Tidal Charges From BraneWorld Black Holes As An Experimental Proof Of The Higher Dimensional Nature Of The Universe.

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Abstract

If the Universe have more than 4 Dimensions then its Extra Dimensional Nature generates in our $4D$ Spacetime a projection of a $5D$ Bulk Weyl Tensor. We demonstrate that this happens not only in the Randall-Sundrum BraneWorld Model where this idea appeared first(developed by Shiromizu,Maeda and Sasaki)but also occurs in the Kaluza-Klein $5D$ Induced Matter Formalism.As a matter of fact this $5D$ Bulk Weyl Tensor appears in every Extra Dimensional Formalism (eg Basini-Capozziello-Wesson-Overduin Dimensional Reduction From $5D$ to $4D$) because this Bulk Weyl tensor is being generated by the Extra Dimensional Nature of the Universe regardless and independently of the Mathematical Formalism used and the Dimensional Reduction From $5D$ to $4D$ of the Einstein and Ricci Tensors in both Kaluza-Klein and Randall-Sundrum Formalisms are similar.Also as in the Randall-Sundrum Model this $5D$ Bulk Weyl Tensor generates in the Kaluza-Klein formalism a Tidal "Electric" Charge "seen" in $4D$ as an Extra Term in the Schwarzschild Metric resembling the Reissner-Nordstrom Metric. We analyze the Gravitational Bending Of Light in this BraneWorld Black Hole Metric(known as the Dadhich,Maartens,Papadopolous and Rezania) affected by an Extra Term due to the presence of the Tidal Charge compared to the Bending Of Light in the Reissner-Nordstrom Metric with the Electric Charge also being generated by the Extra Dimension in agreement with the point of view of Ponce De Leon (explaining in the generation process how and why antiparticles have the same rest mass m_0 but charges of equal modulus and opposite signs when compared to particles)and unlike the Reissner-Nordstrom Metric the terms $G/(c^4)$ do not appear in the Tidal Charge Extra Term.Thereby we conclude that the Extra Term produced by the Tidal Charge in the Bending Of Light due to the presence of the Extra Dimensions is more suitable to be detected than its Reissner-Nordstrom counterpart and this line of reason is one of the best approaches to test the Higher Dimensional Nature of the Universe and we describe a possible experiment using Artificial Satellites and the rotating BraneWorld Black Hole Metric to do so

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1 Introduction-The Kaluza Cylindrical Condition and The Klein Compactification Mechanism

The Physics of Extra Dimensions appeared for the first time in the year of 1918 with the work of the physicists Theodore Kaluza and Oskar Klein[1].Both discovered that a Fifth Dimensional version of the Einstein General Relativity in vacuum [where ${}^{5}G_{AB} = 0$ and $A, B = 0, 1, 2, 3, 4$ being 0 the Time Dimension, $(1, 2, 3)$ the scripts of the $3D$ known Spatial Dimensions x,y,z and 4 the script of an invisible Fifth Dimension] contains all the Four Dimensional Einstein General Relativity in the presence of an Electromagnetic Field [which means to say ${}^{4}G_{\alpha\beta} = {}^{4}T^{EM}_{\alpha\beta}$, with $\alpha, \beta = 0, 1, 2, 3$ as required by a Four-Dimensional theory] plus the equations of the Maxwell Electromagnetic Field. However this formalism implied and raised a fundamental open question still to be solved or explained:

- Why had no Fifth Dimension been observed in Nature until now?[2]

Kaluza suggested that for some unknown reason all known physics happens only in the Four-Dimensional Spacetime $x^0 = t$,$x^1 = x$,$x^2 = y$,$x^3 = z$ leaving the Fifth Dimension x^4 separated from the reality we can observe.Then he created the so-called Cylindrical Condition that in scientific language is:

- All the derivatives with respect to x^4 vanishes.Kaluza did not explained why the observable physics depends on the first Four Spacetime Dimensional coordinates, but not on the Fifth Dimensional one.By making the derivatives disappear Kaluza isolated the Fifth Dimension from the conventional Four Dimensional physics and we dont need to worry any longer with the invisibility of the Fifth Dimension[3]

