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Complex Dynamics and the High-energy Regime of Quantum Field Theory

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Abstract

The standard model embodies our current knowledge of elementary particle physics and represents a well-tested framework for the study of non-gravitational phenomena at low energies. It is built on the foundations of relativistic quantum field theory (QFT), which provides the correct description of electroweak and strong interactions involving leptons and quarks. It is generally believed that, extending the validity of QFT to energies on or beyond the TeV range must include the unavoidable signature of vacuum fluctuations and strong-field gravity. We argue that an effective approach to the high-energy regime of QFT demands the tools of complex dynamics and fractal operators. The unexpected consequences of using fractal operators to model complexity beyond the current range of QFT are outlined and discussed.

Keywords: random walks and Levy flights, quantum field theory, theory and models of chaotic systems, renormalization.

1. Introduction and Motivation

The standard model of elementary particle physics is an effective low-energy framework for the description of non-gravitational phenomena up to several hundreds GeV [1, 13]. It contains a set of quantum field theories (QFT) whose dynamical structure lies on the principles of relativistic invariance and local gauge symmetry. It is generally believed that the standard model is not the "final" theory of nature as it is unable to provide a realistic account of physics above an energy scale of $O(\mathrm{TeV})$. It is reasonable to anticipate that passing this threshold may trigger a mixture of complex processes dominated by large and non-local vacuum fluctuations and various levels of gravitational coupling [14, 15]. The standard model is not properly equipped to handle these phenomena. This is because its framework a) ignores gravity from the outset and b) relies on conventional tools of quantum Hamiltonian dynamics and perturbation theory that fall outside the realm of complexity [10, 29-30, 33]. Let us recall, in this context, that the main tool of QFT for relating observable quantities to theoretic predictions is the field transition amplitude. Feynman diagrams are constructed by expanding the transition amplitude as a power series in the coupling constant. The perturbation technique is likely to break down on or above the TeV threshold whereby fields are expected to fluctuate strongly at all wavelengths and exhibit dynamic patterns that signal the emergence of complex behavior [8, 15, 19-20].

Following the underlying philosophy of [2], the key premise of this work is that adequate modeling of TeV physics demands the tools of fractal operators and fractional calculus. Fractal operators are required when making the transition from a dynamical process characterized by smooth space-time or configuration paths to a process displaying irregular and non-differentiable paths.

This transition is, however, far from being a "trivial" extrapolation. Many phenomena described by fractal operators have multiple time scales and are known to violate the classical paradigms of time symmetry, locality and analyticity [2].

In our work the behavior of fields approaching the TeV threshold is modeled from the standpoint of generic random flows [12]. In particular, we consider stochastic processes with Levy distribution because of their role as representative prototypes of complex dynamics. As it is known, mathematical modeling of Levy flows requires substitution of ordinary derivatives and integrals with fractal operators [2-5]. Levy flows have found numerous applications in science and engineering from the study of turbulence to the physics of plasma, molecular collisions and propagation through disordered media.

Our main findings may be summarized as follows: i) classical gravitation becomes a natural part of the picture for $\beta \neq 1$ and through the use of time fractal operators; ii) the Levy index α may be used to control convergence of Feynman diagrams.

The paper is organized in the following way: section 2 outlines the simplifying assumptions of the model. Section 3 contains a brief review on the theory underlying fractional wave equation (FWE). Section 4 outlines the connection between a generic random flow and the fractal dimension of its trajectory. Formal equivalence of the FWE to the equation of a confined Levy flow is discussed in section 5. Section 6 links the previously developed body of ideas to the theory of complex-scalar fields. The close analogy between FWE and the gravitational time-shift of general relativity is examined in section 7. Use of Levy index α to control the convergence of the perturbation series is detailed in section 8. The paper is summarized in the last section.

2. Assumptions

We list below the set of assumptions underlying the foundation of our model:

A1) fields are treated as classical objects as a result of decoherence induced by steady exposure to randomness [21-23].

A2) all variables are dimensionless upon normalization to their respective unit of measurement.

A3) analysis is limited to a two-dimensional space-time manifold and ultrashort time intervals $t \ll 1$ commensurate with the scale of TeV physics. Exception is made in section 8 where the analysis is carried out in four-dimensional space-time.

A4) spatial fluctuations of the random field are Levy stable processes of index α $(1 \leq \alpha < 2)$.

A5) all scalar functions dependent on space-time are considered analytic.

