mathsf{FCD}$. If R is a set of co-complete funcoids then

$$f \circ \bigcup^{\mathsf{FCD}} R = \bigcup^{\mathsf{FCD}} \langle f\circ\rangle R.$$

Proof. If R is a set of co-complete funcoids then for $X, Z \in \mathscr{P}\mho$

$$\begin{aligned}
& X\left[f \circ \bigcup^{\mathsf{FCD}} R \right] Z \Leftrightarrow \\
\text{(by the theorem 43)} \quad & \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : \left(X \left[\bigcup^{\mathsf{FCD}} R \right] y \wedge y[f]Z \right) \\
& \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : \left(y \cap^{\mathfrak{F}} \left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X \neq \emptyset \wedge y[f]Z \right) \\
\text{(by the theorem 40)} \quad & \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : \left(y \cap^{\mathfrak{F}} \bigcup \{\langle u\rangle X \mid u \in R\} \neq \emptyset \wedge y[f]Z \right) \\
\text{(by the theorem 52 in [5])} \quad & \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\exists u \in R : y \cap^{\mathfrak{F}} \langle u\rangle X \neq \emptyset \wedge y[f]Z) \\
& \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (\exists u \in R : X[u]y \wedge y[f]Z) \\
& \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho, u \in R : (X[u]y \wedge y[f]Z) \\
\text{(by the theorem 43)} \quad & \Leftrightarrow \exists u \in R : X[f \circ u]Z \\
\text{(by the theorem 40)} \quad & \Leftrightarrow X\left[\bigcup^{\mathsf{FCD}} \langle f\circ\rangle R \right] Z.
\end{aligned}$$

□

3.12 Completion of funcoids

I will denote ComplFCD and CoComplFCD the sets of complete and co-complete funcoids correspondingly.

Obvious 92. ComplFCD and CoComplFCD are closed regarding composition of funcoids.

Proposition 93. ComplFCD and CoComplFCD (with induced order) are complete lattices.

Proof. Follows from the corollary 88. □

Theorem 94. $\operatorname{Cor} f = \operatorname{Cor}' f$ for an element f of the filtrator of funcoids.

Proof. From the theorem 26 in [5] and the corollary 89 and theorem 90. □

Definition 95. *Completion* of a funcoid f is the complete funcoid $\operatorname{Compl} f$ defined by the formula $\langle \operatorname{Compl} f \rangle \{\alpha\} = \langle f \rangle \{\alpha\}$ for $\alpha \in \mho$.

Definition 96. *Co-completion* of a funcoid f is defined by the formula

$$\operatorname{CoCompl} f = (\operatorname{Compl} f^{-1})^{-1}.$$

Obvious 97. $\operatorname{Compl} f \subseteq f$ and $\operatorname{CoCompl} f \subseteq f$ for every funcoid f.

Proposition 98. The filtrator $(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})$ is filtered.

Proof. Because the filtrator $(\mathsf{FCD}; \mathscr{P}\mho^2)$ is filtered. □

Theorem 99. $\operatorname{Compl} f = \operatorname{Cor}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f = \operatorname{Cor}'^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f$.

Proof. $\operatorname{Cor}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f = \operatorname{Cor}'^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f$ since (the theorem 26 in [5]) the filtrator $(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})$ is filtered (as a consequence of the theorem 90) and with join closed core (the theorem 88).

Let $g \in \operatorname{up}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f$. Then $g \in \operatorname{Compl}\mathsf{FCD}$ and $g \supseteq f$. Thus $g = \operatorname{Compl} g \supseteq \operatorname{Compl} f$.

Thus $\forall g \in \operatorname{up}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f : g \supseteq \operatorname{Compl} f$.

Let $\forall g \in \operatorname{up}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f : h \subseteq g$ for some $h \in \operatorname{Compl}\mathsf{FCD}$.

Then $h \subseteq \bigcap^{\mathsf{FCD}} \operatorname{up}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f = f$ and consequently $h = \operatorname{Compl} h \subseteq \operatorname{Compl} f$.

Thus $\operatorname{Compl} f = \bigcap^{\operatorname{Compl}\mathsf{FCD}} \operatorname{up}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f = \operatorname{Cor}^{(\mathsf{FCD}; \operatorname{Compl}\mathsf{FCD})} f$. □

Theorem 100. Atoms of the lattice ComplFCD are exactly direct products of the form $\{\alpha\} \times^{\mathsf{FCD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of ComplFCD. Also $\emptyset$ is an element of ComplFCD.

$\{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of ComplFCD because these are atoms of FCD.

Remain to prove that if f is an atom of ComplFCD then $f = \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in \mho$ and an atomic f.o. b.

Suppose f is a non-empty complete funcoid. Then exists $\alpha \in \mho$ such that $\langle f \rangle \alpha \neq \emptyset$. Thus $\{\alpha\} \times^{\mathsf{FCD}} b \subseteq f$ for some atomic f.o. b. If f is an atom then $f = \{\alpha\} \times^{\mathsf{FCD}} b$. □

Theorem 101. $\langle \operatorname{CoCompl} f \rangle X = \operatorname{Cor} \langle f \rangle X$ for every funcoid f and set X.

Proof. $\operatorname{CoCompl} f \subseteq f$ thus $\langle \operatorname{CoCompl} f \rangle X \subseteq \langle f \rangle X$, but $\langle \operatorname{CoCompl} f \rangle X \in \mathscr{P}\mho$ thus $\langle \operatorname{CoCompl} f \rangle X \subseteq \operatorname{Cor} \langle f \rangle X$.

Let $\alpha X = \operatorname{Cor} \langle f \rangle X$. Then $h\emptyset = \emptyset$ and

$$\alpha(X \cup Y) = \operatorname{Cor} \langle f \rangle (X \cup Y) = \operatorname{Cor}(\langle f \rangle X \cup \langle f \rangle Y) = \operatorname{Cor} \langle f \rangle X \cup \operatorname{Cor} \langle f \rangle Y = \alpha X \cup \alpha Y.$$

(used the theorem 64 from [5]). Thus α can be continued till $\langle g \rangle$ for some funcoid g. This funcoid is co-complete.

Evidently g is the greatest co-complete funcoid which is lower than f.

Thus $g = \operatorname{CoCompl} f$ and so $\operatorname{Cor} \langle f \rangle X = \alpha X = \langle g \rangle X = \langle \operatorname{CoCompl} f \rangle X$. □

Theorem 102. $\operatorname{Compl} f \cap^{\mathsf{FCD}} \operatorname{Compl} g = \operatorname{Compl}(f \cap^{\mathsf{FCD}} g)$ for every funcoids f and g.

Proof. $\left\langle \operatorname{CoCompl} f \cap^{\mathsf{FCD}} \operatorname{CoCompl} g \right\rangle x = \langle \operatorname{CoCompl} f \rangle x \cap^{\mathfrak{F}} \langle \operatorname{CoCompl} g \rangle x = \operatorname{Cor} \langle f \rangle x \cap^{\mathfrak{F}} \operatorname{Cor} \langle g \rangle x = \operatorname{Cor} \langle f \rangle x \cap \operatorname{Cor} \langle g \rangle x = \operatorname{Cor}(\langle f \rangle x \cap^{\mathfrak{F}} \langle g \rangle x) = \operatorname{Cor} \left\langle f \cap^{\mathsf{FCD}} g \right\rangle x = \left\langle \operatorname{CoCompl} (f \cap^{\mathsf{FCD}} g) \right\rangle x$ for every atomic f.o. x (used the theorem 63 from [5]).