Klein appeared with a different proposal to explain why we cannot see beyond the Four Dimensional Spacetime:

- Extra Dimensions are invisible because they are compactified to less than an attometer in size (1 am $= 10^{-18}$ m) and this Compactification Mechanism hides the Higher Dimensional Spacetime making the Fifth Dimension not detectable by experimentally accessible Energy Scales. Since we cannot generate the energy output to make the Fifth Dimension visible because this output would be higher than any known Energy Scale except perhaps Energy Scales of the Early Universe[4] we will never be able to see it. Hence like the Kaluza proposal if we cannot see the Fifth Dimension we no longer need to worry about it. This point of view was successful and is the main line of reason of the modern Higher-Dimensional Physics.[5]

Kaluza and Klein made the Fifth Dimension invisible in order to keep compatibility between the known $4D$ conventional physics[6] and the new formalism they invented.However both created different explanations for the invisibility of the Fifth Dimension.At this point one would ask:

[1] see [8] for an excellent account on Kaluza-Klein History

[2] see pg 4 in [8]

[3] see pg 4,5 in [8]

[4] or perhaps Energy Scales generated by the Large Hadron Collider however these Energy Scales could not be kept for the time enough to be detectable

[5] see pg 5,6 in [8]

[6] this "conventional physics" encompasses also Einstein General Relativity and Planck-Schodinger-Dirac Quantum Mechanics

- Which one of these proposed "Invisibility" Mechanisms is the correct?Cylindricity or Compactification?
- Or are both correct?
- Or none are correct and the explanation for the "Invisibility" of the Fifth Dimension lies somewhere else?

The modern scientific community adopted the Klein idea and we can verify that:

- An "industry" [7] of Compactification Mechanisms and Topology of Compact Spaces appeared in the last few years
- Klein Compactification Mechanism evolved to dominate all the Higher Dimensional physics leading to $11D$ Supergravity and $10D$ Strings Theories[8]

Since the Klein Compactification Mechanism known as the Tower Fourier Mode was adopted by many theories we will provide a small explanation [9]:
Klein idealized the following rules for the Compactification Mechanism

- Circular Topology: Any quantity $f(x,y)$ with $x = (x^0, x^1, x^2, x^3)$ and $y = x^4$ becomes periodic or circular $C = 2\pi r$ with $f(x,y) = f(x, y + 2\pi r)$
- Small Scale for the Radius r less than an attometer in size (1 am $= 10^{-18}$ m) but r often is regarded as equal to the Planck length $\ell_{pl} \sim 10^{-35}$ m

Fourier expanding the Spacetime Metric Tensor and the Scalar Field of the Spacetime Metric we get[10] [11]:(see eq 11 pg 16 and eq 20 pg 20 in [8])(see also pg 5 eq 5 in [7])

$$(g_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -\Phi^2 \end{pmatrix} \quad . \tag{1}$$

$$dS^2 = g_{\alpha\beta} dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{2}$$

$$\begin{aligned} g_{\alpha\beta}(x,y) &= \sum_{n=-\infty}^{n=\infty} g^{(n)}_{\alpha\beta}(x) e^{iny/r} \quad , \\ \Phi(x,y) &= \sum_{n=-\infty}^{n=\infty} \Phi^{(n)} e^{iny/r} \quad , \end{aligned} \tag{3}$$

According to Quantum Theory these n states $n = 0, 1, 2, 3....\infty$ known as Fourier Tower Modes carry wave momentum in the y-direction being this momentum of the order $|n|/r$. Note that this according to property (2) if r is small enough, then the y-momenta of even the $n = 1$ modes will be so large and

[7] this word was taken from pg 6 in [8]
[8] see pg 6 in [8]
[9] see pg 20 in [8] for more details
[10] n refers to the nth Fourier mode.
[11] $A, B = 0, 1, 2, 3, 4$ and $\alpha, \beta = 0, 1, 2, 3$

this means a large energy.This energy is beyond the reach of experiment by any known physical process. Therefore only the $n = 0$ modes, which are independent of y, will be observable and the $n = 0$ Mode will make the Fifth Dimension invisible, as required in Kaluza's theory.If r equal to the Planck length $\ell_{pl} \sim 10^{-35}$ m, then the mass of any $n \neq 0$ Fourier modes lies beyond the Planck mass $m_{pl} \sim 10^{19}$ GeV outside the scope of our physics.[12]