3. Fractional Wave Equation

To make the paper self-contained, we review in this section the transport of a generic wave disturbance through a randomly fluctuating medium. Under the most general circumstances, the stochastic flow of the disturbance through a 1+1 space-time is governed by the fractional wave equation (FWE) [2, 24-25]

$$D_t^{\beta}[\rho(x,t)] - \frac{t^{-\beta}\rho_0(x)}{\Gamma(1-\beta)} = \frac{\partial^{\alpha}}{\partial |x|^{\alpha}}[\rho(x,t)] \quad (1)$$

Here $\rho(x,t)$ represents the probability density of locating the center of the wave at x after a time elapse t and $\rho_0(x) = \rho(x,t=0) = \delta(x)$ is the initial probability density. Exponents α and β define the correlation range of incremental steps occurring in space and time, respectively. Because FWE describes wave propagation through a highly disordered medium we are going to characterize it hereafter as a *random flow*.

The definition of the Riemann-Liouville fractal operator acting on the time variable is given by [7]

$$D_t^{-\beta}[\rho(x,t)] \doteq \frac{1}{\Gamma(\beta)} \int_0^t \frac{\rho(x,\eta)}{(t-\eta)^{1-\beta}} d\eta \qquad (2)$$

The Riesz fractional operator is defined through [3-4]

$$\frac{\partial^\alpha}{\partial |x|^\alpha} \doteq \begin{cases} -\dfrac{D_+^\alpha + D_-^\alpha}{2\cos(\pi\alpha/2)} & if\, \alpha \neq 1 \\ -\dfrac{d}{dx} H & if\, \alpha = 1 \end{cases} \qquad (3)$$

in which the left and right Riemann-Liouville derivatives are, respectively

$$D_+^\alpha[\rho(x,t)] \doteq \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_{-\infty}^x \frac{\rho(\xi,t)d\xi}{(x-\xi)^{\alpha-1}}$$

$$D_-^\alpha[\rho(x,t)] \doteq \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int_x^\infty \frac{\rho(\xi,t)d\xi}{(x-\xi)^{\alpha-1}} \qquad (4)$$

and where the Gilbert transform operator is

$$H[\rho(x,t)] = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\rho(\xi,t)d\xi}{x-\xi} \qquad (5)$$

In particular, fractional wave equation with no time memory $(\beta = 1)$ represents the evolution of a flow governed by Levy statistics. The dynamics of an unconstrained Levy flow is represented by the free fractional Fokker-Planck equation (FFPE)

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial^\alpha \rho(x,t)}{\partial |x|^\alpha} \qquad (6)$$

In the presence of an arbitrary driving force that derives from a potential function $U(x)$, FFPE assumes the form [3-4]

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial |x|}[\frac{dU}{d|x|}\rho(x,t)] + \frac{\partial^\alpha \rho(x,t)}{\partial |x|^\alpha} \qquad (7)$$

Balancing out the driving force with Levy noise and the damping term leads to the Langevin equation

$$\frac{d|x|}{dt} = -\frac{dU}{d|x|} + Y_\alpha(t) \qquad (8)$$

in which $Y_\alpha(t)$ denotes the time-dependent random force of Levy index α.

4. Geometrical attributes of the random flow

It can be shown that, to each α, β one can associate a fractal dimension d_F that characterizes the random flow path [2, 26]. The direct consequence of this conjecture is that the dynamics of the random flow may be directly linked to the underlying geometry of the embedding space-time. In this section we briefly review this connection using arguments pertaining to the physics of anomalous diffusion.
The spatial variance of the flow in 1 + 1 dimensions is given by [2]

$$\langle x^2(t) \rangle \sim t^{\frac{2}{d_F}} \qquad (9)$$

The asymptotic limit of the probability density function reads [2], [27]

$$\rho_a(x,t) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)} \sin(\frac{\pi\alpha}{2}) \frac{t^\beta}{|x|^{1+\alpha}} \qquad (10)$$

for

$$t^\beta < x^\alpha \ll 1 \qquad (11)$$

Using (10) leads to

$$\langle x^2(t)\rangle = \int_{x_L}^{x_H} x^2 \rho_a(x,t)\,dx$$
$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\sin(\frac{\pi\alpha}{2})\frac{t^\beta}{(2-\alpha)}[|x_H|^{2-\alpha} - |x_L|^{2-\alpha}] \tag{12}$$

