the general science journal

A COMMENT ON MATHEMATICAL METHODS TO DEAL WITH DIVERGENT SERIES AND INTEGRALS

Jose Javier Garcia Moreta
Graduate Student of Physics (in Solid State) at UPV/EHU
Address: P.O 644 48920 Portugalete , Vizcaya (Spain)
Phone: (00) 34 685 77 16 53
E-mail: josegarc2002@yahoo.es

MSC : 02.30.Lt , 03.65.Sq ,

ABSTRACT: In this paper we study the methods of Borel and Nachbin resummation applied to the solution of integral equation with Kernels K(yx) , the resummation of divergent series and the possible application to Hadamard finite-part integral and distribution theory

1. INTRODUCTION

Divergent series are widely known and appear in many context involving Physics or math, for example if we integrate by parts the error function (Laplace) :

$$erfc(x)=\int_x^\infty dte^{-t^2}\to\frac{e^{-x^2}}{\sqrt{\pi}}\left(1+\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{n!(2x)^{2n}}\right)\qquad(1.1)$$

Or using the 'Saddle point mehtod' for n! when n is big then we have:

$$\Gamma(x+1)\to\sqrt{2\pi}n^{(n+1)/2}\left(1+\frac{1}{12n}+\frac{1}{288n^2}+.....\right)\qquad\int_0^\infty dte^{-t}t^{x-1}=\Gamma(x)\qquad(1.2)$$

But (1.1) and (1.2) are only convergent as $x\to\infty$ for small values of x they both Diverge. Another example with ODE's is the following for the ODE:

$$x^2\frac{dy}{dx}+y=x\quad ,\ y(x)=x\int_0^\infty dt\frac{e^{-t}}{1+xt}\qquad(1.3)\ \text{(exact solution)}$$

For (1.3) Euler gave the series solution: $y(x)=x-(1!)x^2+(2!)x^3-(3!)x^4-...$ (1.4)

Which converges only for x=0 !!! , A similar thing happens with the series:

$$a_0 + a_1 g + a_2 g^2 + a_3 g^3 + \ldots\ldots. \qquad g << 1 \quad (1.5)$$

That apear in QFT and Quantum Mechanics , where g is the 'coupling constant' , in general series of the form (1.5) although divergent are used to calculate the 'mass' or 'charge' (renormalized value), for a given physical theory.

Also as a last example let be the next Taylor series around x=0 :

$$x + x^2 + x^3 + x^4 + \ldots\ldots\ldots = \frac{x}{1-x} \qquad , \qquad x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots\ldots = \frac{x}{(1-x)^2} \qquad (1.6)$$

Convergent for |x| <1 However taking the limit $x \to -1^-$ (-1 by the left) we find the amazing results -1/2 and -1/4 Although we know that in general, any series of the form $1 + 2^m + 3^m + \ldots\ldots$ is divergent, in the last section we discuss a method to deal with divegent integrals with poles on the interval. [a,b] with a and b real numbers for example the integral of $f(x) = x^{-2}$ on the interval (-1,1) .

Of course this paper pretends to be only a kind of introduction to the subject for further references I strongly recommend 'Divergent series' by G.H Hardy or 'Zeta regularization methods' by E.Elizalde and others for historical examples involving divergent series and integrals.

2. ZETA FUNCTION REGULARIZATION

The Zeta function regularization method was used to give a meaning to clearly divergent series in the form:

$$1 + 2^a + 3^a + \ldots \quad a > 0 \qquad (2.1)$$

For any positive and real number 'a' , the series could be re-arranged in the form:

$$1 + 2^{a-s} + 3^{a-s} + \ldots = 1 + 2^{-(s-a)} + 3^{-(s-a)} + \ldots = \zeta(s-a) \qquad (2.2)$$

Where in (2.2) we have used an Analytic prolongation for the Riemann Zeta Function, now letting s tends to 0 we could find the 'sum' of the series (2.1) as the Value $\zeta(-a)$ which can be calculated in general (except if a=1) by the contour Integral on the complex plane:

$$\zeta(-a) = \tfrac{\Gamma(a+1)}{2\pi i} \int_\sigma dq \frac{q^{-(a+1)}}{e^{-q} - 1} \qquad (2.3)$$