While Kaluza did not explained the Cylindrical Condition,the Klein Compactification Mechanism answered the question why we cannot see the Fifth Dimension then it was accepted or adopted by the scientific community and becames the basis of the Strings and Supergravity Theories.However it raises more questions than answers such as:

- Did the Universe when in the Big Bang really opted by the choice of a Compactified Fifth Dimension while the other Spatial Dimensions x,y and z are Uncompactified ????If so then why????
- What generates(or generated in the Big Bang) the Klein Compactification Mechanism in the first place????And how it was generated???
- Do the Klein Compactification Mechanism really exists in the Universe????
- Can we proof that the Universe really adopted the Klein Compactification Mechanism ???
- Is there a logical reason for the Universe choice of $g_{\alpha\beta}(x, y) = \sum_{n=-\infty}^{n=\infty} g_{\alpha\beta}^{(n)}(x)e^{iny/r}$ for the Spacetime Metric Tensor and $\Phi(x, y) = \sum_{n=-\infty}^{n=\infty} \Phi^{(n)}e^{iny/r}$ for the Scalar Field??Or had the Universe when in the Big Bang adopted a different Mathematical Formalism???

To terminate with the Klein Compactification Mechamism a last question must be placed here:

- If the Klein Compactification Mechanism exists then how many of these $n \neq 0$ Fourier Tower Modes really exists???$n = 0, 1, 2....\infty$????Will n stops at a given value or it will grow to ∞ ???? And again why???

The Principle of the Occam Razor[13] states that Nature always choose the simplest solution.The simplest solutions are the more elegant ones.So perhaps the Universe in the Big Bang had adopted a different solution regarding Extra Dimensional Spacetimes and did not followed the Klein Compactification Mechanism.Lets imagine that all the spatial dimensions are uncompressed and not compactified and the Fifth Dimension is as large as x,y and z.This assumption raises the following question:

- If the Fifth Dimension is large like the other spatial dimensions then why we cannot see it??? [14]

In our opinion the Cylindrical Condition of Kaluza must be "reformulated"[15] to explain the "Invisibility" of the Fifth Dimension without large and unreachable Planck Energies from Tower Fourier Modes or

[12] see bottom of pg 20 and top of pg 21 in [8]

[13] William Of Occam was a philosopher of the Century XIV and according to him the Nature always choose the easy way.The Man always want or likes to complicate the things.This statement is known as the Principle of the Occam Razor:Nature(or the Universe)always choose to "cut" what is not necessary

[14] see bottom of pg 29 in [8].To explain why we cannot see the Fifth Dimension is the biggest challenge of Non-Compactified approaches

[15] see pg 23 before paragraph 4.3 in [8].It is mentioned that we can abandon the restrictions of Compactification of the Fifth Dimension.See also top of pg 2 in [4].It is mentioned that the Cylindrical Condition is not sustained or required and we may live in a Universe of Large Extra Dimensions

another(or even more)exotic difficulties arising from the Klein Compactification Mechanism.Our purpose is to "bring back" or "restore" the original 1918 Kaluza idea of a Cylindrical Condition but however with a more modern fashion and an updated shape to make it more "physical".Rewriting the Kaluza Cylindrical Condition with the following modifications:

- 1)-All the derivatives with respect to x^4 do not vanishes although possesses very small values close to zero due to the geometrical shape of the region of Spacetime in which we live.These extremely low values would made the Fifth Dimension looks apparently "Invisible"
- 2)-All the derivatives of the Spacetime Metric Tensor[16] $g_{\alpha\beta}$ from the Spacetime Ansatz of the $5D$ General Relativity $dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2 dy^2$ (see [1] eq 56,[2] eq 42,[6] eq 109,[7] eq 5) with respect to x^4 do not vanishes and behave as described above in the item 1
- 3)-The Human eye have only Tridimensional perception[17],so the known spatial dimensions x,y and z are the ones we can really see and the effects of the Fifth Dimension can only be observed if we change the geometrical shape of a given region of Spacetime[18] affecting the Metric Tensor and its derivatives with respect to x^4 to obtain larger values for these derivatives making the Fifth Dimension "Visible" however these effects will appear and be noticed by indirect ways[19]