Normalization of the probability density function requires

$$\int_{x_L}^{x_H} \rho_a(x,t)\,dx = 1 \tag{13}$$

where $[x_L, x_H]$ denote the spatial boundaries of the flow. A reasonable approximation consistent with (11) is

$$x_L \approx x,\ x_H \gg x_L \tag{14}$$

and (13) yields the following constraint

$$t^\beta = \frac{x^\alpha}{D(\alpha,\beta)} \tag{15}$$

Here

$$D(\alpha,\beta) = \frac{\Gamma(1+\alpha)}{\alpha\Gamma(1+\beta)}\sin(\frac{\pi\alpha}{2}) \tag{16}$$

plays the role of a generalized diffusion constant whose classical value converges to the light speed *in vacuo*, i.e.

$$\lim_{\alpha,\beta\to 1} D(\alpha,\beta) = 1 \Rightarrow \frac{x}{t} = 1 \tag{17}$$

Condition (11) requires that α, β satisfy

$$\frac{\Gamma(1+\alpha)\sin(\frac{\pi\alpha}{2})}{\alpha\Gamma(1+\beta)} > 1 \tag{18}$$

A direct numerical analysis shows that, in order to comply with (18), the choice of α, β is bounded by the following intervals

$$\alpha \approx 1\ (\alpha > 1),\ \beta \approx 0.3 \div 0.8 \tag{19}$$

Plugging (15) and (16) into (12) yields

$$\langle x^2(t)\rangle = \frac{\alpha}{(2-\alpha)} D(\alpha,\beta)^{\frac{2}{\alpha}} t^{\frac{2\beta}{\alpha}} \tag{20}$$

from which we obtain, by comparison to (9), the following definition for the fractal dimension of the flow

$$d_F = \frac{\alpha}{\beta} \tag{21}$$

5. Fractional Wave Equation as a confined Levy flow

Often times it is convenient to study the fractional wave equation (1) in a simplified context where memory effects encoded by β are absent. The Levy flow is a statistical process allowing for such a simplified treatment. It is characterized by $\beta = 1, 1 < \alpha \le 2$ and possesses only long-range space correlations of Levy index α. We are going to show in this section that, up to a first order approximation, (1) may be treated as a Levy flow confined by an equivalent applied potential (EAP). The starting point is the alternative expression of the time operator [7]

$$D_t^\beta[\rho(x,t)] = \frac{\rho(x,\tau)}{\Gamma(1-\beta)}(t-\tau)^{-\beta}$$
$$+ \frac{1}{\Gamma(2-\beta)}\frac{\partial\rho(x,\tau)}{\partial t}(t-\tau)^{1-\beta} + T(\tau,\beta) \tag{22}$$

$$T(\tau,\beta)=\frac{1}{\Gamma(2-\beta)}\int_{\tau}^{t}(t-\eta)^{1-\beta}\frac{\partial^2\rho}{\partial t^2}(x,\eta)d\eta \tag{23}$$

in which τ represents a suitably chosen cutoff scale assigned to the time variable. For $\tau \sim \eta < t \ll 1$, we adopt the following approximations

$$t = q\tau$$

$$\frac{\partial^2\rho(x,\eta)}{\partial t^2}\approx\frac{\partial^2\rho(x,\tau)}{\partial t^2}\approx\frac{1}{(q-1)\tau}\frac{\partial\rho(x,\tau)}{\partial t} \tag{24}$$

$$\int_{\tau}^{t}(t-\eta)^{1-\beta}\frac{\partial^2\rho}{\partial t^2}(x,\eta)d\eta\approx(q-1)^{-\beta}\tau^{1-\beta}\frac{\partial\rho(x,\tau)}{\partial t}$$

where q is a natural number close to 1 $(q \approx 1, 2..)$. Plugging (24) into (1) and (22) yields, after few algebraic manipulations,

$$\frac{\partial\rho}{\partial t}=\frac{\beta-1}{t}[\rho-(1-\frac{1}{q})^{\beta}\rho_0]+\frac{\partial^{\alpha}\rho}{\partial|\xi|^{\alpha}} \tag{25}$$

Here ξ denotes a new spatial variable defined as

$$|\xi|^{\alpha}=\frac{q\tau^{1-\beta}}{(q-1)^{\beta}\Gamma(2-\beta)}|x|^{\alpha} \tag{26}$$

Let us now compare (25) with the fractional Fokker-Planck equation of the confined Levy flow [3- 4]

$$\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial|\xi|}[\frac{dU}{d|\xi|}\rho]+\frac{\partial^{\alpha}\rho}{\partial|\xi|^{\alpha}} \tag{27}$$

in which $U=U(\xi)$ is the equivalent applied potential (EAP). We get

$$\frac{\partial}{\partial|\xi|}[\frac{dU}{d|\xi|}\rho]=\frac{\beta-1}{t}[\rho-(1-\frac{1}{q})^{\beta}\rho_0] \tag{28}$$

