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Title
Equivalent waveguide representation for Dirac plane waves

Abstract
Ideas about the electron as a sort of a bound electromagnetic wave and/or the electron as electromagnetic field trapped in (equivalent) waveguide can be found more or less explicitly in many papers, for example by Zhi-Yong Wang, Roald Ekholdt, David Hestenes, V.A.Induchoodan Menon, J. G. Williamson, M. B. van der Mark.
What we want to show here is that the Dirac equation for electron and positron plane waves admits an equivalent electrical circuit, consisting of an equivalent transmission line.
The same transmission line is representative of a mode in waveguide, so you can also say that the Dirac equation for plane waves includes an implicit representation in terms of an equivalent waveguide.
All the calculation will be done in elementary form, with the usual notations of circuit theory and electromagnetism, without the need to resort to Clifford algebra as in previous papers.

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Equivalent waveguide representation for Dirac plane waves

Introduction and summary

In this paper an equivalence between transmission line and Dirac plane waves is introduced. The same transmission line is representative of TE, TM modes in waveguide, so you can also say that the Dirac equation for plane waves includes an implicit analogy with an equivalent waveguide.
PART ONE: Starting from Maxwell equations, equations in a waveguide for the transverse components are derived.
PART TWO: in these equations we decouple the dependence on x, y introducing an analogue voltage and current V and I equivalent to a waveguide mode (a TE mode). This permits to define an equivalent transmission line for the mode.
PART THREE: there is a degree of freedom in the definition of a scale factor for V and I. With a proper choice of the scale factor for V, I (and the impedance Z) the equations for V, I are reduced to the form of the Dirac equations for plane waves. Thus the plane wave Dirac equations admits the proper equivalent circuit in terms of equivalent transmission line and/or equivalent waveguide.
For simplicity the calculation will be done in extended form only for a TE mode, and shortly for TM.
All the calculation will be done in the classical formalism, with the usual notations of circuit theory and electromagnetism, without the need to resort to Clifford algebra as in [1].

PART ONE: Maxwell's equations in a waveguide for the transverse components

In this section we derive the equations satisfied by the “transverse” component of the $\vec{E}$, $\vec{H}$ guided fields. In particular we consider a cylindrical waveguide (of whatever cross-section) with the axis parallel to the z axis. The non-evanescent $\vec{E}$, $\vec{H}$ fields are therefore assumed to have a dependence on time and z coordinates described by $e^{i\omega t - ik_z z}$. For “transverse” component of $\vec{E}$, $\vec{H}$ we mean the $(E_x + iE_y)$ and $(H_x + iH_y)$ component, transverse to the z-axis.

We start from Maxwell's equations in natural units (c=1):

$$rot\vec{E} = -\frac{\partial \vec{H}}{\partial \tau}, rot\vec{H} = \frac{\partial \vec{E}}{\partial \tau}, div\vec{E} = 0, div\vec{H} = 0$$

and in particular from these two equations:

$$rot\vec{E} = -\frac{\partial \vec{H}}{\partial \tau}, div\vec{E} = 0$$

where

$$\vec{E} \rightarrow E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$
$$\vec{H} \rightarrow H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$$

which, in terms of the individual components, are:

(1) $\partial_y E_z - \partial_z E_y = -\partial_t H_x$

(2) $\partial_z E_x - \partial_x E_z = -\partial_t H_y$

(3) $\partial_x E_y - \partial_y E_x = -\partial_t H_z$

(4) $\partial_x E_x + \partial_y E_y + \partial_z E_z = 0$

Forming –(2)+i(1) i.e. summing i times the equation (1) to minus equation (2) we get:

$$-\partial_z(E_x + iE_y) + (\partial_x + i\partial_y)E_z = -i\partial_t(H_x + iH_y)$$