Or by means of 'Riemann functional equation' :

$$\zeta(s) = \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \pi^{-1} (2\pi)^s \quad (2.4) \qquad \zeta(-2n) = 0 \quad \text{n=1,2,3,4,.....}$$

For 'a' being a Natural number, the calculations can be simplified, first if we introduce the 'Bernoulli Polynomials' using the Generating function:

$$\frac{te^{xt}}{e^{t}-1}=\sum_{n=0}^{\infty}B_{n}(x)\frac{t^{n}}{n!} \qquad (2.5)$$

Then for integer 'n' we find $B_{2n}(0)=-2n\zeta(1-2n)$ (2.6)

The value on the left of (2.5) for the 2n-th Bernoulli Polynomial at x=0 are the 'Bernoulli Numbers' and have a vital importance in Mathematical Analysis, for example the sum of the first N powers of $f(x)=x^r$ for r >0 and integer can be written using the Bernoulli Polynomials in the form:

$$\sum_{n=0}^{N}n^{r}=\frac{B_{r+1}(N+1)-B_{r}(0)}{r+1} \qquad (2.7)$$

Although it may seem we are somhow 'Cheating' to get a finite value, Zeta regularization has been proved frutifull, in many calculations involving theoretical Physics such us the 'Casimir effect' where the value $\zeta(-3)=-1/120$ is used in Calculations, or string theory whenever you need to calculate 1+2+3+4+5.+…..

The first person giving an 'exact' justification to this procedure of Zeta Regularization was Leonhard Euler in XVII century, he desired to calculate (in modern notation and using Operator theory):

$$f(x)+f(x+1)+f(x+2)+...........=S \qquad (2.8)$$

Then he uses the properties of the 'traslation operator':

$$e^{aD}f(x)=f(x+a) \qquad D=\frac{d}{dx} \qquad (2.9)$$

(The formula (2.9) can be proved making the Taylor expansion respect to 'a' and remembering the Taylor series for the Exponential , In Modern Lie Algebra this is just a classical result involving group theory).

Applying (2.9) ot each term in (2.8) we have a geommetric series in e^{aD} for a=0,1,2,3,… multiplying and dividing by 'D' Euler found the relation (involving again Bernoulli numbers):

$$\frac{D}{e^{D}-1}=-\sum_{n=0}^{\infty}\frac{B_{n}}{n!}D^{n}F \qquad \frac{dF}{dx}=f(x) \qquad B_{n}(0)=B_{n} \qquad (2.10)$$

If we put f(x)=x or $f(x)=x^3$ and use (2.10) the series is just a finite one and we get the values -1/12 and -1/120 using tables.

- *Application to divergent integrals:*

A direct calculation for divergent integrals using the Zeta regularization presented here, is (using the rectangle method to calculate integrals):

$$\int_0^\infty dx(x+\beta)^m \approx \sum_{n=0}^{\infty}(\beta+\varepsilon n)^m \varepsilon \to \zeta\left(\frac{\beta}{\varepsilon},-m\right)\varepsilon^{m+1} \quad m>0 \quad (2.11)$$

To give a finite value to the divergent integral (2.11) using the 'Hurwitz Zeta function' (a direct generalization to Riemann zeta function) defined by:

$$\zeta(s,\beta)=\sum_{n=0}^{\infty}(n+\beta)^{-s} \quad \Re\text{e(s)}>1 \qquad (2.12)$$

With 'beta' a real number different from a negative integer, for other values we can use the Functional equation:

$$\zeta(1-s,\frac{\nu}{\mu})=\frac{2\Gamma(s)}{(2\pi\mu)^s}\sum_{r=1}^{\mu}Cos(\frac{\pi s}{2}-\frac{2\pi r\nu}{\mu})\zeta(s,\frac{r}{\mu}) \qquad \mu\geq\nu\geq 1 \qquad (2.13)$$

In case m is a positive integer we can use the easier formula :

$$\zeta(-m,\beta)=-\frac{B_{m+1}(\beta)}{m+1} \qquad (2.14)$$

Hence if taking the limit $\varepsilon \to 0$ (2.11) has a finite value , this value could be considered as the regularized 'sum' or 'Area' for the function $f(x)=x^m$ on the inteerval $[0,\infty)$