We will provide an explanation for the statements above demonstrating why the Fifth Dimension appears "Invisible" and what could be made to turn it "Visible" using the $5D$ General Relativity Ansatz and the $5D$ Schwarzschild Metric to obtain our "modern" version of the Kaluza Cylindrical Condition.[20]

- The $5D$ General Relativity Ansatz is given by the following equations([1] eq 56,[2] eq 42,[6] eq 109,[7] eq 5):

$$dS^2 = g_{\alpha\beta}(x^\rho, y)dx^\alpha dx^\beta - \Phi^2(x^\rho, y)dy^2 \tag{4}$$

- The corresponding $5D$ Ricci Tensors and Scalars are given by [21] ([1] eq 58,[2] eq 44,[8] eq 48 and 49,[6] eq 111 and 112, [7] eq 7 and 8):

$$^5R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} - \frac{1}{2\Phi^2}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{5}$$

$$^5R = R - \frac{\Phi_{,a;b}}{\Phi}g^{\alpha\beta} - \frac{1}{2\Phi^2}g^{\alpha\beta}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{6}$$

[16]see bottom of pg 29 in [8] for the mention of quantities derived from the Metric Tensor being functions of the Fifth Dimension.note also the mention of a relaxed Cylindrical Condition

[17]see the American Physicist Lisa Randall on Wikipedia and Wikiquote.These sites contains wonderful images of what we could see with vision in $5D$ Capability eg:Tesseract Cubes and these are extremely difficult to visualize. Nature adopted a sample way.Lifeforms do not need to "see" the Universe in $5D$,$3D$ is enough.Otherwise the biological evolution of the eye would be un-necessary complicated.A "prey" dont need to see the "predator" in $5D$ so a vision in $3D$ and good legs to escape are enough

[18]we can also move ourselves to a different region of Spacetime where the values of the derivatives becomes more noticeable

[19]more on this in Section 4

[20]in these expressions $g_{\alpha\beta,4} = g_{\alpha\beta,y}$ are the derivatives of the Spacetime Metric Tensor components with respect to x^4 being x^4 or y the Fifth Dimension

[21]$R_{\alpha\beta} =^4 R_{\alpha\beta}$ and $R =^4 R$ are the Fourth Dimensional versions of the Ricci Tensor and Scalar

- The $5D$ Schwarzschild Metric is given by([7] eq 68):

$$dS^2 = [(1 - \frac{2Gm_0}{c^2R})c^2dt^2 - \frac{dR^2}{(1 - \frac{2Gm_0}{c^2R})} - R^2d\eta^2] - \Phi^2(x^\rho, y)dy^2 \tag{7}$$

From the expression above for $5D$ Schwarzschild Metric it is not clear how the Spacetime Metric Tensor components $g_{\alpha\beta}(x^\rho, y)$ are functions of the Extra Coordinate.After all the Spacetime Metric Tensor components for the Schwarzschild Metric are functions of a given rest-mass m_0 and the radius R of the distance from a given point p in space to this mass and we need that at least one of these two quantities must be function of the Extra Coordinate otherwise the values of the derivatives of the Spacetime Metric Tensor components with respect to the Extra Coordinate will be zero and we would recover the original Kaluza Cylindrical Condition.However we know that a given $4D$ rest-mass m_0 is a function of the $5D$ rest-mass M_5 being the relationship between m_0 and M_5 for the $5D$ Schwarzschild Metric above given by the following expression([3] eq 20,[7] eq 1):

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{8}$$

Considering that we live in a region of Spacetime where the Scalar Field $\Phi^2(x^\rho, y)$ is constant[22][23] then all the derivatives of $\Phi^2(x^\rho, y)$ will vanish and we need to worry ourselves only with the derivatives of the Spacetime Metric Tensor components $g_{\alpha\beta,4} = g_{\alpha\beta,y}$

