Einstein's Field Equations in Cosmology Using Harrison's Formula

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Abstract
The most important tool for the study of the gravitational field in Einstein's theory of gravity is his field equations. In this short paper, we demonstrate the derivation of Einstein field equations for the Freedman cosmological model using the Robertson-Walker metric, and furthermore *Harrison's formula* for the Ricci tensor. The difference is that *Harrison's formula* is an actually shorter way of obtaining the field equations. The advantage is that the Cristoffel symbols do not have to be directly calculated one by one. This can actually be a very useful demonstration for somebody who would like to understand a slightly different but faster way of deriving the field equations, something that is actually rarely seen in many of undergraduate and even graduate textbooks.

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1 Introduction
In 1915 Einstein put the finishing touches to the General Theory of Relativity. The famous Schwarzschild solution was the first physically significant solution of the Field Equations of General Relativity. It had showed how space-time is curved around a spherically symmetric distribution of matter. This problem was solved by Schwarzschild and is actually a local problem, in the sense that the distortions of space-time geometry from the Minkowski geometry of Special Relativity gradually diminish to zero as we move further and further away from the gravitating sphere. In general, any space-time geometry generated by such a local distribution of matter is expected to have the same property. It is also known that Newtonian gravity produces an analogous result.

2 The Robertson-Walker metric and the Einstein equations
The Robertson-Walker metric or line element is fundamental in the standard models of cosmology. The mathematical framework in which the Robertson-Walker metric occurs is that of general relativity and takes the form:

$$ds^2 = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}(x)dx^{\mu}dx^{\nu} = \qquad (1)$$

which can be further written as:[1]

$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \qquad (2)$$

where: $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, and R(t) is the scale factor of the universe often called the expansion factor and has the dimensions of length, k can have the values of $k = 0$, -1, 1 corresponding to the three different kind of metrics. One of the most important quantities that they have to be calculated in general relativity is the *Ricci tensor* which is defined below:

$$R_{\mu\nu} = \frac{\partial \Gamma_{\mu\nu}{}^{\lambda}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}{}_{\mu\lambda}}{\partial x^{\nu}} + \Gamma^{\lambda}{}_{\mu\nu}\Gamma^{\sigma}{}_{\lambda\sigma} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} \tag{3}$$

where: Γ's are the so called Cristoffel symbols of the second kind, which are defined as:

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\sigma\lambda}}{\partial x^{\nu}} + \frac{\partial g_{\nu\lambda}}{\partial x^{\sigma}}\right) \tag{4}$$

If we denote the determinant of $g_{\mu\nu}$ as a matrix g then the Ricci tensor can be written as follows:

$$R_{\mu\nu} = \frac{1}{\sqrt{-g}}\left[\Gamma^{\lambda}{}_{\mu\nu}\sqrt{-g}\right]_{\lambda} - \left[ln\left(\sqrt{-g}\right)\right]_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} \tag{5}$$

But our metric $g_{\mu\nu}$ contains only diagonal elements ($\mu = \nu$) (5) it can be futher written as:

$$R_{\mu\mu} = \frac{1}{\sqrt{-g}}\left[\Gamma^{\lambda}{}_{\mu\mu}\sqrt{-g}\right]_{\lambda} - \left[ln\left(\sqrt{-g}\right)\right]_{,\mu\mu} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \tag{6}$$

As a final step we write the $R_{\mu\mu}$ components in terms of the appropriateΓ's and the determinant g with our indices running the values 0,1,2,3. So we have: [2] [3]

$$-R_{oo} = -\frac{\partial \Gamma^{1}_{oo}}{\partial r} + 2\Gamma^{1}_{oo}\Gamma^{o}_{o1} - \Gamma^{1}_{oo}\frac{\partial\, ln\sqrt{-g}}{\partial r}$$

$$-R_{11} = -\frac{\partial \Gamma^{1}_{11}}{\partial r} + \left(\Gamma^{1}{}_{11}\right)^2 + \left(\Gamma^{2}_{12}\right)^2 + \left(\Gamma^{3}_{13}\right)^2 + \left(\Gamma^{o}_{1o}\right)^2 + \frac{\partial^2 ln\sqrt{-g}}{\partial r^2} + \Gamma^{1}_{11}\frac{\partial ln\sqrt{-g}}{\partial r}$$