To solve (28) for $U(\xi)$, we use the closed form expression of the probability density function (10) and substitute the time variable with its spatial counterpart (15). The resulting expression may be written as

$$\frac{\partial}{\partial|\xi|}[\frac{dU}{d|\xi|}\frac{1}{|\xi|}]\sim P(\alpha,\beta)|\xi|^{-(d_F+1)}+Q(\alpha,\beta)|\xi|^{-d_F}\delta(\xi) \tag{29}$$

where $P(\alpha,\beta)$ and $Q(\alpha,\beta)$ are composite functions of α and β, $\rho_0(\xi)=\delta(\xi)$ is the initial probability density function and d_F is the fractal dimension of the flow defined by (21). Neglecting for simplicity the constant term contributed by $\delta(\xi)$ upon integration, we obtain from (29) the following expression for EAP

$$U(|\xi|)\sim|\xi|^{2-d_F} \tag{30}$$

This result indicates that EAP diverges the short-distance regime of $|\xi| \ll 1$ and $d_F > 2$. Recalling that $d_{QM} = 2$ is the fractal dimension of a generic quantum-mechanical path in 1 + 1 dimensions [28], we conclude that $d_F > d_{QM}$ signals the transition from a shallow to a steep potential near $|\xi| \ll 1$. The dynamic implications of this important finding are examined in Appendix A, where we discuss a potential link between confined Levy flows and Feynman diagrams.

6. Random flows and complex scalar field theory

In this section we wish to link the formalism previously developed to the physics of complex scalar field, a traditional precursor of more realistic field models such as quantum electrodynamics and non-abelian gauge theories

[29-30, 33]. To this end, consider a free classical and massless complex scalar field with Lagrangian

$$L = \partial^{\mu}\varphi\partial_{\mu}\varphi^{*} \tag{31}$$

The two conjugate fields φ, φ^{*} are parameterized using their real components φ_1, φ_2, that is

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$$
$$\varphi^{*} = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2) \tag{32}$$

To simplify the argument, let us assume that φ_1 is uniform and slowly varying in time and space $(\frac{\partial\varphi_1}{\partial t} \approx 0, \frac{\partial\varphi_1}{\partial|x|} \approx 0)$, whereas φ_2 is rapidly varying in time. The equations of motion are, respectively

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial|x|^2} = 0$$
$$\frac{\partial^2\varphi^{*}}{\partial t^2} - \frac{\partial^2\varphi^{*}}{\partial|x|^2} = 0 \tag{33}$$

As is it well known, equations (33) determine the evolution of free relativistic spinless fields. The theory contains the globally conserved charge density

$$\rho = i(\frac{\partial\varphi}{\partial t}\varphi^{*} - \frac{\partial\varphi^{*}}{\partial t}\varphi) \tag{34}$$

that may be identified as a standard probability density function [30]. The charge density satisfies the continuity and normalization equations

$$\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial|x|} = 0$$
$$\int_{-\infty}^{\infty}\rho(x,t)dx = 1 \quad \text{for} \quad t > 0 \tag{35}$$

where j stands for the current density

$$j = [i(\frac{\partial\varphi}{\partial|x|}\varphi^{*} - \frac{\partial\varphi^{*}}{\partial|x|}\varphi)] \tag{36}$$

The Levy flow of the complex scalar field represents a natural translation of (33) in the language of fractal operators. Under these circumstances, the conventional gradient operator ∂_{μ} becomes

$$\partial_{\mu} \doteq (\frac{\partial}{\partial t}, \frac{\partial^{\alpha}}{\partial|x|^{\alpha}}) \tag{37}$$

If Levy noise can be expressed as an analytic function in the field observables, Lagrangian (31) and equations (33) may be upgraded to

$$L = \partial^{\mu}\varphi\partial_{\mu}\varphi^{*} + N_{\alpha}(\varphi,\varphi^{*})$$
$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^{\alpha+1}\varphi}{\partial|x|^{\alpha+1}} + \frac{\partial N_{\alpha}}{\partial\varphi^{*}} = 0 \tag{38}$$
$$\frac{\partial^2\varphi^{*}}{\partial t^2} - \frac{\partial^{\alpha+1}\varphi^{*}}{\partial|x|^{\alpha+1}} + \frac{\partial N_{\alpha}}{\partial\varphi} = 0$$