- The Spacetime Metric Tensor components are given by:
- $g_{00}(x^\rho, y) = 1 - \frac{2Gm_0}{c^2R}$
- $g_{11}(x^\rho, y) = -\frac{1}{(1 - \frac{2Gm_0}{c^2R})}$

Note that at a large distance R from the rest-mass m_0 which means to say $Gm_0 << c^2R$ and we must outline that $G = 6,67 \times 10^{-11}\frac{Nm^2}{kg^2}$ and $\frac{1}{c^2} = \frac{1}{9 \times 10^{16}}$ and combining the two powers we would get $10^{-11} \times 10^{-16} = 10^{-27}$ this results in a value almost close to zero[24][25] for the term $\frac{2Gm_0}{c^2R}$ so the Spacetime Metric Tensor components at a large distance R from a given rest-mass m_0 will be close although not equal to the ones of the Minkowsky Metric of Special Relativity $g_{00} = 1$ and $g_{11} = -1$.This is considered "Flat Spacetime" and the Spacetime around Earth is considered "Flat"[26].The derivatives of the Spacetime Metric Tensor in "Flat" Spacetime with respect to the Extra Dimensions are almost close to zero but not exactly zero because m_0 is a function of the Extra Coordinate.Hence this would make the Fifth Dimension looks apparently "Invisible"

[22] more on this in Section 2

[23] or we can have no Scalar Field in the $5D$ General Relativity Ansatz.in this case $\Phi^2(x^\rho, y) = 1$ and the derivatives of Φ components in the $5D$ Ricci Tensor and Scalar will vanish too

[24] Weak-Field Limit

[25] the Mass of the Sun is $m_0 = 1,9891 \times 10^{30}kg$ (see eq 108 in [7]) so dividing 10^{30} by 10^{27} we would get "only" a mere 10^3 so at a large distance R from the Sun the value of the term $\frac{2Gm_0}{c^2R}$ vanishes completely making the Fifth Dimension looks "Invisible"

[26] the mass of Earth is of about $m_0 = 10^{24}kg$ so in this case the term $\frac{2Gm_0}{c^2R}$ would be a division of 10^{24} by 10^{27} giving a small 10^{-3} and also in this case at a large distance R from Earth the term would be close to zero making the Fifth Dimension "Invisible"

This can explain why we cannot "see" the Fifth Dimension and we do not need to use "exotic" Compactification Mechanisms or "unexplainable" Cylindrical Conditions.This more modern version of the Cylindrical Conditions suits better to explain the apparent "Invisibility" of the Fifth Dimension

At this point we would like to outline some questions and answers such as:

- How could the Fifth Dimension be made "Visible"???
- The Fifth Dimension could be made "Visible" if we raise the value of the derivatives of the Spacetime Metric tensor with respect to the Extra Coordinate
- Then how could this be made???
- Look to the term $\frac{2Gm_0}{c^2R}$.At a large distance R from a given rest mass m_0 this term reduces to almost zero but if we move closer to the rest mass and we need to stay too much close in order to make $2Gm_0$ becomes a significant fraction of c^2R.By reducing the distance R between a point p and a gravitational source m_0 and if the mass m_0 is bigger enough to sustain the division by c^2R and bigger enough to sustain the multiplication by G then the term $\frac{2Gm_0}{c^2R}$ would become noticeable and as far as we move close to m_0 reducing the distance R then the values of the derivatives of $\frac{2Gm_0}{c^2R}$ with respect to the Extra Coordinate would become noticeable making the Fifth Dimension "Visible" [27]
- All we have to do is to move ourselves to a region of Spacetime in which the Metric Tensor coefficients g_{00} and g_{11} becomes different than $+1$ and -1 making the derivatives with respect to the Extra Coordinate becomes different than zero [28]
- Earth do not possesses enough mass to make the Fifth Dimension "Visible".It was due to this "Invisibility" of the Fifth Dimension from the Earth that Kaluza created the Cylindrical Condition and Klein created the Compactification Mechanism.[29] The nearest object close to us where this could be accomplished[30] is the Sun
- However in order to notice the effects of the Fifth Dimension we would need to be so close to the Sun where the temperatures would become hazardous
- According to Einstein Classical General Relativity a Light Beam when passing in the neighborhoods of the Sun will suffer a Gravitational Bending.Also we know that Extra Dimensions would generate additional mathematical terms in the formulas of the Gravitational Bending.Then a Laser Beam could perhaps be sent to pass the Sun at a small distance R (since the Laser would not be affected by the Sun temperature) in order to suffer the effects of the Gravitational Bending Of Light and its mathematical extra terms due to the presence of the Fifth Dimension be noticed and detected making at last the Fifth Dimension becomes "Visible". [31]