$$-R_{22} = -\frac{\partial \Gamma^{1}_{22}}{\partial r} + 2\Gamma^{1}_{22}\Gamma^{2}_{21} + \left(\Gamma^{3}_{23}\right)^2 + \frac{\partial^2 ln\sqrt{-g}}{\partial \theta^2} - \Gamma^{1}_{22}\frac{\partial ln\sqrt{-g}}{\partial r}$$

$$-R_{33} = -\frac{\partial \Gamma^{1}_{33}}{\partial r} - \frac{\partial \Gamma^{2}_{33}}{\partial \theta} + 2\Gamma^{1}_{33}\Gamma^{3}_{31} + 2\Gamma^{2}_{33}\Gamma^{2}_{32} - \Gamma^{1}_{33}\frac{\partial ln\sqrt{-g}}{\partial r} - \Gamma^{2}_{33}\frac{\partial ln\sqrt{-g}}{\partial \theta}$$

(6a)

After obtaining the components $R_{\mu\nu}$ the *scalar curvature* invariant can also be determined:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\mu}R_{\mu\mu} \tag{7}$$

Finally the Einstein tensor can be formed:[4],[5]

$$G^{\mu}_{\mu} = R^{\mu}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu}R = \left(\frac{8\pi G}{c^4}\right)T^{\mu}_{\mu},\ \mu = 0,1,2,3 \tag{8}$$

where: $\delta_\mu{}^\mu$ is the Cronecker delta, $T_\mu{}^\mu = T_0{}^0,\ T_1{}^1,\ T_2{}^2,\ T_3{}^3$ are the diagonal elements of the energy momentum tensor, and $R_\mu{}^\mu$ will be defined later. In the case of vacuum $T_{\mu\nu} \neq 0$.

3 Harrison's Formula

In Landau and Lifshitz it is given that for a metric in which $g_{\mu\nu} = 0$ for $\mu \neq \nu$ which we can represent the elements as follows:

$$g_{\mu\mu} = e_{\mu}e^{2F_{\mu}} \text{ and } e_{o} = 1, \text{ and } e_{\ell} = -1 \quad \ell = 1, 2, 3. \tag{9}$$

Harrison's formula now gives the components of the Ricci tensor $R_{\mu\mu}$ in the following relation: [6]

$$R_{\mu\mu} = \sum_{\ell\neq\mu}\left[F_{\mu,\mu}F_{\ell,\mu} - F^2{}_{\ell,\mu} - F_{\ell,\mu,\mu} + e_\mu e_\ell e^{2(F_\mu - F_\ell)}A(\ell)\right] \tag{10}$$

and $A(\ell)$ is given by: [6]

$$A(\ell) = F_{\ell,\ell}F_{\mu,\ell} - F^2{}_{\mu,\ell} - F_{\mu,\ell,\ell} - F_{\mu,\ell}\sum_{m\neq\mu,\ell}F_{m,\ell} \tag{11}$$

subscripts preceded by a comma are denoting ordinary differentiation with respect to the corresponding coordinate.

4. Applying Harisson's Formula

Let us first define the components of the metric tensor appearing in (2):

$$g_{oo} = 1, g_{11} = -\frac{R^2(t)}{1-kr^2}, g_{22} = -r^2R^2(t), g_{33} = -R^2(t)r^2\sin^2\theta \tag{12}$$

Then from (9) we also have that:

$$F_0 = \ln(1) = 0 \ , \ F_1 = \ln\left(\frac{R(t)}{\sqrt{1-kr^2}}\right),\ F_2 = \ln\left(rR(t)\right),\ F_3 = \ln\left(rsin\,\theta sin\theta\right) \tag{13}$$

The R_{00} component of the Ricci tensor can be written as:

$$R_{00} = F_{00}F_{10} - F^2{}_{10} - F_{100} + e_0e_1A(1)e^{2(F_0-F_1)} + F_{00}F_{20} - F^2{}_{20} - F_{200}e_2e_0A(2)e^{2(F_0-F_2)} + \\ + F_{00}F_{30} - F^2{}_{30} - F_{300} + e_0e_3A(3)e^{2(F_0-F_3)} \tag{14}$$