The main importance of divergent integrals arises in the problem of 'Renormalization' when we are forced to give a finite meaning to clearly divergent expressions such as:

$$\int_0^\infty dpp^m \qquad m\in \mathrm{Z} \qquad (2.15)$$

For positive m (UV divergences) we can find a recurrent formula to obtain the values of the different integrals in the form:

$$I(m,\Lambda)=(m/2)I(m-1,\Lambda)+\zeta(-m)-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}(m-2r+1)I(m-2r,\Lambda)$$

$$\int_0^\Lambda p^m dp=\frac{\Lambda^m}{2}+\sum_{n=1}^{\Lambda}n^m-\sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}a_{mr}\Lambda^{m-2r+1} \qquad a_{mr}=\frac{\Gamma(m+1)}{\Gamma(m-2r+2)} \quad (2.16)$$

Where we have called $I(m,\Lambda)=\int_0^\Lambda dpp^m$, here the meaning of 'Lambda' is a momentum cut-off (the maximum value of the modulus for the momentum of the particle) , the Renormalization procedure involves taking the physical limit $\Lambda \to \infty$, at the end the quantities involving these divergent integrals can't depend on the value of the regulator/cut-off 'Lambda' introduced , where we have used the 'Euler-McLaurin sum formula' in the form :

$$\int_0^b f(x)dx = \frac{f(b)}{2} + \sum_{n=1}^{b-1} f(n) - \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(b) \qquad (2.17)$$

(For simplicity we have supposed that $f^{(r)}(0)=0$ for every r , For negative values of m or 'infrared divergences'then make q=1/p and apply the process described above)

the case m=-1,(logarithmic divergence) is slitghly different and involves a method called 'Ramanujan resummation' of a series, a recurrence formula for evaluating this kind of sum is :

$$\sum_{n\ge 1}^{[R]} a(n) = a(1)+a(2)+a(3)+.......+a(N-1)+\sum_{n\ge 0}^{[R]} a(n+N) - \int_1^N dt a(t) \quad \text{N>1}$$

(2.18)

Here the [R] stands for Ramanujan or resummation, applied to a(n)=1/n gives the regularizad value $\sum_{n\ge 1}^{[R]} \frac{1}{n} \to \gamma$ .for the Harmonic series.

The formula in (2.16) describes a method to regularize the divergent integral by recursive calculations, where the divergent series associated to the divergent integral is 'summed' by a zeta regularization method, the case m=0 is simply :

$$\int_0^\Lambda dp = 1+1+1+1+........+\Lambda \to \zeta(0) \quad \text{as} \quad \Lambda \to \infty \qquad (2.19)$$

For more complicate analytic functions F(p) we expand them into a 'Laurent (convergent) series' in the next form:

$$F(p) = \sum_{n=-\infty}^{\infty} c_n p^n \qquad (2.20)$$ and perform term by term integration

Depending on the region of the complex plane we are located (2.20) may include either positive or negative power series, or them both .

Finally another direct application of 'Zeta regularization' involves calculating divergent sums of primes in the form:

$$\sum_{p} p^{s}(\ln p)^{k} \to \sum_{n=1}^{\infty} \mu(n)(-1)^{k} \frac{d^{k-1}}{ds^{k-1}}\left(\frac{\zeta'(-ns)}{\zeta(-ns)}\right) \qquad (2.21)$$

With s >0 a real non-integer number, so the series on the right is convergent, k >0 , used to extend the domain of convergence for P(-s) .

$$\sum_{p} p^{s} = P(s) \quad \text{valid for s> 1} \qquad (2.22)$$

A similar method to 'Zeta regularization' could be stablished using the analytic prolongation of the Gamma function (and hence a similar to integrals involving Euler's Beta function), using the functional equation :

$$\Gamma(x)\Gamma(1-x)\sin(\pi x) = \pi \quad \text{so} \quad \frac{\pi}{\Gamma(\alpha)\sin(\pi\alpha)} \to \int_{0}^{\infty} dt \frac{e^{-t}}{t^{\alpha}} \qquad (2.23)$$