Here

$$N_{\alpha}[\varphi(x,t),\varphi^{*}(x,t)] = Y_{\alpha}(t)[x - x_0] \tag{39}$$

is the random potential term associated with the Levy force $Y_{\alpha}(t)$. Due to the postulated behavior of components φ_1 and φ_2, the sum of the two conjugate fields may be treated as a constant, i.e.

$$\frac{\partial}{\partial t}(\varphi+\varphi^{*})=\frac{\partial}{\partial t}(\sqrt{2}\varphi_{1})\approx 0$$
$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}(\varphi+\varphi^{*})=\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}(\sqrt{2}\varphi_{1})\approx 0 \tag{40}$$

The last condition amounts to

$$\varphi_{1}(x)\sim\frac{|x|^{-\alpha}}{\Gamma(\alpha-1)} \tag{41}$$

On account of (38) and (41), the density function and its time rate become

$$\rho=\varphi_{1}\frac{\partial\varphi_{2}}{\partial t}$$
$$\frac{\partial\rho}{\partial t}=-i[\varphi^{*}\frac{\partial^{\alpha+1}\varphi}{\partial|x|^{\alpha+1}}-\varphi\frac{\partial^{\alpha+1}\varphi^{*}}{\partial|x|^{\alpha+1}}] \tag{42}$$

The last relation reflects the fractional generalization of the continuity equation (35). From (10), (26), (41) and (42) we derive the following scaling behavior for φ_1,φ_2

$$\varphi_{1}\sim|\xi|^{-\alpha},\ \varphi_{2}\sim|\xi|^{2\alpha} \tag{43}$$

To summarize this section, we note that both components of the field as well as EAP exhibit a power-law dependence on the spatial coordinate ξ according to (30) and (43). Such scaling behavior is typical for critical phenomena and far-from equilibrium processes involving long-range interactions and multiple scales [2, 31]. In general, as the dynamics depends on both α and β, it may be concluded that the random flow of the complex scalar field is a direct consequence of space-time geometry. Furthermore, since in very broad terms, space-time geometry may be regarded as manifestation of an underlying gravitational field, it is instructive to explore if a direct connection between α,β and gravitation may be established. This is the object of the next section.

7. Analogy with propagation in a classical gravitational field

The aim of this section is to show that, to a first-order approximation, the dynamics of the complex-scalar field driven by stochastic fluctuations is equivalent to classical motion in a gravitational field. This non-trivial finding is consistent with the guiding philosophy of [8] and may serve as a basis for the high-energy unification of classical gravity with standard model interactions.

For infinitesimal time intervals η near the origin $t=0$, the time rate slows down as [6]

$$\tau(\eta)=\frac{\eta^{\beta}}{\Gamma(1+\beta)} \tag{47}$$

General relativity asserts that the clock rate of proper time at a fixed location in a gravitational field $\tau_G[x(\eta)]$ relates to the proper time η measured sufficiently far from the source through [32]

$$\frac{\tau_{G}[x(\eta)]}{\eta}=\sqrt{g_{00}[x(\eta)]} \tag{48}$$

Equating the effects embodied in (47) and (48) on account of (15) yields the following power law behavior

$$g_{00}(\eta)=\frac{\eta^{2(\beta-1)}}{\Gamma^{2}(1+\beta)}\sim\frac{|\xi|^{2d_{F}(\beta-1)}}{\Gamma^{2}(1+\beta)} \tag{49}$$

This scaling relation gives the metric potential that yields the same time-shift as a fractal manifold with dimension d_F. According to (49), strong-gravity associated with large amplitudes of the metric emerges in the ultra-short distance regime $|\xi| \ll 1$ and for $\beta < 1$. Suppressing memory effects of the flow dynamics in the limit $\beta \to 1$ recovers the Lorentz metric of special relativity ($g_{00} = 1$).

8. Perturbation expansion and renormalization

Contemporary QFT regards particles as zero-dimensional objects experiencing local interactions in space-time. This viewpoint leads to inherent divergences of Feynman diagrams at both ends of the energy scale. The standard way of removing divergences in QFT is through the use of various regularization techniques [29-30, 33]. It is instructive to point out in this context that, unlike conventional QFT, the long-range spatial correlation of Levy flows makes particle interaction a predominantly *non-local* process. This property is fundamentally different from the basic attribute of strings and branes of being extended spatial objects [30, 33]. In our model, fields remain zero-dimensional objects that are subjected to a continuous spectrum of non-local interactions, as implied by the distributed nature of fractal operators [2]. Here we revisit the issue of diagram convergence from the standpoint of fractional dynamics. To illuminate the essentials of the argument, the analysis is limited to the massless Klein-Gordon theory. We caution that our treatment is a preliminary enterprise that does not claim to be either entirely rigorous or exhaustive.