The $A(\ell)$ coefficients becomes:

$$\begin{aligned} A(1) &= F_{11}F_{01} - F^2{}_{01} - \mathrm{F}_{011} - \mathrm{F}_{01}(\mathrm{F}_{21} + \mathrm{F}_{31}) = 0 \\ A(2) &= F_{22}F_{02} - F^2{}_{02} - \mathrm{F}_{022} - \mathrm{F}_{02}(\mathrm{F}_{12} + \mathrm{F}_{31}) = 0 \\ A(3) &= F_{33}F_{03} - F^2{}_{03} - \mathrm{F}_{033} - \mathrm{F}_{03}(\mathrm{F}_{13} + \mathrm{F}_{23}) = 0 \end{aligned} \tag{15}$$

Taking the appropriate derivatives, as symbolized, we can write:

$$R_{00} = F_{00}F_{10} - F^2{}_{10} - F_{100} + F_{00}F_{20} - F^2{}_{20} - F_{200} + F_{00}F_{30} - F_{30}{}^2 - F_{300} \tag{16}$$

Symbolizing the derivatives we obtain:

$$
\begin{aligned}
R_{00} = {} & \frac{\partial \ln (1)}{\partial t}\frac{\partial}{\partial t}\left[\ln\left(\frac{R(t)}{\sqrt{1-kr^2}}\right)\right]-\left[\frac{\partial}{\partial t}\ln\left(\frac{R(t)}{\sqrt{1-kr^2}}\right)\right]^2 \\
& -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\ln\left(\frac{R(t)}{\sqrt{1-kr^2}}\right)\right]+ \\
& +\frac{\partial}{\partial t}\left[\ln (1)\right]\frac{\partial}{\partial t}\left[\ln\left(r\ R(t)\right)\right]-\left[\frac{\partial}{\partial t}\ln\left(r\ R(t)\right)\right]^2 \\
& -\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\ln\left(r\ R(t)\right)\right]+\frac{\partial \ln (1)}{\partial t}\frac{\partial \ln\left(r sin\ \theta s\ (t)\right)}{\partial t}- \\
& -\left[\frac{\partial ln\left(r sin\ \theta\ R(t)\right)}{\partial t}\right]^2-\frac{\partial}{\partial t}\left[\frac{\partial ln\left[r sin\ \theta\ R(t)\right]}{\partial t}\right]
\end{aligned}
\tag{17}
$$

and (17), becomes:

$$
R_{00}=-\left(\frac{\dot{R}}{R}\right)^2-\left(\frac{\ddot{R}R-\left(\dot{R}\right)^2}{R^2}\right)-\left(\frac{\dot{R}}{R}\right)^2-\left(\frac{\ddot{R}R-\left(\dot{R}\right)^2}{R^2}\right)-\left(\frac{\ddot{R}R-\left(\dot{R}\right)^2}{R^2}\right)-\left(\frac{\dot{R}}{R}\right)^2 \tag{18}
$$

and simplifying we obtain:

$$
R_{00} = -3\left(\frac{\ddot{R}}{R}\right) \tag{19}
$$

It should be noted that R stands for R(t) everywhere in our calculations. Similarly the R_{11} component of the Ricci tensor becomes:

$$
\begin{aligned}
R_{11} = {} & F_{11}F_{21} - F_{21}{}^2 - F_{211} + A(2)e^{2(F_1-F_2)} + F_{11}F_{31} \\
& - F^2{}_{31} - F_{311} + A(3)e^{2(F_1-F_3)} + F_{11}F_{01} - \\
& - F^2{}_{01} - F_{011} - A(0)e^{2(F_1-F_0)}.
\end{aligned}
\tag{20}
$$

Similarly as before the $A(\ell)$ coefficients become:

$$A(2)=F_{22}F_{12}-F^2{}_{12}-F_{122}-F_{12}\left(F_{32}+F_{02}\right)=0$$
$$A(3)=F_{33}F_{13}-F^2{}_{13}-F_{133}-F_{13}\left(F_{23}+F_{03}\right)=0$$
$$A(0)=F_{00}F_{10}-F^2{}_{10}-F_{100}-F_{10}\left(F_{23}+F_{03}\right)=$$
$$=-\left(\frac{\dot{R}}{R}\right)^2-\left(\frac{\ddot{R}R-\dot{R}^2}{R^2}\right)-\frac{\dot{R}}{R}\left(\frac{\dot{R}}{R}+\frac{\dot{R}}{R}\right)=-\left(\frac{\dot{R}}{R}\right)^2-\left(\frac{\ddot{R}}{R}\right) \tag{21}$$

