# Threeconnected graphs with only one Hamiltonian circuit[1]

**Emanuels Grinbergs, Dainis Zeps**

**University of Latvia**

We will call graph *1-H-graph* if it is threeconnected and it has only one Hamiltonian circuit *(H-circuit)*. We will say that in the graph *G* three distinct vertices *x, y, z* in the given order comprise *special triplet* – shorter, *s-triplet {x, y, z}* if

1) there is only one Hamiltonian chain *(H-chain) [x...y]* with end vertices *x, y*;
2) there isn't *H*-chain *[x...z]*;
3) and either
    - 3.1) *G* is threeconnected; or
    - 3.2) *G* is not threeconnected, but it becomes threeconnected if vertex *t* and edges *tx*, *ty*, *t*z are added.

*H*-chains *[y...z]* can be of arbitrary number, or be not at all.

Graph *G* satisfying these conditions will be called *preparation.*

If graphs *G* and *G'* without common elements have correspondingly *s*-triplets {x, y, z} and {x',y', z'}, then the linking these graphs by edges xy', yx', zz' will give new graph G'' that is 1-H-graph. Rightly, because of condition 3 G'' is threeconnected. The only H-circuit of G'' is composed from elements [x...y], yx', [x'...y'], y'x.

Indeed, each H-circuit of G'' has just two edges from xy', yx', zz'. Because of the condition 1 first two edges go only into indicated *H*-circuit. Because of the fact that there aren't *H*-chains *[x...z]* in *G* and *[x'...z']* in *G'*, pairs of edges *xy', zz'* and *yx', zz'* do not go in any *H*-circuit of *G''*.

[1] This article is compiled from several fragments from Grinbergs manuscripts by D. Zeps

If $G$ is a graph with only one $H$-circuit we will say that the edges of the $H$-circuit are *strong*, but other edges are *weak*. For each vertex x of $G$ with degree $p \geq 3$ there are at least $2(p-2)$ triplets $x, y, z$ that satisfy condition 1 and 2 (Fig. 1, where strong edges are bold).

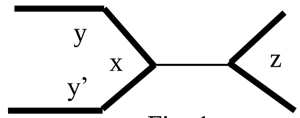
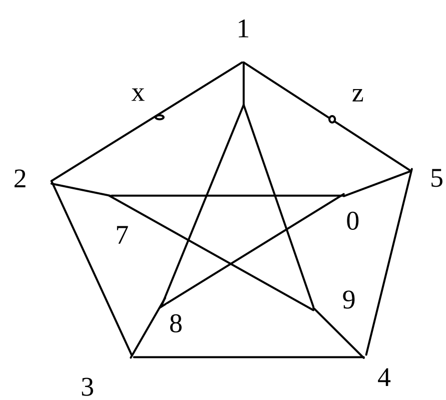


Fig. 1

Vertices $y$ and $z$ are taken correspondingly the end vertices of strong and weak edges $xy$ and $xz$.
If preparation $G$ have vertices of degree $2$ then because of the condition 3.2 they all must go into $s$-triplet. But, if $G$ is $1$-$H$-graph the condition 3 is satisfied, and each triplet of the type of fig. 1 is s-triplet; but there can be other $s$-triplets too. Two such graphs can be linked together in different ways and thus giving new $1$-$H$-graphs.
Thus, it is possible to build $1$-$H$-graphs with arbitrary large number of vertices.
Simplest graphs that we succeeded to find was some modifications of Petersen's graphs: $G_0$ with $n=9$, $G_1$ with $n=11$ and $G_2$, $G_3$ with $n=12$.* [Note of the composer of the article: The matrixes below in $(i, j)$, showing the number of $H$-chains between vertices $i$ and $j$, are computer data and added by us, but in Grinberg's manuscripts indeed were absent. These data allow easy to see that Grinberg characterized all s-triples in considered by him preparations.]