It is well known that the classical Klein-Gordon equation for the free and massless scalar field corresponds to the case $\varphi \equiv \varphi^*$ and may be derived from (31) as

$$\partial^2 \varphi(x,t) \equiv \partial_\mu \partial^\mu \varphi(x,t) = 0 \qquad (60)$$

The model gives rise to the following propagator

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp[ik(x-y)]}{k^2 + i\varepsilon} \qquad (61)$$

In the above $k = \{E, k_i\}$, $i = 1,2,3$ is the four momentum vector and

$$k^2 = k_\mu k^\mu = E^2 - \vec{k}^2 \qquad (62)$$

Consider now the self-interacting φ^4 theory having a small coupling constant $\lambda \ll 1$ [29-30, 33]. Feynman graphs containing the free propagator (61) become singular at large momenta $k \gg 1$ due to either quadratic or logarithmic divergence carried by amplitude terms of the form

$$amplitude \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^{2n}} \qquad (63)$$

in which $n = 1,2$. Consider also the Levy flow of a massless scalar field described by (38). For the sake of simplicity and clarity, we proceed with the following assumptions:

a) contribution of the Levy noise term $N_\alpha(\varphi)$ to the propagator amounts to a constant term.
b) contributions generated by the exponential phase factor $\exp[ik(x-y)]$ and correction term $i\varepsilon$ may be omitted.
c) integration in (63) is carried out in the ultraviolet region ($E, |\vec{k}| \gg 1$) and is bounded to the mass-shell range, that is, $k^2 \approx 0$.
d) all three spatial directions are characterized by the same index α.

Under these circumstances, the corresponding propagator reads

$$D_\alpha(x-y) \sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_\alpha^2} \qquad (64)$$

in which

$$k_\alpha^2 = E^2 - \vec{k}^{2\alpha} \tag{65}$$

To simplify notation, let us define

$$q_\alpha \doteq \left|\vec{k}\right|^\alpha$$
$$d^4k_\alpha \doteq dEd^3q_\alpha \tag{66}$$

From (66) the four-dimensional momentum volume may be expressed as

$$d^4k = dEd^3\left|\vec{k}\right| = \frac{d^4k_\alpha}{\alpha^3(q_\alpha)^{\frac{3(\alpha-1)}{\alpha}}} \tag{67}$$

Thus, propagator (64) becomes

$$D_\alpha(x-y) \sim \int \frac{d^4k_\alpha}{(2\pi)^4}\frac{1}{k_\alpha^2(q_\alpha)^{\frac{3(\alpha-1)}{\alpha}}} \tag{68}$$

Here we have, in light of (65)

$$q_\alpha = \sqrt{E^2 - k_\alpha^2} \tag{69}$$

Using (69) leads to the following extrapolation of (63)

$$amplitude \sim \int \frac{d^4k_\alpha}{(2\pi)^4}\frac{1}{(k_\alpha^{2n})(q_\alpha)^{\frac{3n(\alpha-1)}{\alpha}}} \tag{70}$$

One recovers (64) in the limit $\alpha = n = 1$. The energy variable may be integrated out with the help of the following approximation [33]

$$\frac{1}{k_\alpha^{2n}} \approx \delta(k_\alpha^{2n}) = \frac{\delta(E-q_\alpha)}{2nE^{2n-1}} + \frac{\delta(E+q_\alpha)}{2n\left|E\right|^{2n-1}} \tag{71}$$

Inserting (71) back into (70) and employing only the positive value of E we find

$$amplitude \sim \frac{1}{E^{2n-1}}\int \frac{d^3q_\alpha}{q_\alpha^{\frac{3n(\alpha-1)}{\alpha}}} = \frac{1}{E^{2n-1}}[\int \frac{dq_\alpha}{q_\alpha^{\frac{n(\alpha-1)}{\alpha}}}]^3 \tag{72}$$

On account of assumption c), the amplitude (72) stays finite if

$$\alpha > \frac{3n}{5n-4} \tag{73}$$

Recalling that $1 \le \alpha < 2$ (according to A4)), it follows that (73) is satisfied if $n \ge 2$. To enable convergence of (72) for $n = 1$, we need to relax assumption d) and consider three independent Levy indices, one for each spatial direction. This case may be associated with the geometry of non-commutative field theory and falls beyond the scope of the paper.