Furthermore:

$$e^{2(F_1-F_0)}=e^{2\left[\ln\left(\frac{R(t)}{\sqrt{1-kr^2}}-0\right)\right]}=\frac{R^2(t)}{\left(1-kr^2\right)} \tag{22}$$

Taking the derivatives as indicated and multiplying by the A(0) found above we obtain:

$$\begin{aligned}R_{11}&=\frac{kr}{1-kr^2}\left(\frac{1}{r}\right)-\frac{1}{r^2}+\frac{1}{r^2}\\&+\frac{kr}{1-kr^2}\left(\frac{1}{r}\right)-\frac{1}{r^2}+\frac{1}{r^2}+\frac{kr}{1-kr^2}(0)\\&-\frac{R^2}{\left(1-kr^2\right)c^2}A(0)\end{aligned} \tag{23}$$

which takes the form of:

$$\begin{aligned}R_{11}&=\frac{2k}{\left(1-kr^2\right)}-\frac{R^2}{c^2\left(1-kr^2\right)}\left[-\frac{\ddot{R}}{R}-\left(\frac{\dot{R}}{R}\right)^2\right]\\&=\frac{1}{1-kr^2}\left[\frac{\ddot{R}}{R}+2\left(\frac{\dot{R}}{R}\right)^2+\frac{2kc^2}{R^2}\right]\end{aligned} \tag{24}$$

In the same way we can write down the two remaining components of the Ricci tensor:

$$\begin{aligned}R_{22}&=F_{22}F_{12}-F^2{}_{12}-F_{122}+A(1)e^{2(F_2-F_1)}\\&+F_{22}F_{32}-F^2{}_{32}-F_{322}+A(3)e^{2(F_2-F_3)}+F_{22}F_{02}-\\&-F^2{}_{02}-F_{022}-A(0)e^{2(F_2-F_0)}\end{aligned} \tag{25}$$

and

$$A(1)=F_{11}F_{21}-F_{21}{}^{2}-F_{211}-F_{21}\left(F_{31}+F_{01}\right)=\frac{k}{1-kr^{2}}-\frac{1}{r^{2}}$$
$$A(2)=F_{33}F_{23}-F_{23}{}^{2}-F_{233}-F_{23}\left(F_{03}+F_{13}\right)=0 \tag{26}$$
$$A(0)=F_{00}F_{20}-F^{2}{}_{20}-F_{200}-F_{20}\left(F_{10}+F_{30}\right)=-\frac{\ddot{R}}{R}-2\left(\frac{\dot{R}}{R}\right)^{2}.$$

Finally the R_{22} component becomes:

$$R_{22}=\left(\frac{k}{1-kr^{2}}-\frac{1}{r^{2}}\right)e^{2\ln\left(\frac{R\sqrt{1-kr^{2}}}{R}\right)}$$
$$+\left(-\frac{\cos^{2}\theta}{\sin^{2}\theta}+\left(\frac{\sin^{2}\theta+\cos^{2}\theta}{\sin^{2}\theta}\right)\right)\times$$
$$e^{2\ln\left(\frac{Rr}{c}\right)}\left(-\frac{\ddot{R}}{R}-2\left(\frac{\dot{R}}{R}\right)^{2}\right)=2kr^{2}-\frac{R^{2}r^{2}}{c^{2}}\left[-\frac{\ddot{R}}{R}-2\left(\frac{\dot{R}}{R}\right)^{2}\right]= \tag{27}$$
$$=r^{2}\left[\frac{\ddot{R}}{R}+2\left(\frac{\dot{R}}{R}\right)^{2}+\frac{2kc^{2}}{R^{2}}\right]$$