Similarly, forming (4) + i(3), we get:

$$(\partial_x - i\partial_y)(E_x + iE_y) + \partial_z E_z = -i\partial_t H_z$$

We can repeat the procedure for the other two Maxwell equations, i.e. $rot\vec{H} = \frac{\partial \vec{E}}{\partial \tau}$ and $div\vec{H} = 0$. The resulting 4 equations are:

$$-\partial_z(E_x + iE_y) + (\partial_x + i\partial_y)E_z = -i\partial_t(H_x + iH_y)$$
$$(\partial_x - i\partial_y)(E_x + iE_y) + \partial_z E_z = -i\partial_t H_z$$

$$-\partial_z(H_x + iH_y) + (\partial_x + i\partial_y)H_z = i\partial_t(E_x + iE_y)$$
$$(\partial_x - i\partial_y)(H_x + iH_y) + \partial_z H_z = i\partial_t E_z$$

Now we specialise to the waveguide case and we examine first the TE mode ($E_z = 0$). The above equations become:

(5) $$-\partial_z(E_x + iE_y) = -i\partial_t(H_x + iH_y)$$
(6) $$(\partial_x - i\partial_y)(E_x + iE_y) = -i\partial_t H_z$$
(7) $$-\partial_z(H_x + iH_y) + (\partial_x + i\partial_y)H_z = i\partial_t(E_x + iE_y)$$
(8) $$(\partial_x - i\partial_y)(H_x + iH_y) + \partial_z H_z = 0$$

Suppose now a z propagation with an exponential $e^{i\omega t - ik_z z}$ (IEEE convention). Replace everywhere $\partial_t \to i\omega$:

(5') $$\partial_z(E_x + iE_y) = -\omega(H_x + iH_y)$$
(6') $$(\partial_x - i\partial_y)(E_x + iE_y) = \omega H_z$$
(7') $$-\partial_z(H_x + iH_y) + (\partial_x + i\partial_y)H_z = -\omega(E_x + iE_y)$$
(8') $$(\partial_x - i\partial_y)(H_x + iH_y) + \partial_z H_z = 0$$

We want equations expressed in terms of the transverse components $(E_x + iE_y)$ and $(H_x + iH_y)$ only. Take equation (7') and use equation (6') to eliminate the component H_z as follows.
From (6') we get:

$$(\partial_x + i\partial_y)H_z = \frac{1}{\omega}(\partial_x + i\partial_y)(\partial_x - i\partial_y)(E_x + iE_y)$$

and then, as it is well know from the theory of waveguides, being:

$$(\partial_x + i\partial_y)(\partial_x - i\partial_y)(E_x + iE_y) = (\partial_x^2 + \partial_y^2)(E_x + iE_y) = -k_c^2(E_x + iE_y)$$

we arrive at:

$$(\partial_x + i\partial_y)H_z = -\frac{1}{\omega}k_c^2(E_x + iE_y)$$

which can be substituted in (7'), obtaining:

$$-\partial_z(H_x + iH_y) - \frac{1}{\omega}k_c^2(E_x + iE_y) = -\omega(E_x + iE_y)$$

or:

$$\partial_z(H_x+iH_y)=(\omega-\frac{1}{\omega}k_c^2)(E_x+iE_y)$$

But (in natural units c = 1):

$$k_c^2=\omega_0^2=\omega^2-k_z^2$$

so that:

$$(\omega-\frac{1}{\omega}k_c^2)=\omega(1-\frac{\omega_0^2}{\omega^2})$$

From (5') and (7') we then have:

(9) $$\partial_z(E_x+iE_y)=-\omega(H_x+iH_y)$$

(10) $$\partial_z(H_x+iH_y)=\omega(1-\frac{\omega_0^2}{\omega^2})(E_x+iE_y)$$

To establish a more direct correspondence with the transmission line equations:

(11) $$\frac{dV}{dz}=-i\omega\mu I$$

(12) $$\frac{dI}{dz}=-i\omega\varepsilon\left(1-\frac{\omega_0{}^2}{\omega^2}\right)V$$

we rewrite the equations (9) and (19) as:

(13) $$\partial_z i(E_x+iE_y)=-i\omega(H_x+iH_y)$$

(14) $$\partial_z(H_x+iH_y)=-i\omega(1-\frac{\omega_0^2}{\omega^2})i(E_x+iE_y)$$

<u>Note</u>: this means that equations similar to those of the transmission involve the quantities $i(E_x+iE_y)$ and (H_x+iH_y) and **<u>not</u>** (E_x+iE_y) and (H_x+iH_y), ie there is an imaginary i between.