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | x | y |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 2 |
| 0 | 0 | 0 | 0 | 1 | 3 | 3 | 0 | 1 | 2 | 1 | 5 |
| 0 | 1 | 0 | 0 | 0 | 0 | 3 | 2 | 1 | 2 | 5 | 1 |
| 0 | 0 | 1 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 2 | 0 |
| 0 | 2 | 3 | 3 | 2 | 0 | 3 | 0 | 0 | 3 | 2 | 2 |
| 0 | 0 | 2 | 2 | 1 | 3 | 0 | 1 | 0 | 0 | 1 | 5 |
| 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 4 | 4 |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 4 |
| 0 | 1 | 2 | 2 | 0 | 3 | 0 | 0 | 1 | 0 | 5 | 1 |
| 0 | 0 | 1 | 5 | 2 | 2 | 1 | 4 | 4 | 5 | 0 | 0 |
| 0 | 2 | 5 | 1 | 0 | 2 | 5 | 4 | 4 | 1 | 0 | 0 |

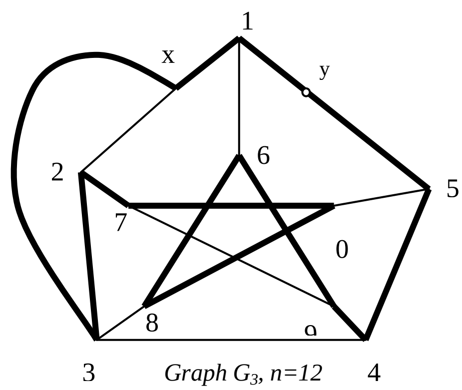


Graph $G_2$, n=12

Fig. 2

Here (in fig. 2) is *s*-triple *{x, 3, z}* (that by automorphisms of $G_2$ transforms into equivalent *s*-triples *{x, 7, z}*, *{z, 4, x}*, *{z, 0, x}*). Indeed, there are not *H*-chains [x...z], otherwise there were *H*-circuit in the Petersen's graph. If we add edge *x3*, we get graph isomorphic to $G_3$ (in Fig. 3). In Fig.3 the only *H*-circuit of the graph $G_3$ is drawn bold, which has in correspondence the only *H*-chain of $G_2$ , namely, *[x...3]*.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | x | y |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 0 | 0 | 1 | 1 | 1 | 2 | 1 | 1 |
| 0 | 0 | 1 | 5 | 2 | 6 | 1 | 3 | 1 | 2 | 0 | 7 |
| 1 | 1 | 0 | 0 | 1 | 3 | 2 | 0 | 2 | 3 | 1 | 6 |
| 2 | 5 | 0 | 0 | 1 | 5 | 4 | 3 | 1 | 5 | 5 | 4 |
| 0 | 2 | 1 | 1 | 0 | 2 | 1 | 1 | 1 | 0 | 2 | 1 |
| 0 | 6 | 3 | 5 | 2 | 0 | 3 | 1 | 1 | 4 | 2 | 3 |
| 1 | 1 | 2 | 4 | 1 | 3 | 0 | 2 | 0 | 1 | 1 | 6 |
| 1 | 3 | 0 | 3 | 1 | 1 | 2 | 0 | 2 | 1 | 4 | 8 |
| 1 | 1 | 2 | 1 | 1 | 1 | 0 | 2 | 0 | 3 | 5 | 6 |
| 2 | 2 | 3 | 5 | 0 | 4 | 1 | 1 | 3 | 0 | 6 | 3 |
| 1 | 0 | 1 | 5 | 2 | 2 | 1 | 4 | 5 | 6 | 0 | 1 |
| 1 | 7 | 6 | 4 | 1 | 3 | 6 | 8 | 6 | 3 | 1 | 0 |