With α real number bigger than 0 and different from an integer, clearly (2.23) is a form to give a finite meaning to otherwise divergent integral due to the pole at t=0, using the analytic prolongation to negative values for the Gamma function, in both cases involving (2.1) and (2.23) we consider that The 'sum' or 'Area' for the series or integral is just the value of $\Gamma(\alpha)$, $\zeta(-a)$, this process of 'regularization' avoids the pole for the Gamma and Riemann zeta function at the points $\alpha = 0$ (Gamma) and s=1 (Riemann Zeta function). Also for alternating series:

$$1 - 2^{m} + 3^{m} - 4^{m} + ... = \eta(-m) \qquad \text{since} \quad \eta(s) = \sum_{n=1}^{\infty} (-1)^{n} n^{-s} = (1-2^{1-s})\zeta(s)$$

(2.24)

Appart from Euler the formalism of 'Zeta regularization' (but not with this name) was introduced by Ramanujan in his Notebooks using the Euler-McLaurin sum formula defining the 'constant' of the series:

$$c = -\frac{1}{2} f(0) - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} f^{2k-1}(0) \qquad (2.25)$$

So for $f(x) = x^{n}$, with n integer he found the relations (R=resummation)

$$1 + 2^{2n+1} + 3^{2n+1} + ... = -\frac{B_{2n+1}}{2n} (\Re) \qquad (2.26)$$

$$1 + 2^{2n} + 3^{2n} + ... = 0 (\Re)$$

3. BOREL RESUMMATION FOR SERIES AND INTEGRALS

Let be the divergent (Numerical) series:

$$S = a_0 + a_1 + a_2 + a_3 + ... \qquad (3.1)$$

Borel gave a very ingenious method to calculate it, first we multiply and divide each term by n! getting :

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} \int_0^{\infty} dt t^n e^{-t} \qquad \sum_{n=0}^{\infty} \frac{a_n}{n!} t^n = f(t) \qquad (3.2)$$

Using (3.1) and (3.2) and supposing that $f(t) = O(e^{bt})$ for a real positive number b then we can writte the 'sum' of the series in (3.1) as:

$$\int_0^{\infty} dt f(t) e^{-t} = \mathrm{B(S)} \rightarrow \sum_{n=0}^{\infty} a_n \quad \text{or} \quad s\int_0^{\infty} dt f(t) e^{-st} \rightarrow \sum_{n=0}^{\infty} a_n s^{-n} \qquad s>0 \ (3.3)$$

As a 'toy model' of our Borel resummation method we have:

$$1-1+1-1+1-1+1-1+ .... \rightarrow 1/2 \ \text{ since } \ f(t) = \exp(-t) \qquad (3.4)$$

Unfortunately we can't always know an exact expression for f(t), to give an approximate evaluation of our Borel transform, we can use the 'Euler-Abel' transform widely known for power series, to calculate a power series:

$$\sum_{n=0}^{\infty} a(n) x^n \approx \sum_{k=0}^{p} \frac{(-1)^k x^k \Delta^k [(-1)^n a_n]_{n=0}}{(1+x)^{k+1}} + \left(\frac{x}{1+x}\right)^{p+1} \sum_{n=0}^{\infty} (-1)^{k+p+1} \Delta^{p+1} [(-1)^n a_n]_{n=0} x^n$$

$$\Delta^n a(0) = \sum_{n=0}^{n} (-1)^m \frac{n!}{n!(n-m)!} a_{n-m} \qquad (3.5)$$

Also we apply another known property of the 'Laplace transform'

$$\int_0^{\infty} dt \frac{t^p}{(t+c)^{p+1}} e^{-st} = \frac{1}{p!} \frac{\partial^p}{\partial s^p} \frac{\partial^p}{\partial c^p} E_1(cs) e^{cs} \qquad E_1(x) = \int_x^{\infty} dt \frac{e^{-t}}{t} = -E_i(-x) \quad (3.6)$$

The first expression in (3.5) is an approximate evaluation for f(x) setting x=t then the B(S) 'Borel sum' for our divergent series (3.1) is:

$$B(S) \approx \frac{1}{s} \sum_{k=0}^{p} (-1)^k \frac{1}{k!} \frac{\partial^k}{\partial s^k} \frac{\partial^k}{\partial c^k} \Bigg|_{c=1,s=1} \Delta^k [(-1)^n b_n]_{n=0} E_1(s) e^{cs} \qquad (3.7)$$