PART TWO: decouple the dependence on x, y

In this section we establish a clear correspondence of equations (13) and (14) with the transmission line equations.
The meaning of the imaginary unit i which multiplies the second member of the equations (13) and (14) is well expressed by equation (5):

(5) $$\partial_z (E_x + iE_y) = i\partial_t (H_x + iH_y)$$

which shows that $(E_x + iE_y)$ and $(H_x + iH_y)$ are each other at 90° in the x, y plane; it's $i(E_x + iE_y)$ and $(H_x + iH_y)$ which are "parallel”. Their quotient, as well as V / I in a transmission line, it is purely ohmic (or better purely real) such that V = ZI with Z real, just as in a lossless transmission line-
More precisely, as the z,t dependence is given by the exponential:

$$e^{i\omega t - ik_z z}$$

making the two derivatives ∂_z and ∂_t:

$$\partial_z \to -ik_z$$
$$\partial_t \to i\omega$$

we get from (5):

$$ik_z (E_x + iE_y) = \omega (H_x + iH_y)$$

This shows again and explicitly that $i(E_x + iE_y)$ and $(H_x + iH_y)$ are "parallel", and their quotient is real:

$$\frac{\omega}{k_z} = \frac{i(E_x + iE_y)}{(H_x + iH_y)}$$

or:

$$i(E_x + iE_y) = \frac{\omega}{k_z}(H_x + iH_y)$$

Write now the transverse fields $(E_x + iE_y)$ and $(H_x + iH_y)$ in the form:

(15) $$\begin{aligned} \vec{E}_t &= \vec{e}(x, y)V(z) \\ \vec{H}_t &= \vec{h}(x, y)I(z) \end{aligned}$$

Note that if there are not physical conditions which determine V and I, the amplitudes to be assigned individually to $\vec{e}(x,y),V(z)$ as well as to $\vec{h}(x,y),I(z)$ are arbitrary, provided their product remains constant and equal to the amplitude of $\vec{E}_t$ and respectively $\vec{H}_t$.
We can rewrite the previous equation:

(16) $$i(E_x + iE_y) = \frac{\omega}{k_z}(H_x + iH_y)$$

as:

(17) $$i\vec{E}_t = \frac{\omega}{k_z}\vec{H}_t$$

or even:
(18) $$\grave{i}\vec{e}(x,y)V(z) = \frac{\omega}{k_z}\vec{h}(x,y)I(z)$$

The (18) shows what we need right now, a parallelism between $\grave{i}\vec{e}(x,y)$ and $\vec{h}(x,y)$.
Express the parallelism in the form:

(19) $$\grave{i}\vec{e}(x,y) = A\vec{h}(x,y)$$

This allows to eliminate the dependence on x, y, as shown below.
Thanks to the definition (15), equations (13) (14) become:

$$\partial_z i\vec{E}_t = -i\omega\vec{H}_t$$
$$\partial_z \vec{H}_t = -i\omega(1-\frac{\omega_0^2}{\omega^2})i\vec{E}_t$$

or

$$\partial_z i\vec{e}V = -i\omega\vec{h}I$$
$$\partial_z \vec{h}I = -i\omega(1-\frac{\omega_0^2}{\omega^2})i\vec{e}V$$

but being:

$$\grave{i}\vec{e} = A\vec{h}$$

we obtain:

$$\partial_z V = -i\omega\frac{I}{A}$$
$$\partial_z I = -i\omega(1-\frac{\omega_0^2}{\omega^2})AV$$

if there are not physical conditions that uniquely determine V and I (as it happens for example for TE and TM in waveguide) you might make for A the choice which is most convenient , e.g. $A = 1$.
With this choice the above equations are written in the final form:

(20) $\partial_z V = -i\omega I$

(21) $\partial_z I = -i\omega(1 - \frac{\omega_0^2}{\omega^2})V$

Compare it with the usual equations of the transmission lines.
Since V and I depend only on z we can write the equations (20) and (21) in the usual form of transmission line equivalent to a TE mode in MKSA units (see for example Ramo Whinnery [2]):

(22)
$$\frac{dV}{dz} = -i\omega\mu I$$
$$\frac{dI}{dz} = -i\omega\varepsilon\left(1 - \frac{\omega_0^{\;2}}{\omega^2}\right)V$$

The equations of a transmission line are:

(23)
$$\frac{dV}{dz} = -ZI$$
$$\frac{dI}{dz} = -YV$$

where Z and Y depend on the transmission line and $\sqrt{Z/Y}$ is the characteristic impedance of the line.
Equations (22) then implicitly assume as the characteristic line impedance:

(24)
$$Z = i\omega\mu$$
$$Y = i\omega\varepsilon\left(1 - \frac{\omega_0^{\;2}}{\omega^2}\right)$$

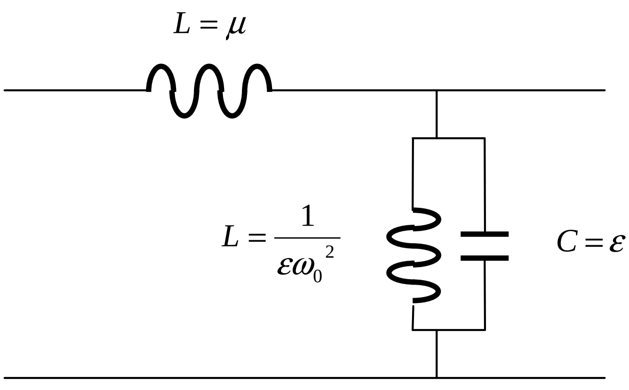
Note also that

(25) $$\sqrt{Z/Y} = \sqrt{\frac{\mu/\varepsilon}{1 - \frac{\omega_0^{\;2}}{\omega^2}}}$$

is equal to the mode impedance ("Schelkunoff choice"):

(26) $$\sqrt{Z/Y} = \frac{Z_0}{\sqrt{1-\frac{\omega_0^2}{\omega^2}}} = Z_{TE}$$

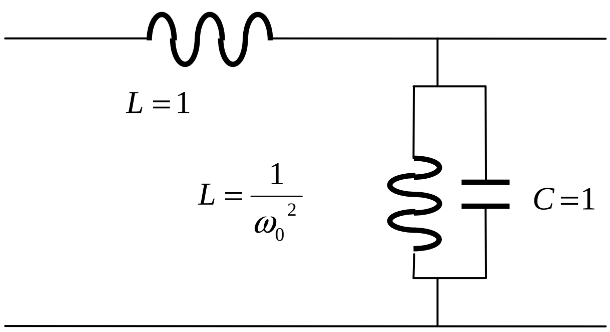
The equivalent line has inductance and capacitance as in the following figure:



The transmission line is dispersive because the characteristic impedance (29) is frequency dependent and it resonates when:

(27) $$\omega = \frac{1}{\sqrt{LC}} = \omega_0$$

In natural units (see (20), (21)) we have instead:



Remains, for the waveguide, all the remaining arbitrariness in the definition of impedance and thus V, I discussed [1], which we summarize here in the following.

PART THREE: reduction to the form of the Dirac equation

As we have seen in the previous section, in the theory of waveguides, we can introduce an equivalent voltage and current, V and I. For all modes but the TEM one the definition of V and I leaves the freedom in the choice of a scale factor, as shown below.

We remember the definition of transverse fields in terms of V and I:

(28)
$$\vec{E}_t(x,y,z) = V(z)\vec{e}(x,y))$$
$$\vec{H}_t(x,y,z) = I(z)\vec{h}(x,y))$$

with the condition:

(29) $$P = \frac{1}{2}\mathrm{Re}\int_S \vec{E}_t \times \vec{H}_t \cdot \hat{n}dS = \frac{1}{2}\mathrm{Re}\left(VI^*\right)$$

The physical meaning of (28) is that V and I deliberately ignore the detailed configuration of $\vec{E}_t$ and $\vec{H}_t$ on the transverse plane.