Thus $\operatorname{CoCompl} f \cap^{\mathsf{FCD}} \operatorname{CoCompl} g = \operatorname{CoCompl} (f \cap^{\mathsf{FCD}} g)$. □

Corollary 103. If f, g are (co-)complete funcoids then $f \cap^{\mathsf{FCD}} g$ is a (co-)complete funcoid.

Proof. Let f and g are complete funcoids.

$f \cap^{\mathsf{FCD}} g = \operatorname{Compl} f \cap^{\mathsf{FCD}} \operatorname{Compl} g = \operatorname{Compl}(f \cap^{\mathsf{FCD}} g) \in \operatorname{Compl}\mathsf{FCD}$. □

Corollary 104. If f, g are discrete funcoids then $f \cap^{\mathsf{FCD}} g$ is a discrete funcoid.

Proof. Let f, g are discrete funcoids. Then $f \cap^{\mathsf{FCD}} g$ is both complete and co-complete. □

Corollary 105. If f, g are discrete funcoids then $f \cap^{\mathsf{FCD}} g = f \cap g$.

Theorem 106. $\operatorname{Compl}\mathsf{FCD}$ is an atomistic lattice.

Proof. Let $f \in \operatorname{Compl}\mathsf{FCD}$. $\langle f \rangle X = \bigcup^{\mathfrak{F}} \{ \langle f \rangle x \mid x \in X \} = \bigcup^{\mathfrak{F}} \left\{ \left\langle f|_{\{x\}} \right\rangle x \mid x \in X \right\}$, thus $f = \bigcup^{\mathsf{FCD}} \left\{ f|_{\{x\}} \mid x \in X \right\}$. It is trivial that every $f|_{\{x\}}$ is a union of atoms of $\operatorname{Compl}\mathsf{FCD}$. □

Theorem 107. A funcoid is complete iff it is a join (on the lattice FCD) of atomic complete funcoids.

Proof. Follows from the theorem 88 and the previous theorem. □

Corollary 108. $\operatorname{Compl}\mathsf{FCD}$ is join-closed.

Theorem 109. $\operatorname{Compl}(\bigcup^{\mathsf{FCD}} R) = \bigcup^{\mathsf{FCD}} \langle \operatorname{Compl} \rangle R$ for every set R of funcoids.

Proof. $\left\langle \operatorname{Compl}(\bigcup^{\mathsf{FCD}} R) \right\rangle X = \bigcup^{\mathfrak{F}} \left\{ \left\langle \bigcup^{\mathsf{FCD}} R \right\rangle \{\alpha\} \mid \alpha \in X \right\} = \bigcup^{\mathfrak{F}} \left\{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{\alpha\} \mid f \in R \} \mid \alpha \in X \right\} = \bigcup^{\mathfrak{F}} \left\{ \bigcup^{\mathfrak{F}} \{ \langle f \rangle \{\alpha\} \mid \alpha \in X \} \mid f \in R \right\} = \bigcup^{\mathfrak{F}} \{ \langle \operatorname{Compl} f \rangle X \mid f \in R \} = \langle \bigcup^{\mathsf{FCD}} \langle \operatorname{Compl} \rangle R \rangle X$ for every set X. □

Lemma 110. Co-completion of a complete funcoid is complete.

Proof. Let f is a complete funcoid.

$\langle \operatorname{CoCompl} f \rangle X = \operatorname{Cor} \langle f \rangle X = \operatorname{Cor} \bigcup^{\mathfrak{F}} \{ f x \mid x \in X \} = \bigcup \{ \operatorname{Cor} f x \mid x \in X \} = \bigcup \{ \langle \operatorname{CoCompl} f \rangle x \mid x \in X \}$ for every set X. Thus $\operatorname{CoCompl} f$ is complete. □

Theorem 111. $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{CoCompl} \operatorname{Compl} f = \operatorname{Cor} f$ for every funcoid f.

Proof. $\operatorname{Compl} \operatorname{CoCompl} f$ is co-complete since (used the lemma) $\operatorname{CoCompl} f$ is co-complete. Thus $\operatorname{Compl} \operatorname{CoCompl} f$ is a discrete funcoid. Trivially [TODO: Detailed proof] it is the greatest discrete funcoid under f. Thus $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{Cor} f$. Similarly $\operatorname{CoCompl} \operatorname{Compl} f = \operatorname{Cor} f$. □

Theorem 112. $\operatorname{Cor} f \cap^{\mathsf{FCD}} \operatorname{Cor} g = \operatorname{Cor}(f \cap^{\mathsf{FCD}} g)$.

Proof. From the previous theorem and the theorem 102. □

Question 113. Is $\operatorname{Compl}\mathsf{FCD}$ a co-brouwerian lattice?

3.13 Monovalued funcoids

Following the idea of definition of monovalued morphism let's call *monovalued* such a funcoid f that $f \circ f^{-1} \subseteq I_{\mathrm{im}\, f}$.

Obvious 114. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of funcoids is monovalued iff the funcoid f is monovalued.

Theorem 115. The following statements are equivalent for a funcoid f:

1. f is monovalued.
2. $\forall a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}: \langle f \rangle a \in \mathrm{atoms}^{\mathfrak{F}} \mho \cup \{\emptyset\}$.
3. $\forall \mathcal{I}, \mathcal{J} \in \mathfrak{F}: \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J}$.
4. $\forall I, J \in \mathscr{P} \mho: \langle f^{-1} \rangle (I \cap J) = \langle f^{-1} \rangle I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J$.

Proof.

(2) ⇒ (3). Let $a \in \mathrm{atoms}^{\mathfrak{F}} \mho$, $\langle f \rangle a = b$. Then because $b \in \mathrm{atoms}^{\mathfrak{F}} \mho \cup \{\emptyset\}$

$$(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} b \neq \emptyset \Leftrightarrow \mathcal{I} \cap^{\mathfrak{F}} b \neq \emptyset \wedge \mathcal{J} \cap^{\mathfrak{F}} b \neq \emptyset;$$
$$a [f] (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \Leftrightarrow a [f] \mathcal{I} \wedge a [f] \mathcal{J};$$
$$(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) [f^{-1}] a \Leftrightarrow \mathcal{I} [f^{-1}] a \wedge \mathcal{J} [f^{-1}] a;$$
$$a \cap^{\mathfrak{F}} \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \neq \emptyset \Leftrightarrow a \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{I} \neq \emptyset \wedge a \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J} \neq \emptyset;$$
$$\langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J}.$$

(4) ⇒ (1). $\langle f^{-1} \rangle a \cap^{\mathfrak{F}} \langle f^{-1} \rangle b = \emptyset$ for any two distinct atomic filter objects a and b. This is equivalent to $\neg (b [f^{-1}] \langle f^{-1} \rangle a)$; $\neg (\langle f^{-1} \rangle a [f] b)$; $b \cap^{\mathfrak{F}} \langle f \rangle \langle f^{-1} \rangle a = \emptyset$; $b \cap^{\mathfrak{F}} \langle f \circ f^{-1} \rangle a = \emptyset$; $\neg (a [f \circ f^{-1}] b)$. So $a [f \circ f^{-1}] b \Rightarrow a = b$ for any atomic filter objects a and b. This is possible only when $f \circ f^{-1} \subseteq I_{\mathrm{Dst}\, f}$.