[27]for the Sun $10^{30} \times 10^{-11} = 10^{19}$ and dividing this by 10^{16} we would get 10^3.

[28]see footnote 17 in this work.if we can move ourselves to a different region of Spacetime where the values of the derivatives becomes more noticeable we as a matter of fact changing the Shape of the Spacetime

[29]Note that Earth is placed in a region of the Universe where the Scalar Field $\Phi^2(x^\rho, y) = 1$,then all the derivatives of the Scalar Field with respect to the Extra Coordinate will vanish and in a region of Flat Spacetime the derivatives of the Spacetime Metric Tensor with respect to the Extra Coordinate will "vanish" because $g_{00} \simeq 1$ and $g_{11} \simeq -1$ and $g_{\alpha\beta,4} = g_{\alpha\beta,y} \simeq 0$.Then the $5D$ Components of the Ricci Tensors and Ricci Scalars becomes approximately equal to its $4D$ counterparts and we cannot tell if we live in a $5D$ or a $4D$ Universe.more of this in section 2

[30]To make the Fifth Dimension "Visible"

[31]see abstract and bottom of pg 18 and pg 19 to top of pg 21 in [7]. see also eqs 156 to 158 pg 70 section 8.7 in [8] and between page 70 and 71 also in [8] the comment that the shift is physically measurable

- This could prove the real existence of the Fifth Dimension in the Universe
- This is as a matter of fact the purpose of this work

This Section by itself contains all the scientific arguments developed in this work but in order to give a more physical and mathematical concise treatment of the subsequent material involved we divided the remaining scientific context into the following Sections:

- Section 2)-Dimensional Reduction from a $5D$ Spacetime to a $4D$ Spacetime in both Induced Kaluza-Klein and Randall-Sundrum Formalisms. The approaches of Shiromizu-Maeda-Sasaki and Basini-Capozziello-Overduin-Wesson.The "Hypotheses Non Fingo" of Isaac Newton
- Section 3)-Rest-Masses and Electric Charges seen in a $4D$ Spacetime but being generated by a $5D$ Spacetime due to the Geometrical Nature of the Hamilton-Jacobi Equation.Why does the electron and the positron possesses the same rest mass but different charges of equal modulus and opposite signs??.And why both annihilates??Is our $4D$ Universe the intersection point between two different $5D$ BraneWorld Universes ?:The approach of Ponce de Leon
- Section 4)-The Structure of a BraneWorld Star according to the Metric of Dadhich,Maartens,Papadopolous and Rezania: What happens when the Schwarzschild Radius is reached ?:The Tidal Charge changes its sign:The approaches of Germani-Maartens and Kotrlova-Stuchlik-Torok
- Section 5)-Gravitational Bending Of Light in both BraneWorld Black Hole and Reissner-Nordstrom Spacetime Metrics: The approaches of Briet-Hobill,Kar-Sinha,Gergely-Darazs-Keresztes-Dwornik,Aliev-Talazan and Bohmer-Harko-Lobo
- Section 6)-Experimental Detection of Extra Dimension in Outer Space using Artificial Satellites and Laser Beams:The Rotating BraneWorld Black Hole Metric of Dadhich,Maartens,Papadopolous and Rezania as defined by Kotrlova-Stuchlik with the $\delta\phi$ defined by Aliev-Talazan and the European Space Agency Satellite GAIA
- Section 7)-Tidal Charges From BraneWorld Black Holes As An Experimental Proof Of The Higher Dimensional Nature Of The Universe.