With $a_n = b_n n!$, only in case that the coefficients of our initial series (3.1) were of the form $(-1)^n$ n!P(n) with P(n) a Polynomial (3.7) is exact. The error term is given by the expression:

$$E = O\left(\frac{1}{s}\frac{1}{p!}\frac{\partial^{p+1}}{\partial s^{p+1}}\frac{\partial^{p}}{\partial c^{p}} E_1(cs)e^{cs}\bigg|_{c=1,s=1} \right) \qquad (3.8)$$

In case (3.1) were convergent then its ‘ Borel sum’ is equivalent to the term-by-term Laplace transform at s=1, in that case if we had f(t) and g(t) analytic series near x=0 ::

$$\sum_{n=0}^{\infty} a_n x^n = g(x) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = f(x) \qquad (3.9)$$

If we define $g(s) = \int_0^\infty dtG(t)e^{-st}$,since f(t) and g(t) are convergent series then we should have as a main property of Borel sums:

$$g(x) = \int_0^\infty dt \sum_{n=0}^{\infty} \frac{a_n}{n!}(xt)^n e^{-t} = \frac{1}{x}\int_0^\infty dt f(xt)e^{-(t/x)} \qquad (3.10)$$

Setting s=1/x , and using the a known property for the Laplace inverse transform of $g(1/s)s^{-1}$ we find the relation:

$$f(t) = 2\int_0^\infty duuG(u^2)J_0(2\sqrt{tu}) \qquad F_0(k = 2\sqrt{t})(2G(u^2) = f(t) \qquad (3.11)$$

The expression (3.10) above is the ‘Zero-th order Hankell transform’ evaluated at the point $k = 2\sqrt{t}$ with $J_0(x) = \sum_{i=0}^{\infty} \frac{(-1)^i (x/2)^{2i}}{(i!)^2}$

- *Application to divergent integrals:*

The formalism of Borel resummation for integrals is inmediatly acomplished if we define the Riemann sum multiplying and dividing each term by a Gamma function we have:

$$\sum_{n=0}^{\infty} \frac{f(a+n\Delta x)s^{\Delta x + a}}{\Gamma(a+n\Delta x+1+\alpha)} \int_0^\infty dtt^{a+n\Delta x+} e^{-t} t^{\alpha} \Delta x \qquad (3.12)$$

Taking the limit as $\Delta x \to 0$ the sums becomes the double-integral :

$$s\int_0^\infty dt \left(\int_0^\infty dx \frac{f(x)t^x}{\Gamma(x+1+\alpha)} \right) t^{\alpha} e^{-st} \qquad (3.13)$$

Of course in general, unless $\int_0^\infty dx f(x)$ is convergent ‘Fubini’s theorem’ does not hold for (3.12) and (3.13) so:

$$\int_0^\infty dt\int_0^\infty dx\sigma(x,t) \neq \int_0^\infty dx\int_0^\infty dt\sigma(x,t) \qquad \sigma(x,t)=\frac{f(x)}{\Gamma(x+1+\alpha)}t^{x+\alpha}e^{-t}$$

(3.14)

Now if we define the integral transform

$$H_\alpha(t)=\int_0^\infty dx\frac{f(x)}{\Gamma(x+1+\alpha)}t^x \qquad \text{With } H_\alpha(t)=O(e^{b_2 t}) \quad (3.15)$$

If the 2 conditions on (3.16) holds then the 'Borel integral' is just the Laplace transform of $H_\alpha(t)t^\alpha$, $\alpha>0$

But. Can a 'Borel sum' be the real sum of the series?, let's take :

$$\mathrm{E_i}(x)=-\int_{-x}^\infty dt\frac{e^{-t}}{t}\approx\frac{e^{-x}}{x}\sum_{n=0}^\infty\frac{(-1)^n}{x^n}n! \qquad (3.16)$$

The alternating series has the Borel transform:

$$x\int_0^\infty dte^{-xt}\frac{1}{t+1}\to\sum_{n=0}^\infty\frac{(-1)^n}{x^n}n! \qquad (3.17)$$

Using the result for the Laplace transform of 1/(t+1) ,we find:

$$L\left\{\frac{1}{t+1}\right\}=e^sE_i(s) \qquad (3.18)$$

Setting s=1 we find that the 'asymptotic' expansion (3.16) can be 'summed' even for high values of x.