(3) ⇒ (4). Obvious.

¬(2) ⇒ ¬(1). Suppose $\langle f \rangle a \notin \mathrm{atoms}^{\mathfrak{F}} \mathcal{B} \cup \{\emptyset\}$ for some $a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}$. Then there exist two atomic filter objects $p \neq q$ such that $\langle f \rangle a \supseteq p \wedge \langle f \rangle a \supseteq q$. Consequently $p \cap^{\mathfrak{F}} \langle f \rangle a \neq \emptyset$; $a \cap^{\mathfrak{F}} \langle f^{-1} \rangle p \neq \emptyset$; $a \subseteq \langle f^{-1} \rangle p$; $\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \supseteq \langle f \rangle a \supseteq q$; $\langle f \circ f^{-1} \rangle p \nsubseteq p$. So it cannot be $f \circ f^{-1} \subseteq I_{\mathrm{Dst}\, f}$. □

Corollary 116. A function is a monovalued funcoid.

Remark 117. This corollary can be reformulated as follows: For binary relations the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

Proof. Because $\forall I, J \in \mathscr{P} \mho: \langle f^{-1} \rangle (I \cap J) = \langle f^{-1} \rangle I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J$ is true for any function f. □

3.14 T_1- and T_2-separable funcoids

For funcoids can be generalized T_0-, T_1- and T_2- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 118. Let call T_1-*separable* such funcoid f that for any $\alpha, \beta \in \mho$ is true

$$\alpha \neq \beta \Rightarrow \neg (\{\alpha\} [f] \{\beta\})$$

Definition 119. Let call T_0-*separable* such funcoid f that $f \cap^{\mathsf{FCD}} f^{-1}$ is T_1-separable.

Definition 120. Let call T_2-*separable* such funcoid f that the funcoid $f^{-1} \circ f$ is T_1-separable.

For symmetric transitive funcoids T_1- and T_2-separability are the same (see theorem 13).

3.15 Filter objects closed regarding a funcoid

Definition 121. Let's call *closed* regarding a funcoid f such filter object $\mathcal{A}$ that $\langle f\rangle\mathcal{A}\subseteq\mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 122. If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding some funcoid), S is a set of closed filter objects, then

1. $\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J}$ is a closed filter object;
2. $\bigcap^{\mathfrak{F}}S$ is a closed filter object.

Proof. Let denote the given funcoid as f. $\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})=\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}\subseteq\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J}$, $\langle f\rangle\bigcap^{\mathfrak{F}}S\subseteq\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle S\subseteq\bigcap^{\mathfrak{F}}S$. Consequently the filter objects $\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J}$ and $\bigcap^{\mathfrak{F}}S$ are closed. □

Proposition 123. If S is a set of closed regarding a complete funcoid filter objects, then the filter object $\bigcup^{\mathfrak{F}}S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle\bigcup^{\mathfrak{F}}S=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle S\subseteq\bigcup^{\mathfrak{F}}S$ where f is the given funcoid. □

4 Reloids

Definition 124. I will call a *reloid* a filter object on the set of binary relations.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations.

Definition 125. The *reverse* reloid of a reloid f is defined by the formula

$$\mathrm{up}\, f^{-1}=\left\{F^{-1}\mid F\in\mathrm{up}\, f^{-1}\right\}.$$

Reverse reloid is a generalization of conjugate quasi-uniformity.

I will denote RLD either the set of reloids or the category of reloids (defined below), dependently on context.

4.1 Composition of reloids

Definition 126. Composition of reloids is defined by the formula

$$g\circ f=\bigcap^{\mathsf{RLD}}\{G\circ F\mid F\in\mathrm{up}\, f,G\in\mathrm{up}\, g\}.$$

Composition of reloids is a reloid.

Lemma 127. $(h\circ g)\circ f=h\circ(g\circ f)$ for any reloids f, g, h.

Proof. For two nonempty collections A and B of sets I will denote

$$A\sim B\Leftrightarrow(\forall K\in A\exists L\in B\colon L\subseteq K)\wedge(\forall K\in B\exists L\in A\colon L\subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B\circ A=\{L\circ K|K\in A,L\in B\}$.

Let first prove that for any nonempty collections of relations A, B, C

$$A\sim B\Rightarrow A\circ C\sim B\circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B: K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for any $P \in A \circ C$ exist $P' \in B \circ C$ such that $P' \subseteq P$; vice verse is analogous. So $A \circ C \sim B \circ C$.

$\mathrm{up}((h \circ g) \circ f) \sim \mathrm{up}(h \circ g) \circ \mathrm{up}\, f$, $\mathrm{up}(h \circ g) \sim (\mathrm{up}\, h) \circ (\mathrm{up}\, g)$. By proven above $\mathrm{up}((h \circ g) \circ f) \sim (\mathrm{up}\, h) \circ (\mathrm{up}\, g) \circ (\mathrm{up}\, f)$.

Analogously $\mathrm{up}(h \circ (g \circ f)) \sim (\mathrm{up}\, h) \circ (\mathrm{up}\, g) \circ (\mathrm{up}\, f)$.

So $\mathrm{up}((h \circ g) \circ f) \sim \mathrm{up}(h \circ (g \circ f))$ what is possible only if $\mathrm{up}((h \circ g) \circ f) = \mathrm{up}(h \circ (g \circ f))$. □

Theorem 128.

1. $f \circ f = \bigcap^{\mathsf{RLD}} \{F \circ F \mid F \in \mathrm{up}\, f\}$;
2. $f^{-1} \circ f = \bigcap^{\mathsf{RLD}} \{F^{-1} \circ F \mid F \in \mathrm{up}\, f\}$;
3. $f \circ f^{-1} = \bigcap^{\mathsf{RLD}} \{F \circ F^{-1} \mid F \in \mathrm{up}\, f\}$.

Proof. I will prove only (1) and (2) because (3) is analogous to (2).

1. Enough to show that $\forall F, G \in \mathrm{up}\, f \exists H \in \mathrm{up}\, f: H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.
2. Enough to show that $\forall F, G \in \mathrm{up}\, f \exists H \in \mathrm{up}\, f: H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq G^{-1} \circ F$. □

Conjecture 129. If f, g, h are reloids then

1. $f \circ (g \cup^{\mathsf{RLD}} h) = f \circ g \cup^{\mathsf{RLD}} f \circ h$;
2. $(g \cup^{\mathsf{RLD}} h) \circ f = g \circ f \cup^{\mathsf{RLD}} h \circ f$.