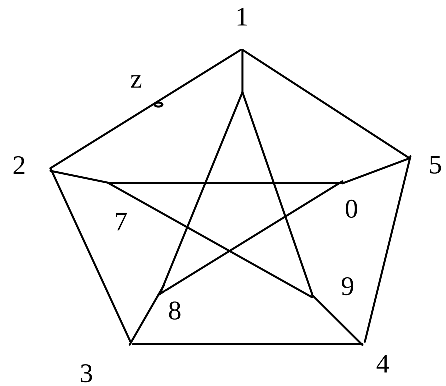

*Graph $G_3$, n=12*
Fig. 3

In the graph $G_3$ , because of the condition 3.2, vertex *y* goes into each *s*-triple. From *y* goes out *H*-chains with ends in each other vertex of $G_3$ , but only in vertices *1, 5* or *x* exactly once. Thus, one of these vertices can be first vertex of *s*-triple, but *y* must be the second in any case.

Good are both trivial *s*-triples *{1, y, 6}* and *{5, y, 0}*. It can be established that there are two more *s*-triples, *{1, y, 2}* and *{x, y, 2}* - giving together four *s*-triples . Triples *{1, y, 5}* and *{5, y, 1}* are not *s*-triples because of condition 3.2. Because $G_3$ has only identical automorphism, these *s*-triples are essentially different.

One more simple preparation ($G_1$, fig. 4) with *s*-triple *{1, 4,z}*. Equivalent with vertex *4* are *8, 9* and *0*, because automorphisms by *(1)(2)(z)* are two: *(37)(40)(5)(6)(8 9)* and *(3)(7)(5 6)(4 8)(9 0)*.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0 | z |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 1 | 0 | 0 | 2 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 1 | 2 | 2 | 0 | 1 | 1 | 1 | 0 |
| 2 | 0 | 0 | 0 | 4 | 4 | 4 | 0 | 3 | 3 | 2 |
| 1 | 1 | 0 | 0 | 0 | 3 | 3 | 2 | 0 | 2 | 6 |
| 0 | 2 | 4 | 0 | 0 | 4 | 4 | 3 | 3 | 0 | 2 |
| 0 | 2 | 4 | 3 | 4 | 4 | 0 | 4 | 0 | 3 | 2 |
| 2 | 0 | 4 | 3 | 4 | 4 | 0 | 3 | 0 | 0 | 2 |
| 1 | 1 | 0 | 2 | 3 | 0 | 3 | 0 | 2 | 0 | 6 |
| 1 | 1 | 3 | 0 | 3 | 0 | 0 | 2 | 0 | 2 | 6 |
| 1 | 1 | 3 | 2 | 0 | 3 | 0 | 0 | 2 | 0 | 6 |
| 0 | 0 | 2 | 6 | 2 | 2 | 2 | 6 | 6 | 6 | 0 |

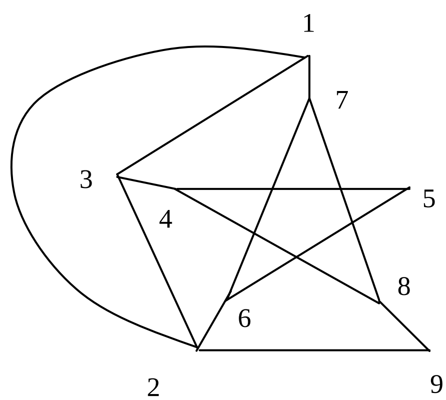



Graph $G_1$, n=11

Fig. 4

Preparation with *n=9* is $G_0$ (fig. 5) with *s*-triple *{1, 9, 5}*. Thus we get *1-H*-graph with *18* vertices (fig. 6).