In closing, we remark that the regularization method discussed above is conceptually similar to the procedure developed in [11]. The main argument of [11] is that convergence is obtained upon integrating each term of the perturbation series on a D-dimensional space-time endowed with fractal support $(D \le 4)$.

9. Summary and conclusions

The basic assertion of our work is that mathematical tools of fractional calculus and complexity theory are necessary to properly describe the high-energy regime of QFT. The dynamic effect of the fractal space-time geometry encoded in the pair (α, β) has been studied with the help of fractional wave equation and the Levy flow model. We have found that i) classical gravitation becomes a natural part of the picture for $\beta \neq 1$ and through the use of fractal time operators; ii) Levy index α may be used to control convergence of Feynman diagrams.

In Appendix A we indicate that bifurcations of the probability density function may act as a natural source for the creation–annihilation events in particle physics. The formal connection between (α, β) and quantum mechanical spin is briefly discussed in Appendix B. According to this viewpoint, it seems conceivable that bosons and fermions are condensates of space-time geometry produced by cooling from the high temperature regime of TeV physics to the low-energy regime of the standard model.

Appendix A

Despite its impressive utility and predictive power, the technique of Feynman diagrams does not explain the underlying physics of vertices and loops in particle interactions. Feynman graphs are primarily employed as an effective tool for computing amplitudes and quantum corrections associated with various channels [29-30, 33]. It was recently shown that Levy flows confined by steep potentials undergo natural bifurcations of the probability density function [3-4]. Following this finding, we suggest here that decay vertices of Feynman graphs may be identified with *bifurcations in the flow of charge density* carried by gauge bosons, leptons and quarks. We elaborate below on this point.

A distinctive property of a generic Levy flow confined by anharmonic potentials of the type

$$U(x) = \lambda \frac{|x|^c}{c} \tag{A1}$$

where $c \geq 4$, is that its probability density function $\rho(x,t)$ bifurcates from an initial mono-modal state to a stationary bimodal state. Fig. 1 below illustrates this type of behavior for the case of a stationary quartic oscillator characterized by $c = 4$, $\alpha = 1$ and the probability density function

$$\rho_{st}(x) = \frac{1}{\pi(1 - ax^2 + x^4)} \tag{A2}$$

where a plays the role of a control parameter (see [3-4] for additional details).

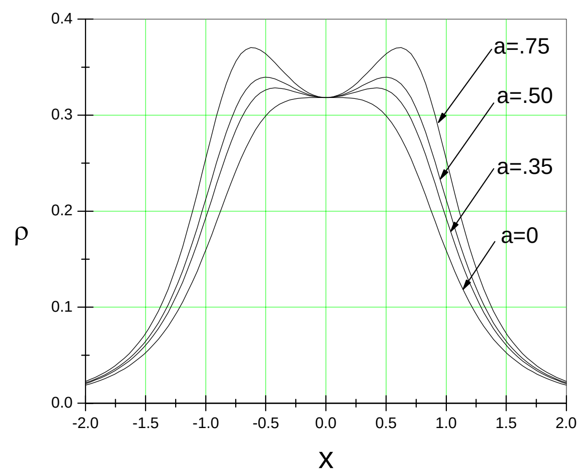

Fig. 1: Probability density function ρ of the stationary quartic oscillator for $\alpha = 1$ and for various values taken by the control parameter a.

In section 6 it was indicated that the analog of probability density function in field theory is the concept of charge density. It follows that bifurcation of probability density function, arisen from coupling the field to steep potentials, corresponds to splitting the charge density flow. In field theoretic terms, bifurcation of the charge density may be viewed as a creation-annihilation event involving emission and absorption of particles at each interaction vertex [29-30, 33]. In section 5 it was shown that random flows with $d_F > 2$ are characterized by steep equivalent applied potentials (EAP) as the spatial variable goes to zero $(|x| \ll 1)$. In this context, the emergence of interaction vertices in Feynman graphs appears to be a natural consequence of the bifurcation scenario presented above.