4.2 Direct product of filter objects

In theory of reloids direct product of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \overset{\text{def}}{=} \bigcap^{\mathfrak{F}} \{A \times B \mid A \in \mathrm{up}\, \mathcal{A}, B \in \mathrm{up}\, \mathcal{B}\}.$$

Theorem 130. $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = \bigcup^{\mathfrak{F}} \{a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{B}\}$ for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. Obviously

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \supseteq \bigcup^{\mathfrak{F}} \{a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{B}\}$$

Reversely, let $K \in \mathrm{up} \bigcup^{\mathfrak{F}} \{a \times^{\mathsf{RLD}} b \mid a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{B}\}$. Then $K \in \mathrm{up}(a \times^{\mathsf{RLD}} b)$ for every $a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}$, $b \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{B}$; $K \supseteq X_a \times^{\mathsf{RLD}} Y_b$ for some $X_a \in \mathrm{up}\, a$, $Y_b \in \mathrm{up}\, b$; $K \supseteq \bigcup \{X_a \times Y_b \mid a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}, b \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{B}\} = \bigcup \{X_a \mid a \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}\} \times \bigcup \{Y_b \mid b \in \mathrm{atoms}^{\mathfrak{F}} \mathcal{A}\} = A \times B$ where $A \in \mathrm{up} \mathcal{A}$, $B \in \mathrm{up} \mathcal{B}$; $K \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Theorem 131. $\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1 = (\mathcal{A}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{B}_1)$ for any $\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}$.

Proof.

$$\begin{aligned}
\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1 &= \bigcap^{\mathsf{RLD}} \{P \cap Q \mid P \in \mathrm{up}(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0), Q \in \mathrm{up}(\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1)\} \\
&= \bigcap^{\mathsf{RLD}} \{A_0 \times B_0 \cap A_1 \times B_1 \mid A_0 \in \mathrm{up}\mathcal{A}_0, B_0 \in \mathrm{up}\mathcal{B}_0, A_1 \in \mathrm{up}\mathcal{A}_1, \\
&\quad B_1 \in \mathrm{up}\mathcal{B}_1\} \\
&= \bigcap^{\mathsf{RLD}} \{(A_0 \cap A_1) \times (B_0 \cap B_1) \mid A_0 \in \mathrm{up}\mathcal{A}_0, B_0 \in \mathrm{up}\mathcal{B}_0, A_1 \in \mathrm{up}\mathcal{A}_1, \\
&\quad B_1 \in \mathrm{up}\mathcal{B}_1\} \\
&= \bigcap^{\mathsf{RLD}} \{K \times L \mid K \in \mathrm{up}(\mathcal{A}_0 \cap \mathcal{A}_1), L \in \mathrm{up}(\mathcal{B}_0 \cap \mathcal{B}_1)\} \\
&= (\mathcal{A}_0 \cap^{\mathsf{RLD}} \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap^{\mathsf{RLD}} \mathcal{B}_1).
\end{aligned}$$

□

Theorem 132. If $S \in \mathscr{P}\mathfrak{F}^2$ then

$$\bigcap^{\mathsf{RLD}}\{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A};\mathcal{B}) \in S\} = \bigcap^{\mathfrak{F}}\mathrm{dom}\, S \times^{\mathsf{RLD}} \bigcap^{\mathfrak{F}}\mathrm{im}\, S.$$

Proof. Let $\mathcal{P} = \bigcap^{\mathfrak{F}}\mathrm{dom}\, S$, $\mathcal{Q} = \bigcap^{\mathfrak{F}}\mathrm{im}\, S$; $l = \bigcap^{\mathsf{RLD}}\{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A};\mathcal{B}) \in S\}$.

$\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q} \subseteq l$ is obvious.

Let $F \in \mathrm{up}(\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q})$. Then exist $P \in \mathrm{up}\,\mathcal{P}$ and $Q \in \mathrm{up}\,\mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \cap \ldots \cap P_n$ where $P_i \in \langle \mathrm{up} \rangle \mathrm{dom}\, S$ and $Q = Q_1 \cap \ldots \cap Q_m$ where $Q_i \in \langle \mathrm{up} \rangle \mathrm{im}\, S$.

$P \times Q = \bigcap_{i,j} (P_i \times Q_i)$.

$P_i \times Q_i \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some $(\mathcal{A};\mathcal{B}) \in S$. $P \times Q = \bigcap_{i,j} (P_i \times Q_i) \supseteq l$. $F \in \mathrm{up}\, l$. □

Conjecture 133. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{RLD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice RLD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Definition 134. I will call a reloid *convex* iff it is a union of direct products.

I will call two filter objects *isomorphic* when the corresponding filters are isomorphic (in the sense defined in [5]).

Theorem 135. The reloid $\{a\} \times^{\mathsf{RLD}} \mathcal{F}$ is isomorphic to the filter object $\mathcal{F}$ for every $a \in \mho$.

Proof. Consider $B = \{a\} \times \mho$ and $f = \{(x;(a;x)) \mid x \in \mho\}$. Then f is a bijection from $\mho$ to B.

If $X \in \mathrm{up}\,\mathcal{F}$ then $\langle f \rangle X \in B$ and $\langle f \rangle X = \{a\} \times X \in \mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$.

For every $Y \in \mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P}B$ we have $Y = \{a\} \times X$ for some $X \in \mathrm{up}\,\mathcal{F}$ and thus $Y = \langle f \rangle X$.

So $\langle f \rangle|_{\mathrm{up}\,\mathcal{F} \cap \mathscr{P}\mho} = \langle f \rangle|_{\mathrm{up}\,\mathcal{F}}$ is a bijection from $\mathrm{up}\,\mathcal{F} \cap \mathscr{P}\mho$ to $\mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P}B$.

We have $\mathrm{up}\,\mathcal{F} \cap \mathscr{P}\mho$ and $\mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P}B$ directly isomorphic and thus $\mathrm{up}\,\mathcal{F}$ is isomorphic to $\mathrm{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$. □

4.3 Restricting reloid to a filter object. Domain and image

Definition 136. I call restricting a reloid f to a filter object $\mathcal{A}$ as $f|_{\mathcal{A}} = f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho$.

Definition 137. *Domain* and *image* of a reloid f are defined as follows:

$$\mathrm{dom}\, f = \bigcap^{\mathfrak{F}}\langle \mathrm{dom} \rangle \mathrm{up}\, f; \quad \mathrm{im}\, f = \bigcap^{\mathfrak{F}}\langle \mathrm{im} \rangle \mathrm{up}\, f.$$

Proposition 138. $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$.

Proof.

Direct implication. Follows from $\mathrm{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \mathrm{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.

Reverse implication. $\mathrm{dom}\, f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \mathrm{up}\mathcal{A}\, \exists F \in \mathrm{up}\, f : \mathrm{dom}\, F \subseteq A$. Analogously

$$\mathrm{im}\, f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \mathrm{up}\mathcal{B}\, \exists G \in \mathrm{up}\, f : \mathrm{im}\, G \subseteq B.$$

Let $\mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$, $A \in \mathrm{up}\mathcal{A}$, $B \in \mathrm{up}\mathcal{B}$. Then exist $F \in \mathrm{up}\, f, G \in \mathrm{up}\, f$ such that $\mathrm{dom}\, F \subseteq A \wedge \mathrm{im}\, G \subseteq B$. Consequently $F \cap G \in \mathrm{up}\, f$, $\mathrm{dom}(F \cap G) \subseteq A$, $\mathrm{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. We have exists $H \in \mathrm{up}\, f$ such that $H \subseteq A \times B$ for any $A \in \mathrm{up}\mathcal{A}$, $B \in \mathrm{up}\mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Definition 139. I call *identity reloid* for a filter object $\mathcal{A}$ the reloid $I_{\mathcal{A}} \overset{\mathrm{def}}{=} (=)|_{\mathcal{A}}$.