Also, if the integral is convergent then using the property of Laplace transform with s=1 $L\left\{t^x\right\}_{s=1}=\Gamma(x+1)$ then the definition of 'integral' (3.13) is the same as the usual definition for the integral in terms of convergent Riemann sums.

The relationship of this 'Borel resummation' for integrals can be written as this, using the next property for Laplace transforms:

$$L\left\{\int_0^\infty dxt^x\frac{f(x)}{\Gamma(x+1}\right\}=\frac{F(\ln s)}{s\ln s} \quad \text{and} \quad L\left\{t^nf(t)\right\}=(-1)^nD^nF(s) \quad (3.19)$$

Then we can write (3.14) in terms of Laplace transforms:

$$\int_0^\infty dx f(x) s^{-x} \to s(-1)^\alpha \frac{d^\alpha}{ds^\alpha}\left(\frac{F_\alpha(\ln s)}{s \ln s}\right) \qquad (3.20)$$

Where $F_\alpha(s)$ is the Laplace transform of $g(x) = \frac{f(x)}{(x+1)(x+2).....(x+\alpha)}$

Valid for $\alpha > 0$ and integer

For $\alpha \notin Z$,we must apply the analytic prolongation of the Gamma function $\Gamma(z)$ and use the definition of the differintegral $D_x^\alpha f$.

The method of Borel resummation can be used to calculate divergent Fourier series that don't belong to an $L^2(-\infty,\infty)$, but may have a defined Fourier transform re-defined as $\int_0^\infty dx f(x) e^{-i\omega x} \to (i\omega)^{-1} e^{i\omega} F(s = i\omega)$. F(s) is the Laplace transform of f(x). Another deep relationship between the Borel resummation for series and for integrals can be obtained using the Euler-Mclaurin summation formula and using the property (3.20) :

$$\frac{F(\ln(s))}{\ln(s)} = \int_0^\infty dt \left(\sum_{n=1}^\infty \frac{a_n}{n!}\left(\frac{t}{s}\right)^n \right) e^{-t} + \int_0^\infty dt \left(\sum_{r=1}^\infty \frac{B_{2r}}{(2r)!} \frac{\partial^{2r-1}}{\partial x^{2r-1}} \left(\frac{a(x)\left(\frac{t}{s}\right)^x}{\Gamma(x+1)} \right) \right) e^{-t} \qquad (3.21)$$

All the derivatives ∂_x^{2r-1} must be evaluated at x=0, with $\lim_{x\to\infty} \frac{a(x)t^x}{\Gamma(x+1)} = 0$

4. NACHBIN RESUMMATION AND INTEGRAL EQUATIONS

We could write a generalization to (3.3) as the integral expressions

$$B(a_n) = \int_0^\infty dt f(t) h(t) \qquad f(t) = \sum_{n=0}^\infty \frac{a_n}{M(n+1)} t^n \qquad (4.1)$$

With $M(n+1) = \int_0^\infty dt h(t) t^n$, in case $h(t) = e^{-t}$ and M(n+1)=n! , expression (4.1) is just the Borel transform of the sequence $\{a_n\}$, with the advantage that now f(t) can grow faster than e^{bt} so $f(t) \neq O(e^{bt})$, we will study the applications of this Nachbin/Generalized Borel resummtion to solve integral equations and to the Riesz criterion for Riemann Hypothesis

- *Applications of Nachbin resummation to integral equations with Kernel of the form K(st)and to the Riesz function Riesz(x):*

In order to apply the Borel generalized resummation to integral equations, let be the Fredholm equation of first kind :

$$g(s)=s\int_0^\infty dtK(st)y(t) \qquad \hat{K}(n+1)=\int_0^\infty dtK(t)t^n \qquad (4.2)$$

Here K(st) is the Kernel of the integral equation , and g(s) has the form of a Z-transform involving inverse powers of 's' :

$$g(s)=c_0+\frac{c_1}{s}+\frac{c_2}{s^2}+\frac{c_3}{s^3}+........ \qquad c_n=\frac{1}{2\pi i}\int_\gamma dzg(z)z^{n-1} \qquad (4.3)$$