In Section 2 we analyze the $5D$ to $4D$ Dimensional Reduction made by Shiromizu-Maeda and Sasaki according to [14] and [15] for the Einstein Equations on the Bulk[32] being reduced to the Einstein Equations on the Brane[33] according to pg 2 in [14] but we will not consider the Confinement Mechanisms (see pg1 and 2 in [14]) [34][35][36].We consider a $5D$ Einstein Equation without vacuum energy and without Cosmological Constant described by eq 6 pg 2 in [14] or eq 1 pg 2 in [15] or eq 3.1 pg 8 in [13][37]

$$^{(5)}G_{\alpha\beta} = {}^{(5)}R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}{}^{(5)}R = \kappa_5^2\,{}^{(5)}T_{\alpha\beta}\,, \tag{9}$$

[32]for unfamiliar readers the Bulk is the other name given to the Fifth Dimension

[33]again for unfamiliar readers the Brane is the other name given to the ordinary $4D$ Spacetime

[34]these are evolutions of the original Klein Compactification Mechanisms

[35]recall pg 6 in [8] for the "Industry" of Compactification Mechanisms

[36]the Modified Kaluza Cylindrical Condition can absorb the Shiromizu-Maeda-Sasaki Dimensional Reduction from $5D$ to $4D$ and provide similar results with perhaps more simplicity

[37]the $^{(5)}T_{\mu\nu}$ is the stress energy momentum tensor related to the $5D$ rest-mass M_5 according to Ponce De Leon.see eq 20 in [3]

with

$$\kappa_5^2 = 8\pi G_5 \tag{10}$$

and a $5D$ to $4D$ Dimensional Reduction of a Riemann Tensor given by the Gauss Equation described by eq 1 pg 2 in [14] (see also eqs 3.5 and eqs 3.7 to 3.9 in [13])

$$^{(4)}R^{\alpha}{}_{\beta\gamma\delta} = {}^{(5)}R^{\mu}{}_{\nu\rho\sigma} q_{\mu}{}^{\alpha} q_{\beta}{}^{\nu} q_{\gamma}{}^{\rho} q_{\delta}{}^{\sigma} + K^{\alpha}{}_{\gamma} K_{\beta\delta} - K^{\alpha}{}_{\delta} K_{\beta\gamma}\,, \tag{11}$$

with the Extrinsic Curvature written in terms of the the Covariant Derivative being defined by pg 2 in [14](see also eq 3.3 pg 8 in [13])

$$K_{\mu\nu} = q_{\mu}{}^{\alpha} q_{\nu}{}^{\beta} \nabla_{\alpha} n_{\beta} \tag{12}$$

The Spacetime Metric for the equations above is described by eq 12 pg 3 in [14] or eq 2 pg 2 in [15](see also eq 3.2 pg 8 in [13])

$$dS^2 = d\chi^2 + q_{\mu\nu} dx^{\mu} dx^{\nu}\,. \tag{13}$$

$$dS^2 = (n_{\mu} n_{\nu} + q_{\mu\nu})\, dx^{\mu} dx^{\nu} = (n_{\mu\nu} + q_{\mu\nu})\, dx^{\mu} dx^{\nu} = d\chi^2 + q_{\mu\nu} dx^{\mu} dx^{\nu}\,, \tag{14}$$

$$^{(5)}g_{\mu\nu} = q_{\mu\nu} + n_{\mu\nu} \tag{15}$$

with according to pg 2 in [14] or pg 2 in [15](see also pg 8 in [13])

$$n_{\mu} dx^{\mu} = d\chi \tag{16}$$

The "Electric"[38] part in $5D$ is given by eq 5 pg 2 in [14].It is exactly equal to the $5D$ Ricci Tensor[39].We will examine this in details.This will be fundamental when examining the Tidal Charges From BraneWorld Black Holes.

$$\tilde{E}_{\mu\nu} = {}^{(5)}E_{\mu\nu} \equiv {}^{(5)}R^{\alpha}{}_{\beta\rho\sigma} n_{\alpha} n^{\rho} q_{\mu}{}^{\beta} q_{\nu}{}^{\sigma} = {}^{(5)}R_{\mu\nu} \tag{17}$$