Theorem 140. $I_{\mathcal{A}} = \bigcap^{\mathfrak{F}}\{I_A \mid A \in \mathrm{up}\mathcal{A}\}$ where I_A is the identity relation on a set A.

Proof. Let $K \in \mathrm{up} \bigcap^{\mathfrak{F}}\{I_A \mid A \in \mathrm{up}\mathcal{A}\}$, then exists $A \in \mathrm{up}\mathcal{A}$ such that $K \supseteq I_A$. Then $(=)|_{\mathcal{A}} = (=) \cap^{\mathsf{RLD}} \mathcal{A} \times \mho \subseteq (=) \cap A \times \mho = I|_A \subseteq K$; $K \in \mathrm{up}\; I_A$. Reversely let $K \in \mathrm{up}\; I_{\mathcal{A}} = \mathrm{up}((=) \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho)$, then exists $A \in \mathrm{up}\mathcal{A}$ such that $K \in \mathrm{up}((=) \cap A \times \mho) = \mathrm{up}\, I_A \subseteq \mathrm{up}\, I_{\mathcal{A}}$. □

Proposition 141. $I_{\mathcal{A}}{}^{-1} = I_{\mathcal{A}}$.

Proof. Follows from the previous theorem. □

Theorem 142. $f|_{\mathcal{A}} = f \circ I_{\mathcal{A}}$ for any reloid f and filter object $\mathcal{A}$.

Proof. We need to prove that $f \cap^{\mathsf{RLD}} \mathcal{A} \times \mho = f \circ \bigcap^{\mathsf{RLD}}\{I_A \mid A \in \mathrm{up}\mathcal{A}\}$. $f \circ \bigcap^{\mathsf{RLD}}\{I_A \mid A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}}\{F \circ I_A \mid F \in \mathrm{up}\, f, A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}}\{F|_A \mid F \in \mathrm{up}\, f, A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}}\{F \cap A \times \mho \mid F \in \mathrm{up}\, f, A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}}\{F \mid F \in \mathrm{up}\, f\} \cap \bigcap^{\mathsf{RLD}}\{A \times \mho \mid A \in \mathrm{up}\mathcal{A}\} = f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho$. □

Theorem 143. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for any reloids f and g and filter object $\mathcal{A}$.

Proof. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ I_{\mathcal{A}} = g \circ (f \circ I_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

Theorem 144. $f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$ for any reloid f and filter objects $\mathcal{A}$ and $\mathcal{B}$.

Proof. $f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho \cap^{\mathsf{RLD}} \mho \times^{\mathsf{RLD}} \mathcal{B} = f|_{\mathcal{A}} \cap^{\mathsf{RLD}} \mho \times \mathcal{B} = f \circ I_{\mathcal{A}} \cap^{\mathsf{RLD}} \mho \times \mathcal{B} = ((f \circ I_{\mathcal{A}})^{-1} \cap^{\mathsf{RLD}} (\mho \times^{\mathsf{RLD}} \mathcal{B})^{-1})^{-1} = (I_{\mathcal{A}} \circ f^{-1} \cap^{\mathsf{RLD}} \mathcal{B} \times^{\mathsf{RLD}} \mho)^{-1} = (I_{\mathcal{A}} \circ f^{-1} \circ I_{\mathcal{B}})^{-1} = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$. □

4.4 Category of reloids

I will define the category RLD of reloids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a reloid such that $\mathrm{dom}\, f \subseteq \mathcal{A}$, $\mathrm{im}\, f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

4.4.1 Monovalued reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f \circ f^{-1} \subseteq I_{\mathrm{im}\, f}$.

Obvious 145. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloids is monovalued iff the reloid f is monovalued.

Conjecture 146. If a reloid is monovalued then it is a monovalued function restricted to some filter object.

Conjecture 147. A reloid f is monovalued iff $\forall g \in \mathsf{RLD}: (g \subseteq f \Rightarrow \exists \mathcal{A} \in \mathfrak{F}: g = f|_{\mathcal{A}})$.

Conjecture 148. A monovalued reloid restricted to an atomic filter object is atomic or empty.

A weaker conjecture:

Conjecture 149. A (monovalued) function restricted to an atomic filter object is atomic or empty.

4.5 Complete reloids and completion of reloids

Definition 150. A *complete* reloid is a reloid representable as join of direct products $\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Definition 151. A *co-complete* reloid is a reloid representable as join of direct products $a \times^{\mathsf{RLD}} \{\beta\}$ where $\beta \in \mho$ and a is an atomic f.o.

I will denote the sets of complete and co-complete reloids correspondingly as $\mathrm{Compl}\mathsf{RLD}$ and $\mathrm{CoCompl}\mathsf{RLD}$.

Obvious 152. Complete and co-complete are dual.

Obvious 153. Complete and co-complete reloids are convex.

Obvious 154. Discrete reloids are complete and co-complete.

Conjecture 155. If a reloid is both complete and co-complete then it is discrete.

Conjecture 156. Composition of complete reloids is complete.

Obvious 157. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 158. $\mathrm{Compl}\mathsf{RLD}$ (with the induced order) is a complete lattice.

Conjecture 159. Let $f \in \mathsf{RLD}$. If R is a set of co-complete reloids then

$$f \circ \bigcup^{\mathsf{RLD}} R = \bigcup^{\mathsf{RLD}} \langle f \circ \rangle R.$$

Definition 160. *Completion* and *co-completion* of a reloid f are defined by the formulas:

$$\mathrm{Compl}\, f = \mathrm{Cor}^{(\mathsf{RLD};\mathrm{Compl}\mathsf{RLD})} f \quad \text{and} \quad \mathrm{CoCompl}\, f = \mathrm{Cor}^{(\mathsf{RLD};\mathrm{CoCompl}\mathsf{RLD})} f.$$

Theorem 161. Atoms of the lattice $\mathrm{Compl}\mathsf{RLD}$ are exactly direct products of the form $\{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in \mho$ and b is an atomic f.o.

Proof. First, easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\mathrm{Compl}\mathsf{RLD}$. Also $\emptyset$ is an element of $\mathrm{Compl}\mathsf{RLD}$.

$\{\alpha\} \times^{\mathsf{RLD}} b$ are atoms of $\mathrm{Compl}\mathsf{FCD}$ because these are atoms of RLD.

Remain to prove that if f is an atom of $\mathrm{Compl}\mathsf{RLD}$ then $f = \{\alpha\} \times^{\mathsf{RLD}} b$ for some $\alpha \in \mho$ and an atomic f.o. b.

Suppose f is a non-empty complete reloid. Then $\{\alpha\} \times^{\mathsf{RLD}} b \subseteq f$ for some $\alpha \in \mho$ and atomic f.o. b. If f is an atom then $f = \{\alpha\} \times^{\mathsf{FCD}} b$. □

Obvious 162. $\mathrm{Compl}\mathsf{RLD}$ is an atomistic lattice.

Conjecture 163. $\mathrm{Compl}\, f \cap^{\mathsf{RLD}} \mathrm{Compl}\, g = \mathrm{Compl}(f \cap^{\mathsf{RLD}} g)$ for every reloids f and g.