The bifurcation time may be computed from the general solution of the random flow equation by imposing [3-4]

$$\frac{\partial^2 \rho(x,t_{12})}{\partial x^2} = 0 \text{ as } x \to 0 \qquad \text{(A3)}$$

An equivalent condition is given by

$$\int_0^\infty k^2 \hat{\rho}(k,t)dk = 0 \qquad \text{(A4)}$$

where the characteristic function of the symmetric Levy law is [2]

$$\hat{\rho}(k,t) = \exp(-k^\alpha t) \qquad \text{(A5)}$$

In the most general case of time-correlated dynamics $\beta \neq 1$ and to the first order approximation of infinitesimal time intervals ($k^\alpha t^\beta \ll 1$), we keep only the first two terms of the Mittag-Leffler expansion to produce [2, 7]

$$\hat{\rho}(k,t) = 1 - \frac{1}{\Gamma(1+\beta)}(k^\alpha t^\beta) \qquad \text{(A6)}$$

and the bifurcation time equation becomes

$$\int_0^\mu k^2[1 - \frac{1}{\Gamma(1+\beta)}(k^\alpha t_{12}^\beta)]dk = 0 \qquad \text{(A7)}$$

where μ is the upper bound of the momentum range. We arrive at

$$t_{12} = [\frac{(\alpha+3)\Gamma(1+\beta)}{3\mu^\alpha}]^{1/\beta} \qquad \text{(A8)}$$

Fig. 2 below displays the variation of the decay width $\Gamma \doteq t_{12}^{-1}$ as a function of both α and μ.

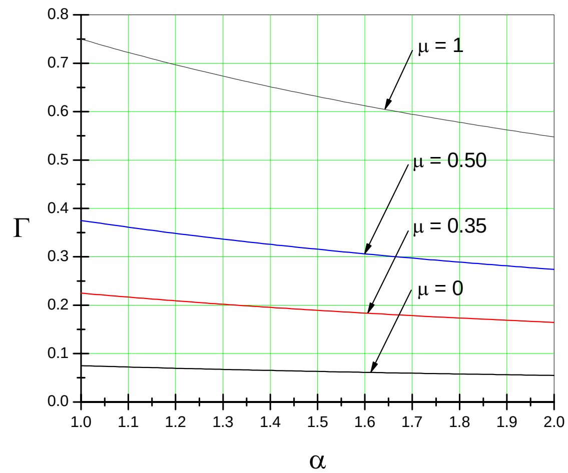

Fig. 2: Decay width Γ as a function of the Levy index α and the upper bound of momentum range μ.

Appendix B

Let us restate that (1) embodies the stochastic evolution of a free scalar wave whose dynamics is correlated in both space and time. Since exponents α, β span a continuous range of values, a natural interpretation of FWE is that it describes the flow of a free classical particle having an *arbitrary spin*. Equation (1) settles to a "classical" dynamic pattern by fixing α and β. In particular:

i) Dirac equation for free massless fermions (where the density function is approximated by the Lorentz scalar $\bar{\psi}\psi$) may be mapped to $\alpha = \beta = 1$.

ii) Schrödinger equation may be obtained for $\alpha = 2, \beta = 1$.

iii) Klein-Gordon equation corresponds to $\alpha = \beta = 2$.

We have repeatedly argued that α, β represent a metric for the underlying geometry of space-time fluctuations. In light of this interpretation, the spin eigenvalues of bosons and fermions may be regarded as *symmetry broken vacua of the space-time geometry*. By analogy with the condensation process, the spin symmetry breaking occurs when the temperature drops from the TeV bound of the high-energy regime to the energy scale of the standard model.

Furthermore, since the distinction between bosons and fermions vanishes if $\alpha, \beta \neq 1, 2$ and all interactions are mediated via gauge bosons, it follows that the distinction between “matter” particles and “force” particles disappears in the TeV regime. Both emerge as a dynamic manifestation of space-time geometry encoded in α and β. This finding is consistent with the analysis carried out in [9].

It is also instructive to point out the difference between our interpretation of spin as a continuous random variable in α, β space and supersymmetry (SUSY). Under some general assumptions, it is possible to show that SUSY – a conjectured symmetry of nature relating fermions to bosons – represents the maximum extension of the Poincare group of space-time symmetries consisting of translations, rotations and boosts. However, since the boson-fermion transposition generates only a spin change of ½, SUSY is a discrete symmetry. In contrast, our approach treats the spin transformation as a continuous operation. A continuous transformation group in α, β space leads to a symmetry that is more comprehensive than SUSY. Moreover, it is known that local SUSY theory (alternatively called supergravity) is thought to provide means for unification of gravitation with standard model interactions. As FWE is not restricted to integer values for α and β, a continuous SUSY transformation offers additional grounds for integrating classical gravity in our model, as advocated in section 7.

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