Since the Mellin transform for Kernel K(u) exists, we will apply Nachbin resummation to solve the integral equation given in (4.2)

$$g(s)=\sum_{n=0}^\infty c_n s^{-n}=\int_0^\infty dt\left(\sum_{n=0}^\infty \frac{c_n}{\hat{K}(n+1)}.\frac{t^n}{s^n}\right)K(t)=s\int_0^\infty dtK(st)y(t) \qquad (4.4)$$

Here 'γ' is a closed path on the complex plane, using (4.4) we have proven that a infinite power series in the form $\sum_{n=0}^\infty \frac{c_n}{\hat{K}(n+1)}t^n=y(t)$ can solve an integral equation similar to (4.2) , as an example if we consider the function $y(t)=\pi\left(e^t\right)$, and the Kernel $K(t)=\left(e^t-1\right)^{-1}$ with $\hat{K}(n+1)=\Gamma(n+1)\zeta(n+1)$ we have the identity for the prime counting function in terms of an infinite series as:

$$\pi(x)=\sum_{p\le x}1=\sum_{k=1}^\infty \frac{d_k}{\zeta(k+1)\Gamma(k+1)}\ln^k(x) \qquad d_k=\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{1-k}}\ln\zeta(z) \quad (4.5)$$

expression (4.5) is the number of primes less than 'x' in terms of an infinite series of powers involving ln(x), in general this idea can be extended to include Dirichlet generating functions $H(s)=\sum_{n=1}^\infty \frac{a_n}{n^s}$ with $A(x)=\sum_{n\le x}a_n$, then A(x) satisfies an integral equation similar to (4.2) with K(t)=exp(-t) so we can find the Nachbin resumed solution

$$A(x)=\sum_{n\le x}a_n=\sum_{m=0}^\infty u_m\frac{\ln^m(x)}{m!} \qquad \mathrm{u_m}=\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^{1-m}}H(z) \qquad (4.6)$$

For the case of the Riesz function involved in Riesz criterion, the Riemann Hypothesis can be stated as the condition

$$Riesz(x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{(k-1)!\zeta(2n)} = O(x^{1/4+\varepsilon}) \qquad (4.7)$$

We will try to give an integral equation for Riesz(x) , using Borel resummation and the Taylor expansion for the exp(-x) and the identity $1 = \frac{n}{\zeta(2n)} \int_0^\infty dt \left[\sqrt{\frac{1}{t}} \right] t^{n-1}$, the integral equation for Riesz function reads

$$1 - e^{-x} = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n!} = -\int_0^\infty dt \left(\sum_{n=1}^{\infty} \frac{(-x)^n t^{n-1}}{\zeta(2n)(n-1)!} \right) \left[\sqrt{\frac{1}{t}} \right] = \int_0^\infty \frac{dt}{t} \left[\sqrt{\frac{x}{t}} \right] Riesz(t) \qquad (4.8)$$

then (4.8) is a proposed integral equation for the Riesz(x) ,from the definition of floor function [x] when x=0 both sides on (4.8) are 0

5. HADAMARD FINITE-PART INTEGRAL

In sections 2 and 3 we have discussed divergent integrals when $f(x) \to \infty$ as $x \to 0$, but let's suppose that for a < c < b then $f(c) \to \infty$ as $x \to c$ we can define the 'Cauchy's Principal value' P.V of the integral in the form of a limit:

$$\lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} dx f(x) + \int_{c+\varepsilon}^b dx f(x) \right) \qquad (5.1)$$

Where c is a point so the function f(x) behaves in the form:

$$\int_a^c dx f(x) = \pm\infty \qquad \int_c^b dx f(x) = \mp\infty \qquad (5.2)$$

As for example $f(x) = x^{-(2n+1)}$ $n \in N$, with a < c < b

Hadamard introduced the 'Finite part' of an integral in the form:

$$F.P \int_{-a}^{a} dx f(x) = \left. \frac{d^{k-1} g(x)}{dx^{k-1}} \right|_{-\varepsilon}^{\varepsilon} \qquad f(x) = \frac{d^k g(x)}{dx^k} \qquad (5.3)$$

Where we split the interval $[-a, a] = [-a, -\varepsilon] \cup (-\varepsilon, \varepsilon) \cup [\varepsilon, a]$ so all the possible singularities of the integral lie on $(-\varepsilon, \varepsilon)$ for k=1 we get the usual Leibniz's rule for integral calculus.