Conjecture 164. $\mathrm{Compl}(\bigcup^{\mathsf{RLD}} R) = \bigcup^{\mathsf{RLD}} \langle \mathrm{Compl} \rangle R$ for every set R of reloids.

Conjecture 165. $\mathrm{Compl}\,\mathrm{CoCompl}\, f = \mathrm{CoCompl}\,\mathrm{Compl}\, f = \mathrm{Cor}\, f$ for every reloid f.

Question 166. Is $\mathrm{Compl}\mathsf{RLD}$ a distributive lattice? Is $\mathrm{Compl}\mathsf{RLD}$ a co-brouwerian lattice?

5 Relationships of funcoids and reloids

5.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD})f$ by the following formulas:

$$\mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}\, f : \mathcal{X} [F] \mathcal{Y}$$
$$\langle (\mathsf{FCD}) f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \{ \langle F \rangle \mathcal{X} \mid F \in \mathrm{up}\, f \}.$$

We should prove that $(\mathsf{FCD})f$ is really a funcoid. For this purpose we will additionally define

$$\langle (\mathsf{FCD}) f^{-1} \rangle \mathcal{Y} = \bigcap^{\mathfrak{F}} \{ \langle F^{-1} \rangle \mathcal{Y} \mid F \in \mathrm{up}\, f \}.$$

Proof. We need to prove that

$$\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle(\mathsf{FCD})f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle(\mathsf{FCD})f^{-1}\right\rangle\mathcal{Y} \neq \emptyset.$$

The above formula is equivalent to:

$$\forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}}\{\langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}}\left\{\left\langle F^{-1}\right\rangle\mathcal{Y} \mid F \in \operatorname{up} f\right\} \neq \emptyset.$$

We have $\mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}}\{\langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\} = \bigcap^{\mathfrak{F}}\left\{\mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\right\}$.

Let's denote $W = \left\{\mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\right\}$.

We need to prove that $\bigcap^{\mathfrak{F}} W \neq \emptyset \Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \neq \emptyset$. (The rest follows from symmetry.)

Let's prove that W is a generalized filter base. For this enough to prove that $V = \{\langle F\rangle\mathcal{X} \mid F \in \operatorname{up} f\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \langle P\rangle\mathcal{X}$, $\mathcal{B} = \langle Q\rangle\mathcal{X}$ where $P, Q \in \operatorname{up} f$. Then for $\mathcal{C} = \langle P \cap Q\rangle\mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base.

From this by the corollary 4 follows that $\bigcap^{\mathfrak{F}} W \neq \emptyset \Leftrightarrow \emptyset \notin W \Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle\mathcal{X} \neq \emptyset$. □

Theorem 167. $\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \cap f \neq \emptyset$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$ and $f \in \mathsf{RLD}$.

Proof.

$$\begin{aligned}
\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \cap^{\mathsf{RLD}} f \neq \emptyset &\Leftrightarrow \forall P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap^{\mathsf{RLD}} f \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: X \times^{\mathsf{RLD}} Y \cap^{\mathsf{RLD}} F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: X[F]Y \\
&\Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \\
&\Leftrightarrow \mathcal{X}[(\mathsf{FCD})f]\mathcal{Y}.
\end{aligned}$$

□

Theorem 168. $(\mathsf{FCD})f = \bigcap^{\mathsf{FCD}} \operatorname{up} f$ for any reloid f.

Proof. Let a is an atomic filter object.

$((\mathsf{FCD})f)a = \bigcap^{\mathfrak{F}}\{\langle F\rangle a \mid F \in \operatorname{up} f\}$ by the definition of (FCD).

$\left\langle\bigcap^{\mathsf{FCD}} \operatorname{up} f\right\rangle a = \bigcap^{\mathfrak{F}}\{\langle F\rangle a \mid F \in \operatorname{up} f\}$ by the theorem 56.

So $\langle(\mathsf{FCD})f\rangle a = \left\langle\bigcap^{\mathsf{FCD}} \operatorname{up} f\right\rangle a$ for any atomic filter object a. □

Lemma 169. $\langle g\rangle\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle S$ if g is a funcoid and S is a filter base.

Proof. $\operatorname{up}\bigcap^{\mathfrak{F}} S = \bigcup \langle\operatorname{up}\rangle S$ by the theorem 3.

$\langle g\rangle\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle\operatorname{up}\bigcap^{\mathfrak{F}} S$ by the theorem 33.

$\bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle\operatorname{up}\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle \bigcup \langle\operatorname{up}\rangle S$.

Easy to see that $\bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle \bigcup \langle\operatorname{up}\rangle S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle S$ because $S \subseteq \bigcup \langle\operatorname{up}\rangle S$.

Combining these equalities we produce $\langle g\rangle\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}}\langle\langle g\rangle\rangle S$. □

Lemma 170. For two sets of binary relations S and T and a set A

$$\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T \Rightarrow \bigcap^{\mathfrak{F}}\{\langle F\rangle A \mid F \in S\} = \bigcap^{\mathfrak{F}}\{\langle G\rangle A \mid G \in T\}$$

Proof. Let $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T$. Suppose $X \in \operatorname{up}\bigcap^{\mathfrak{F}}\{\langle F\rangle A \mid F \in S\}$. Then $X' \in \{\langle F\rangle A \mid F \in S\}$ where $X \supseteq X'$. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$. So $Y' = \langle G\rangle A \subseteq X' \subseteq X$. $Y' \in \{\langle G\rangle A \mid G \in T\}$; $Y' \in \operatorname{up}\bigcap^{\mathfrak{F}}\{\langle G\rangle A \mid G \in T\}$; $X \in \operatorname{up}\bigcap^{\mathfrak{F}}\{\langle G\rangle A \mid G \in T\}$. The reverse is symmetric. □

Theorem 171. $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ for any reloids f and g.

Proof.

$$\begin{aligned}\langle(\mathsf{FCD})(g\circ f)\rangle X &= \bigcap^{\mathfrak{F}}\{\langle H\rangle X \mid H\in\operatorname{up}(g\circ f)\}\\ &= \bigcap^{\mathfrak{F}}\left\{\langle H\rangle X \mid H\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}\right\}.\end{aligned}$$

Obviously

$$\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\} = \bigcap^{\mathsf{RLD}}\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\};$$

from this by the lemma 170

$$\bigcap^{\mathfrak{F}}\left\{\langle H\rangle X \mid H\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}\right\} = \bigcap^{\mathfrak{F}}\{\langle G\circ F\rangle X \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}.$$

On the other side

$$\begin{aligned}\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X &= \langle(\mathsf{FCD})g\rangle\langle(\mathsf{FCD})f\rangle X\\ &= \langle(\mathsf{FCD})g\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X \mid F\in\operatorname{up}f\}\\ &= \bigcap^{\mathfrak{F}}\left\{\langle G\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X \mid F\in\operatorname{up}f\} \mid G\in\operatorname{up}g\right\}.\end{aligned}$$

Let's prove that $\{\langle F\rangle X \mid F\in\operatorname{up}f\}$ is a filter base. If $A,B\in\{\langle F\rangle X \mid F\in\operatorname{up}f\}$ then $A=\langle F_1\rangle X$ and $B=\langle F_2\rangle X$ where $F_1,F_2\in\operatorname{up}f$. $A\cap B\supseteq\langle F_1\cap F_2\rangle X\in\{\langle F\rangle X \mid F\in\operatorname{up}f\}$. So $\{\langle F\rangle X \mid F\in\operatorname{up}f\}$ is really a filter base.