Next the R_{33} component will be:

$$R_{33}=F_{33}F_{13}-F_{13}{}^{2}-F_{133}+A(1)e^{2(F_{3}-F_{1})}+F_{33}F_{23}$$
$$-F_{23}{}^{2}-F_{233}+A(2)e^{2(F_{3}-F_{2})}+F_{33}F_{03}- \tag{28}$$
$$-F_{03}{}^{2}-F_{033}-A(0)e^{2(F_{3}-F_{0})}$$

Similarly the A coefficients for this component are given by:

$$A(1)=F_{11}F_{31}-F_{31}{}^{2}-F_{311}-F_{31}\left(F_{21}+F_{01}\right)=\frac{k}{1-kr^{2}}-\frac{1}{r^{2}}$$
$$A(2)=F_{22}F_{32}-F^{2}{}_{32}-F_{322}-F_{32}\left(F_{12}+F_{02}\right)=-\frac{\cos^{2}\theta}{\sin^{2}\theta}-\left(\frac{-1}{\sin^{2}\theta}\right)=\left(\frac{1-\cos^{2}\theta}{\sin^{2}\theta}\right)=1$$
$$A(0)=-\frac{\ddot{R}}{R}-2\left(\frac{\dot{R}}{R}\right)^{2} \tag{29}$$

Therefore:

$$R_{33}=r^{2}\sin^{2}\theta-\frac{r^{2}R^{2}\sin^{2}\theta}{c^{2}}\left[-\frac{\ddot{R}}{R}-\frac{2\dot{R}^{2}}{R^{2}}\right]=r^{2}\sin^{2}\theta\left[\frac{2kc^{2}}{R^{2}}+2\left(\frac{\dot{R}}{R}\right)^{2}+\frac{\ddot{R}}{R}\right] \tag{30}$$

5 Calculation of the scalar *R*

Now the scalar quantity R can be calculated:

$$R = R_\mu^\mu = g^{\mu\mu} R_{\mu\mu} = R_o^o + R_1^1 + R_2^2 + R_3^3 \\ = g^{oo} R_{oo} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \tag{31}$$

and so we obtain:

$$R = -\frac{6k}{\dot{R}^2} - \frac{6\ddot{R}}{c^2 R} - \frac{6}{c^2}\left(\frac{\dot{R}}{R}\right)^2 \tag{32}$$

6 Calculation of the Einstein Tensor Components

The mixed components of the Einstein tensor are given by:

$$G_\mu^\mu = R_\mu^\mu - \frac{\delta_\mu^\mu}{2} R = \frac{8\pi \quad GT_\mu^\mu}{c^4} \tag{33}$$

Therefore we obtain them for the four different components:

$$G_0^0 = R_0^0 - \frac{R}{2} = -3\frac{\ddot{R}}{R} + \frac{3}{c^2}\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{R}}{R} + \frac{kc^2}{R^2}\right]$$

$$= \frac{3}{c^2}\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{kc^2}{R^2}\right] = \frac{8\pi G T_0^0}{c^4}[7],$$

$$G_1^1 = R_1^1 - \frac{R}{2} = -\frac{2}{c^2}\left(\frac{\dot{R}}{R}\right)^2 - \frac{\ddot{R}}{c^2 R} - \frac{2k}{R^2} + \frac{3}{c^2}\left[\left(\frac{\dot{R}}{R}\right)^2 + \frac{\ddot{R}}{R} + \frac{kc^2}{R^2}\right]$$

$$G_1^1 = \frac{2\ddot{R}}{c^2 R} + \frac{1}{c^2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{Rc^2} = \frac{8\pi G T_1^1}{c^4}, \quad \mathrm{T}_0^0 = \varepsilon = \rho c^2 [7]$$

$$G_1^1 = G_2^2 = G_3^3 \quad \text{and} \quad \mathrm{T}_1^1 = T_2^2 = T_3^3 = -p \ [7] \tag{34}$$

7 Conclusions

By making use of *Harrison's formula* the covariant components of the Ricci tensor were derived. Once their expressions were obtained the mixed components of the Einstein tensor were also calculated. The metric used was the standard Robertson-Walker metric of the Freedman cosmological model. In my opinion this method is faster and has the advantage of allowing for the calculation of the components of the Ricci tensor in such a way that calculation of every individual Cristoffel symbol is directly avoided. This method leaves less room for computational error in an otherwise more lengthy calculation. Finally it can be said that the equations derived agree with those of standard cosmology.

References

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