According to (29) V and I correctly reproduce the value of the total energy that propagates.

The impedance Z is given of course by:

(30) $$\frac{V}{I} = Z$$

Equation (28) leaves a degree of freedom in the definition of V and I: we can alter V and I and simultaneously $\vec{e},\vec{h}$ as follows:

(31)
$$V' = \alpha V, \vec{e}' = \frac{1}{\alpha}\vec{e}$$
$$I' = \frac{1}{\alpha}I, \vec{h}' = \alpha\vec{h}$$

which leaves condition (29) invariant:

(32) $$P = \frac{1}{2}\mathrm{Re}\left(VI^*\right) = \frac{1}{2}\mathrm{Re}\left(V'I'^*\right)$$

Accordingly the value of the impedance Z becomes:

(33) $$Z' = \frac{V'}{I'} = \alpha^2\frac{V}{I} = \alpha^2 Z$$

This freedom does not change the value of quantities related to energy storage and propagation, such as:

$$\frac{V^2}{Z}, ZI^2, VI^*$$

We can now derive an explicit form of the equivalent transmission line which is implied by the Dirac equation for plane wave.
Select the scale factor in (31) as:

$$\alpha = \frac{\sqrt{\omega + \omega_0}}{\sqrt{\omega}} \tag{34}$$

Substituting in (20), (21) we get:

$$\begin{aligned} &\frac{dV'}{dz} + (i\omega + i\omega_0) I' = 0 \\ &\frac{dI'}{dz} + (i\omega - i\omega_0) V' = 0 \end{aligned} \tag{35}$$

and the new Z is:

$$Z' = \frac{V'}{I'} = \frac{\omega + \omega_0}{\omega} Z_{TE} \tag{36}$$

It is immediate now to see that the equations (35) for voltage and current are actually the Dirac equation for ψ_3 and ψ_1.
To see this we refer to the Dirac equation written in extended form, as can be found for example in Schiff [3]:

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_4 + \frac{\partial}{\partial z}\psi_3 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_1 = 0$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\psi_3 - \frac{\partial}{\partial z}\psi_4 + \left(\frac{\partial}{\partial \tau} + im\right)\psi_2 = 0$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\psi_2 + \frac{\partial}{\partial z}\psi_1 + \left(\frac{\partial}{\partial \tau} - im\right)\psi_3 = 0$$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\psi_1-\frac{\partial}{\partial z}\psi_2+\left(\frac{\partial}{\partial \tau}-im\right)\psi_4=0$$

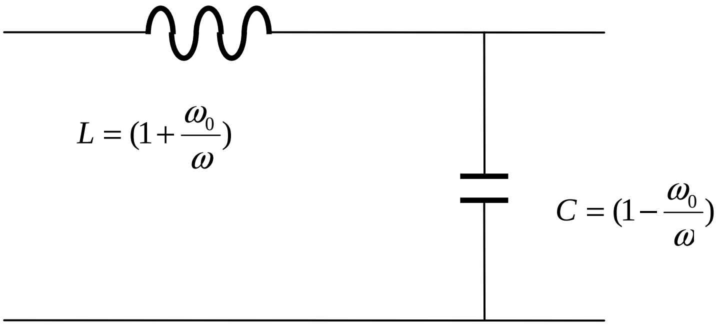
Here $\psi_1,\psi_2,\psi,\psi_4$ are complex functions such as V, I in the usual circuit theory. By setting $\psi_2=\psi_4=0$ and assuming the $e^{+i\omega t}$ dependence on t, as it is for the Dirac's plane wave solution we have for the two components different from zero:

(37)
$$\frac{\partial\psi_3}{\partial z}+(i\omega+i\omega_0)\psi_1=0$$
$$\frac{\partial\psi_1}{\partial z}+(i\omega-i\omega_0)\psi_3=0$$

which coincide with (35).
Thus the Dirac equations (37) are perfectly analogous to the waveguide-transmission line equations (35), once we select the choice (34) for the scale factor α. In particular the characteristic impedance for the Dirac equation is not the "Schelkunoff choice" (26), but it's that determined by (35), i.e.:

(38) $$\sqrt{Z/Y}=\sqrt{\frac{\omega+\omega_0}{\omega-\omega_0}}$$

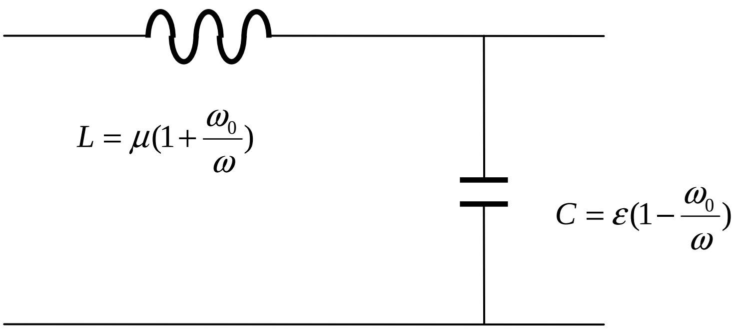
The equivalent transmission line has inductances and capacitance as shown in the following figure:



The natural units employed to write (35) and (37) are convenient but they may mask the true meaning of the electrical parameters of inductance, capacitance and impedance.
Rewrite the equations (35) in MKSA units:

$$
\begin{aligned}
\frac{dV'}{dz} &= -\left(i\omega + i\omega_0\right)\mu I' = -i\omega\mu(1+\frac{\omega_0}{\omega})I' \\
\frac{dI'}{dz} &= -\left(i\omega - i\omega_0\right)\varepsilon V' = -i\omega\varepsilon(1-\frac{\omega_0}{\omega})V'
\end{aligned}
\tag{39}
$$

Comparing (39) with (23) we deduce the parameters of the transmission line.
The equivalent transmission line has serial Z and parallel Y like this:



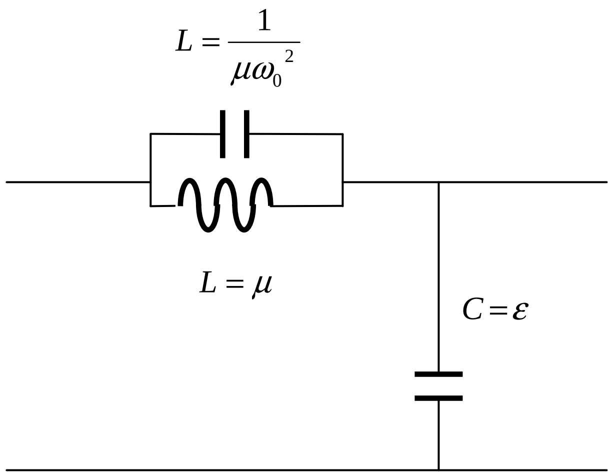
The "Dirac choice" for the characteristic impedance is then:

$$
\sqrt{Z/Y} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{\omega+\omega_0}{\omega-\omega_0}} = Z_0\sqrt{\frac{\omega+\omega_0}{\omega-\omega_0}} \tag{40}
$$

For $\omega \to \infty$ this impedance tends to the impedance Z_0 of empty space.
The same holds for $\omega_0 = 0$, which holds for a TEM-like propagation (which means also no waveguide and neutrino equations).

The equivalent circuit shows that for $\omega = \omega_0$ there is voltage V while the current I is zero, as can be seen also by the impedance (40).
This seems physically reasonable by the fact that the whole calculation from the beginning has been developed for a TE.
For $\omega \leq \omega_0$ exponentially damped evanescent waves propagate.
Until now the calculation was done in extended form for a TE, from the Maxwell equations (5) and (7).
This will be now briefly repeated for a TM with the pair of equations (6), (8).

With similar procedure, for TM mode we get the following equivalent circuit



and equations similar to (22):

$$\begin{aligned} \frac{dV}{dz} &= -i\omega\mu\left(1 - \frac{\omega_0^{\ 2}}{\omega^2}\right)I \\ \frac{dI}{dz} &= -i\omega\varepsilon V \end{aligned} \tag{41}$$

where we use the MKSA units.