By the lemma 169 $\langle G\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X \mid F\in\operatorname{up}f\}=\bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X \mid F\in\operatorname{up}f\}$. So continuing the above equalities,

$$\begin{aligned}\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X &= \bigcap^{\mathfrak{F}}\left\{\bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X \mid F\in\operatorname{up}f\} \mid G\in\operatorname{up}g\right\}\\ &= \bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}\\ &= \bigcap^{\mathfrak{F}}\{\langle G\circ F\rangle X \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}.\end{aligned}$$

Combining these equalities we get $\langle(\mathsf{FCD})(g\circ f)\rangle X=\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X$ for any set X. □

5.2 Reloids induced by funcoid

Every funcoid f induces a reloid in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}}f\overset{\mathrm{def}}{=}\bigcap^{\mathsf{RLD}}\left\{F \mid F\in\mathscr{P}\mho^2, F\supseteq f\right\}$$
$$(\mathsf{RLD})_{\mathrm{in}}f\overset{\mathrm{def}}{=}\bigcup^{\mathsf{RLD}}\left\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B} \mid \mathcal{A},\mathcal{B}\in\mathfrak{F}, \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\right\}$$

Proposition 172. $\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f=\left\{F \mid F\in\mathscr{P}\mho^2, F\supseteq f\right\}$.

Proof. Easy to prove. □

Theorem 173. $(\mathsf{RLD})_{\mathrm{in}}f=\bigcup^{\mathsf{RLD}}\left\{a\times^{\mathsf{RLD}}b \mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho, a\times^{\mathsf{FCD}}b\subseteq f\right\}$.

Proof. Follows from the theorem 130. □

Lemma 174. $F\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f\Leftrightarrow\forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a[f]b\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b)$ for a funcoid f.

Proof.

$$\begin{aligned}F\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f &\Leftrightarrow F\in\operatorname{up}\bigcup^{\mathfrak{F}}\left\{a\times^{\mathsf{RLD}}b \mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho, a\times^{\mathsf{FCD}}b\subseteq f\right\}\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a\times^{\mathsf{FCD}}b\subseteq f\Rightarrow F\in\operatorname{up}(a\times^{\mathsf{RLD}}b))\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a\times^{\mathsf{FCD}}b\cap^{\mathsf{FCD}}f\neq\emptyset\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b)\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a[f]b\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b).\end{aligned}$$

□

Surprisingly a funcoid is greater inward than outward:

Theorem 175. $(\mathsf{RLD})_{\mathrm{out}} f \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ for a funcoid f.

Proof. We need to prove

$$\bigcap^{\mathsf{RLD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \} \subseteq \bigcup^{\mathsf{RLD}} \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \}.$$

Let

$$K \in \mathrm{up} \bigcup^{\mathfrak{F}} \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \}.$$

Then

$$\begin{aligned} K &= \bigcup \{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \} \\ &= \bigcup^{\mathsf{RLD}} \{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \} \\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}} \in \mathrm{up}\, \mathcal{A}$, $Y_{\mathcal{B}} \in \mathrm{up}\, \mathcal{B}$. $K \in \mathrm{up} \bigcap^{\mathsf{RLD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \}$. □

Theorem 176. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} f = f$ for any funcoid f.

Proof. $\mathrm{up}\, (\mathsf{RLD})_{\mathrm{out}} f = \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \}$

$(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} f = \bigcap^{\mathsf{FCD}} \mathrm{up} (\mathsf{RLD})_{\mathrm{out}} f = \bigcap^{\mathsf{FCD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \}$.

$\bigcap^{\mathsf{FCD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \} = f$ by the theorem 90. So $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} f = f$. □

Conjecture 177. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for any funcoid f.

Conjecture 178. For any funcoid f and reloid g

$$(\mathsf{RLD})_{\mathrm{out}} f \subseteq g \subseteq (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow (\mathsf{FCD}) g = f.$$

Conjecture 179. For a convex reloid f

1. $(\mathsf{RLD})_{\mathrm{out}} (\mathsf{FCD}) f = f$;
2. $(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f = f$.

6 Continuous morphisms

This section will use the apparatus from the section "Partially ordered dagger categories".

6.1 Traditional definitions of continuity

6.1.1 Pre-topology

Let μ and ν are funcoids representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall \epsilon \in \mathrm{up} \langle \nu \rangle f a \exists \delta \in \mathrm{up} \langle \mu \rangle \{a\} : \langle f \rangle \delta \subseteq \epsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \mathrm{up} \langle \nu \rangle f a : \langle f \rangle \langle \mu \rangle \{a\} \subseteq \epsilon;$$
$$\langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle f a;$$
$$\langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle \langle f \rangle \{a\}.$$

f is a continuous map from μ to ν in every point of its domain iff $\langle f \rangle \langle \mu \rangle \subseteq \langle \nu \rangle \langle f \rangle$ what is equivalent to $f \circ \mu \subseteq \nu \circ f$.

6.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a nearness-continuous map from μ to ν iff

$$\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow (\langle f\rangle X)[\nu](\langle f\rangle Y)).$$

Equivalently transforming this formula we get:

$$\begin{gathered}
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y \cap \langle\nu\rangle\langle f\rangle X \neq \emptyset);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y \cap \langle\nu\circ f\rangle X \neq \emptyset);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X[\nu\circ f]\langle f\rangle Y);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y\left[(\nu\circ f)^{-1}\right]X);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow \langle f\rangle Y\left[f^{-1}\circ\nu^{-1}\right]X);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X \cap \left\langle f^{-1}\circ\nu^{-1}\right\rangle\langle f\rangle Y \neq \emptyset);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X \cap \left\langle f^{-1}\circ\nu^{-1}\circ f\right\rangle Y \neq \emptyset);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow Y\left[f^{-1}\circ\nu^{-1}\circ f\right]X);\\
\forall X, Y \in \mathscr{P}\mho: (X[\mu]Y \Rightarrow X\left[f^{-1}\circ\nu\circ f\right]Y);\\
\mu \subseteq f^{-1}\circ\nu\circ f.
\end{gathered}$$

So a function f is nearness-continuous iff $\mu \subseteq f^{-1}\circ\nu\circ f$.

6.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu\forall(x; y) \in \delta: (fx; fy) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\begin{gathered}
\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu\forall(x; y) \in \delta: \{(fx; fy)\} \subseteq \epsilon\\
\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu\forall(x; y) \in \delta: f\circ\{(x; y)\}\circ f^{-1} \subseteq \epsilon\\
\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu: f\circ\delta\circ f^{-1} \subseteq \epsilon\\
\forall\epsilon \in \operatorname{up}\nu: f\circ\mu\circ f^{-1} \subseteq \epsilon\\
f\circ\mu\circ f^{-1} \subseteq \nu.
\end{gathered}$$

So a function f is uniformly continuous iff $f\circ\mu\circ f^{-1} \subseteq \nu$.