As an example let be $f(x) = x^{-r}$ r>1 then :

$$g(x)=\frac{P_k}{x^{m-k}}\quad k,m>1 \qquad\qquad P_k=(-1)^k\prod_{i=0}^{k-1}\frac{1}{(m+i)} \qquad (5.4)$$

Hadamard finite part is:

$$\sum_{i=0}^{k-1}(-1)^{i+1}\frac{d^i}{dx^i}\frac{1}{r^{m-k}}\frac{d^{k-1-i}\varphi(x)}{dx^{k-1-i}}\Bigg|_{-a}^{a}=FP\int_{-a}^{a}dx\,\frac{\varphi(x)}{x^m} \qquad (5.5)$$

Here $\varphi(x)$ is an infinitely differentiable function, the Hadamard integral uses the properties of distributions, for a test function , so $\varphi(\pm\infty)=0$ then if we introduce the scalar product in the form:

$$\langle S,\varphi\rangle=\int_{-\infty}^{\infty}dxS(x)\varphi(x)\ \text{ so }\ \langle S',\varphi\rangle=-\langle S,\varphi'\rangle \qquad (5.6)$$

For the special case of Dirac delta function then:

$$\delta[\varphi]=\varphi(0)\quad\text{and}\quad\delta^{(n)}[\varphi]=(-1)^n\frac{d^n\varphi(0)}{dx^n} \qquad (5.7)$$

Using the definition of Dirac delta as a Fourier transform then:

$$2\pi\delta^{(n)}(x)=\int_{-\infty}^{\infty}dx(ix)^n e^{ikx} \qquad (5.8)$$

So for every Analytic function f(x) we can define its Fourier transform in term of 'test function' as the linear functional:

$$I[f,\varphi]\to 2\pi f(-i\partial_x)\delta(x) \qquad (5.9)$$

The 'Borel integral' (3.) and 'Hadamard finite-part integral' can be useful to deal with divergences in the form $\int d^4p\,|p|^r$ for r >0 and r<0 , $r\neq -1$ and the modulus of the momentum $|p|\to 0,\infty$ that appear in QFT at long or short distances (wavelength), since in natural units $\lambda|p|=1$, hence we could use these 2 techniques to get rid of the divergences, introducing a cut-off or regulator Lambda and making $\Lambda\to\infty$.

6. REFERENCES

[1] Abramowitz and Stegun "*Handbook of Mathematical functions*" New York: Dover (1972)

[2] Brendt Bruce "*Ramanujan´s Notebooks: Parts I-II*" Springer-Verlag New York (1985-1989)

[4] Demidovich B.P and Maron I.A “*Fundamentos de Analisis Numerico” (Spanish)* Ed: Parninfo Madrid (1985) , ISBN: 84-283-0887-X

[5] Elizalde .E, Romeo .A “Zeta regularization techniques with applications” World Scientific Publishing company. (1994)

[6] Hardy G.H *“Divergent series”* Oxford Claredon Press (1949)

[7] Mostepanenko, V. M. and Trunov, N. N. §2.2 in “*The Casimir Effect and Its Applications.* Oxford” , England: Clarendon Press, 1997.

[8] Nachbin L. "*An extension of the notion of integral functions of the finite exponential type*", *Anais Acad. Brasil. Ciencias.* **16** (1944)

[9] Peskin M.E and Schröeder D. V “Introduction to Quantum Field Theory” Addison-Wesley (1995)

[10] Riesz M. "*Sur l'hypothèse de Riemann*", *Acta Mathematica*, **40** (1916), pp.185-90.

[11] Rudin, W.: *Functional Analysis*, McGraw-Hill Science, 1991

[12] Strichartz R . “*A Guide to Distribution Theory and Fourier Transforms*.” CRC Press. ISBN 0849382734 (1994)

[13] Zozulya V.V “ *Regularization of divergent integrals”* Electronic Journal of Boundary elements Vol 4 Nº 2 pp 49-57 (2006)