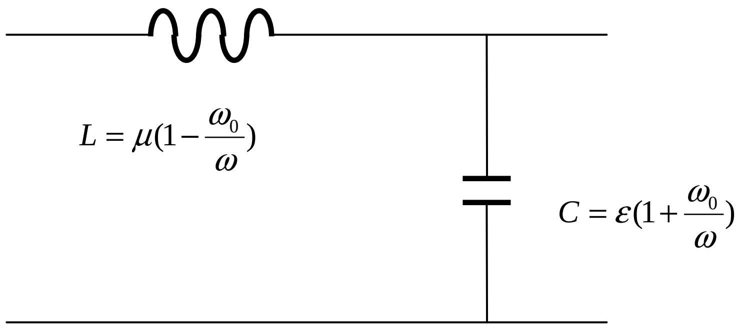
Taking advantage of the arbitrariness inherent in voltage and current and proceeding as for (34) (35) we arrive at:

(42)
$$\frac{dV'}{dz}+\left(i\omega-i\omega_0\right)\mu I'=0$$
$$\frac{dI'}{dz}+\left(i\omega+i\omega_0\right)\varepsilon V'=0$$

You can now see that the equations (42) for voltage and current are now corresponding to the Dirac equation for ψ_4 and ψ_2. These are, in a form similar to (37):

(43)
$$-\frac{\partial}{\partial z}\psi_4+\left(i\omega+i\omega_0\right)\psi_2=0$$
$$-\frac{\partial}{\partial z}\psi_2+\left(i\omega-i\omega_0\right)\psi_4=0$$

Identify with (42) except for a complex conjugate operation, which is interpreted as wave propagation $e^{-i\omega t+ik_z z}$ instead of $e^{i\omega t-ik_z z}$. The equivalent circuit, deductible from (42), is the following



The " Dirac choice" for the characteristic impedance is then:

(44) $$\sqrt{Z/Y}=\sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{\omega-\omega_0}{\omega+\omega_0}}=Z_0\sqrt{\frac{\omega-\omega_0}{\omega+\omega_0}}$$

The characteristic impedance assumes a highly symmetric form between the TE and TM cases, see (40) and (44).
Also the equivalent circuit is very symmetrical: it is always the same apart for a change of sign in ω_0.

For $\omega \to \infty$ the impedance tends to the impedance Z_0 of empty space.
The same holds for $\omega_0 = 0$, which still means a TEM-like propagation (which means also no waveguide and neutrino equations, but now with opposite polarization).
The equivalent circuit shows now that for $\omega = \omega_0$ there is current I, while voltage V is zero, as can be seen also by the impedance (40).
This seems physically reasonable by the fact that now the calculation has been developed for a TM.

CONCLUSIONS

We have thus shown that the Dirac equation for plane waves can be put in correspondence with an electrical circuit, equivalent to a transmission line.
The same transmission line is representative of a mode in waveguide, so you can also say that the Dirac equation for plane waves includes an implicit representation of an equivalent waveguide.
The equivalence is embedded in the usual V and I description.
To quote Hestenes "we want to emphasize that this interpretation is by no means a radical speculation; it is a fact! The interpretation has been implicit in the Dirac theory all the time. All we have done is make it explicit". (Hestenes here refers to the interpretation of the imaginary "i").
The calculation was done in extended form for a TE, from the Maxwell equations (5) and (7). This was briefly repeated for a TM with the pair of equations (6), (8).
Doing so, the full set of plane wave Dirac equations can be interpreted in terms of appropriate equivalent transmission line circuits and/or equivalent waveguide.
Obviously solutions with opposite spin are represented by opposite polarization in the waveguide.
The equivalent transmission line shares all the usual properties of the transmission lines, including the dispersive character, and evanescent waves.
The evanescent waves may be the correspondent of electrons propagating through a potential barrier.

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[3] L. I. Schiff, “Quantum Mechanics”, McGraw-Hill (1968)