6.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \operatorname{Mor}(\operatorname{Ob}\mu; \operatorname{Ob}\nu) \wedge f\circ\mu \subseteq \nu\circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$\begin{aligned}
f \in \mathrm{C}'(\mu; \nu) &\Leftrightarrow f \in \operatorname{Mor}(\operatorname{Ob}\mu; \operatorname{Ob}\nu) \wedge \mu \subseteq f^{\dagger}\circ\nu\circ f;\\
f \in \mathrm{C}''(\mu; \nu) &\Leftrightarrow f \in \operatorname{Mor}(\operatorname{Ob}\mu; \operatorname{Ob}\nu) \wedge f\circ\mu\circ f^{\dagger} \subseteq \nu.
\end{aligned}$$

Remark 180. In the examples about funcoids and reloids the "dagger functor" is the inverse of a funcoid or reloid, that is $f^{\dagger} = f^{-1}$.

Proposition 181. Every of these three definitions of continuity forms a sub-precategory (subcategory if the original precategory is a category).

Proof.

C. Let $f \in \mathrm{C}(\mu; \nu)$, $g \in \mathrm{C}(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}(\mu; \mu)$ is obvious.

C′. Let $f \in \mathrm{C}'(\mu; \nu)$, $g \in \mathrm{C}'(\nu; \pi)$. Then $\mu \subseteq f^{\dagger} \circ \nu \circ f$, $\nu \subseteq g^{\dagger} \circ \pi \circ g$;

$$\mu \subseteq f^{\dagger} \circ g^{\dagger} \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^{\dagger} \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}'(\mu; \mu)$ is obvious.

C″. Let $f \in \mathrm{C}''(\mu; \nu)$, $g \in \mathrm{C}''(\nu; \pi)$. Then $f \circ \mu \circ f^{\dagger} \subseteq \nu$, $g \circ \nu \circ g^{\dagger} \subseteq \pi$;

$$g \circ f \circ \mu \circ f^{\dagger} \circ g^{\dagger} \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^{\dagger} \subseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}''(\mu; \mu)$ is obvious. □

Proposition 182. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}'(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof. Let $f \in \mathrm{C}'(\mu; \nu)$. Then $\mu \subseteq f^{\dagger} \circ \nu \circ f$; $f \circ \mu \subseteq f \circ f^{\dagger} \circ \nu \circ f \subseteq 1_{\mathrm{Dst}\,f} \circ \nu \circ f = \nu \circ f$; $f \in \mathrm{C}(\mu; \nu)$.

Let $f \in \mathrm{C}(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f \circ \mu \circ f^{\dagger} \subseteq \nu \circ f \circ f^{\dagger} \subseteq \nu \circ 1_{\mathrm{Dst}\,f} = \nu$; $f \in \mathrm{C}''(\mu; \nu)$. □

Proposition 183. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}''(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}'(\mu; \nu).$$

Proof. Let $f \in \mathrm{C}''(\mu; \nu)$. Then $f \circ \mu \circ f^{\dagger} \subseteq \nu$; $f \circ \mu \circ f^{\dagger} \circ f \subseteq \nu \circ f$; $f \circ \mu \circ 1_{\mathrm{Src}\,f} \subseteq \nu \circ f$; $f \circ \mu \subseteq \nu \circ f$; $f \in \mathrm{C}(\mu; \nu)$.

Let $f \in \mathrm{C}(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f^{\dagger} \circ f \circ \mu \subseteq f^{\dagger} \circ \nu \circ f$; $1_{\mathrm{Src}\,f} \circ \mu \subseteq f^{\dagger} \circ \nu \circ f$; $\mu \subseteq f^{\dagger} \circ \nu \circ f$; $f \in \mathrm{C}'(\mu; \nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 184. If f is a monovalued and entirely defined morphism then

$$f \in \mathrm{C}'(\mu; \nu) \Leftrightarrow f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 185. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu; \nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD}) f \in \mathrm{C}'((\mathsf{FCD})\mu; (\mathsf{FCD})\nu)$.

Exercise 1. I leave a simple exercise for the reader to prove the last theorem.

6.3 Continuousness of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to *identity element* I_A of the semigroup (for example identity funcoid or identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is "dagger" (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define *restricting* an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 186. For any elements f, μ, ν of our semigroup and an object A

1. $f \in \mathrm{C}(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu)$;
2. $f \in \mathrm{C}'(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu)$;
3. $f \in \mathrm{C}''(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof.

1. $f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu; \nu)$.
2. $f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^\dagger \circ \nu \circ f|_A \Leftarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A \Leftarrow \mu \subseteq f^\dagger \circ \nu \circ f \Leftrightarrow f \in \mathrm{C}'(\mu; \nu)$.
3. $f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^\dagger \subseteq \nu \Leftarrow f \circ \mu \circ f^\dagger \subseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu; \nu)$. □

7 Postface

7.1 Misc

I deem that now two most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids.

Also a future research topic are n-ary (where n is an ordinal, or more generally an index set) funcoids and reloids (plain funcoids and reloids are binary by analogy with binary relations).

We should also research relationships between complete funcoids and complete reloids.

7.2 Pointfree funcoids and reloids

I have set wiki site http://funcoids.wikidot.com to write on that site the pointfree variant of the theory of funcoids and reloids (that is generalized funcoids on arbitrary lattices rather than funcoids on a lattice of sets as in this work).

However I consider for me research of pointfree funcoids and pointfree reloids a low priority project. (There are yet enough research topics in the point-set topology and I don't want to meddle into pointfree topology in foreseeable future.)

The work about pointfree funcoids and reloids seems being largely technical and boring. Pointfree theory of funcoids and reloids seems being a trivial generalization of the theory of point-set funcoids and reloids. It is not similar to the traditional pointfree topology which is not an obvious generalization of point-set topology.

But if someone indeed wishes to treat pointfree funcoids, please use the above menioned wiki.

Appendix A Some counter-examples

[TODO: More counter-examples similar to examples in [5].]

Theorem 187. For a f.o. a we have $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ only in the case if $a = \emptyset$ or a is a trivial atomic f.o. (that is an one-element set).

Proof. If $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ then exists $m \in \operatorname{up}(a \times^{\mathsf{RLD}} a)$ such that $m \subseteq (=)|_{\mho}$. Consequently exist $A, B \in \operatorname{up} a$ such that $A \times B \subseteq (=)|_{\mho}$ what is possible only in the case when $A = B = a$ is an one-element set or empty set. □

Corollary 188. Direct product (in the sense of reloids) of non-trivial atomic filter objects is non-atomic.

Proof. Obviously $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \neq \emptyset$ and $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \subset a \times^{\mathsf{RLD}} a$. □

Example 189. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object and $x \in \mho$. Then

$$(a \times \{x\}) \circ (\{x\} \times a) = \bigcap^{\mathfrak{F}} \{(A \times \{x\}) \circ (\{x\} \times A) \mid A \in \operatorname{up} a\} = \bigcap^{\mathfrak{F}} \{A \times A \mid A \in \operatorname{up} a\} = a \times a$$

is non-atomic despite of $a \times \{x\}$ and $\{x\} \times a$ are atomic. □

Example 190. There exists non-monovalued atomic reloid.

Proof. From the previous example follows that the atomic reloid $\{x\} \times a$ is not monovalued. □

[TODO: Example of $(\mathsf{RLD})_{\mathrm{in}} \neq (\mathsf{RLD})_{\mathrm{out}}$.]

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