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 1 | 0 | 1 | 2 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 3 |
| 3 | 1 | 0 | 2 | 2 | 0 | 1 | 1 | 3 |
| 1 | 0 | 2 | 0 | 3 | 0 | 0 | 1 | 1 |
| 0 | 1 | 2 | 3 | 0 | 3 | 1 | 1 | 3 |
| 1 | 1 | 0 | 0 | 3 | 0 | 2 | 0 | 2 |
| 2 | 0 | 1 | 0 | 1 | 2 | 0 | 2 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 2 | 0 | 3 |
| 1 | 3 | 3 | 1 | 3 | 2 | 1 | 3 | 0 |




Graph $G_0$, n=9

Fig. 5

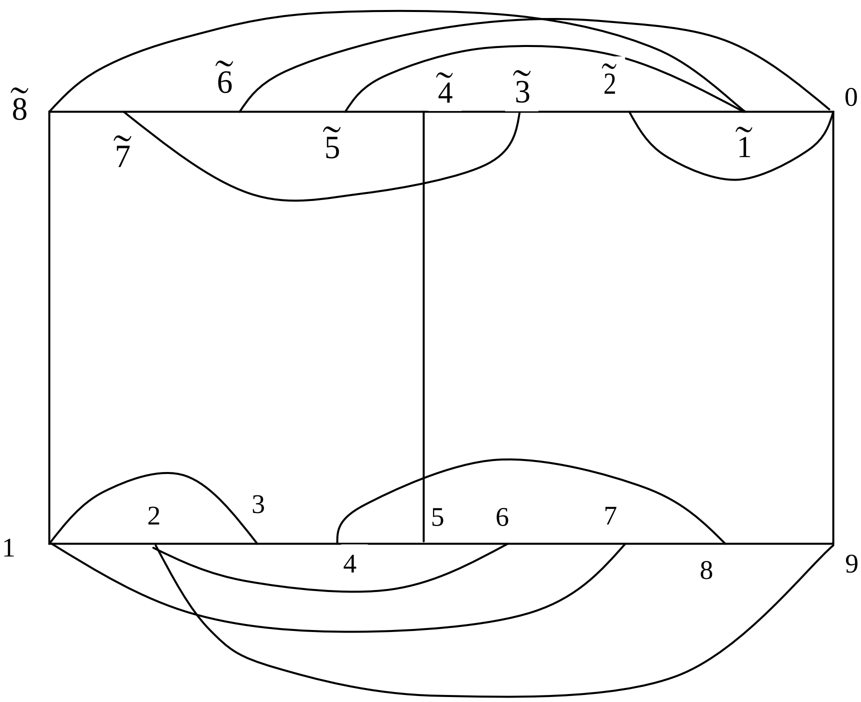



Fig. 6

Thus we get threeconnected *1-H*-graph with *n=18* vertices. Vertices 1, 2, 0, $\tilde{1}$ are with degree four, other of degree three. It seams that at least four edge crossings. The only non-trivial automorphism is symmetry (1 0)(2 $\tilde{1}$)(3 $\tilde{2}$)(4 $\tilde{3}$)(5 $\tilde{4}$)(6 $\tilde{5}$)(7 $\tilde{6}$)(8 $\tilde{7}$)(9 $\tilde{8}$).

The graph constructed from preparations with 9 vertices is possibly minimal threeconnected graph with only one Hamiltonian circuit. Our construction gives only nonplanar graphs. Existence of planar such graphs remains as unsolved problem.

E. Grinberg found these results in years before 1981. In that time still hypothesis lived that triangulation could be found with only one Hamiltonian circuit that was disproved in [1,2,3]. Question about existence of planar threeconnected graph with only one Hamiltonian circuit is still open.

**References**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2. Kratochvil, Jan; Zeps, Dainis. *On the Number of Hamiltonian Cycles in Triangulations*, Journal of Graph Theory, Vol. 12, No. 2, 191-194, 1988.
3. Kratochvíl, Jan ; Zeps, Dainis. *On Hamiltonian Cycles in Two-Triangle Graphs*, Proceedings of the 15th Winter School on Abstract Analysis. Charles University, Praha, 1987. Acta Universitatis Carolinae - Mathematica et Physica, Vol. 28, No. 2. pp. 73–78.