Applying the $5D$ formula for the Weyl Tensor eq 7 pg 2 in [14] (see also eq $A1$ pg 5 in [14],eq 3.10 pg 9 in [13],eq 3.97 pg 89 in [9])

$$^{(5)}R_{\mu\alpha\nu\beta} = \frac{2}{3}(g_{\mu[\nu}{}^{(5)}R_{\beta]\alpha} - g_{\alpha[\nu}{}^{(5)}R_{\beta]\mu}) - \frac{1}{6} g_{\mu[\nu} g_{\beta]\alpha}{}^{(5)}R + {}^{(5)}C_{\mu\alpha\nu\beta}, \tag{18}$$

Remember that according to eq 3.76 pg 85 in [9]

$$^{(5)}R_{\rho\sigma\mu\nu} = g_{\rho\lambda}{}^{(5)}R^{\lambda}{}_{\sigma\mu\nu} \tag{19}$$

or better(see eq 3.90 pg 88 in [9])

$$^{(5)}R_{\rho\sigma\lambda\nu} = g_{\rho\lambda}{}^{(5)}R^{\lambda}{}_{\sigma\lambda\nu} = g_{\rho\lambda}{}^{(5)}R_{\sigma\nu} \tag{20}$$

Then from the $5D$ Riemann Tensor with one upper script we can obtain the $5D$ Ricci Tensor and the $5D$ Riemann Tensor with all lower scripts and the $5D$ Weyl Tensor[40].

[38] the world "Electric" appears in pg 6 before eq $A2$ in [14]
[39] script α equal to script ρ
[40] with a proper manipulation of the scripts as for example in diagonalized metrics

The $4D$ "Eletrical" part is a contracted $5D$ Weyl Tensor and will affect the Schwarzschild Black Hole Metric with a mathematical extra term due to the presence of the Fifth Dimension converting the Schwarzschild Metric into the Tidal Charge BraneWorld Black Hole Metric also known as the Dadhich,Maartens,Papadopolous and Rezania Metric and is given by eq 9 pg 2 in [14])(see also eq $A2$ pg 6 in [14],eq 7 pg 2 in [15],eq 3.12 pg 9 in [13],eq 6 pg 2 in [12], eq 47 pg 7 in [17] and eq 21 pg 6 in [16])

$$E_{\mu\nu} \equiv {}^{(5)}C^{\alpha}{}_{\beta\rho\sigma} n_{\alpha} n^{\rho} q_{\mu}{}^{\beta} q_{\nu}{}^{\sigma} = {}^{(5)}C_{\mu\nu}. \tag{21}$$

Note that $5D$ Riemann and Ricci Tensors or Weyl Tensors appears also in the $5D$ to $4D$ Dimensional Reduction according to Basini-Capozziello-Overduin-Wesson already presented as eqs 4 to 6 in this section.Then making equal the Spacetime Ansatzs described in this section by eq 4 and eqs 13 and 14 the same Tidal Charge as a function of the $5D$ Ricci and Weyl Tensors will appear in both formalisms because regardless and independently of the mathematics being used this is a Geometrical Property of the Fifth Dimensional Nature of the Universe.Note that the Spacetime Ansatz defined by eq 1 pg 3 in [16] and eq 1 pg 3 in [17] have the mathematical structure of the Basini-Capozziello-Overduin-Wesson Ansatz and even so these will also generate the $4D$ "Eletrical" Weyl Tensor according to eq 47 pg 7 in [17] and pg 6 eq 21 in [16]

In Section 3 we will analyze how the $5D$ generates the rest-masses and the electric charges seen in $4D$ using the Hamilton-Jacobi Equation according to the formalism of Ponce De Leon.Masses and Charges are Geometrical Effects of a Hidden Fifth Dimension[41][42].Again we will avoid Confinement and Compactification Mechanisms[43] and we will adopt the Ponce de Leon point of view[44] of Space-Time-Matter theory where matter in $4D$ is purely geometric in nature and a Large $5D$ Extra Dimension is needed to get a consistent description of the properties of matter observed in $4D$.According to Ponce De Leon the mathematical support for Space-Time-Matter theory is given by the theorem of Campbell-Magaard.[45] All the matter fields seen in $4D$ are generated by a geometrical effect due to the presence of the $5D$.[46].The variation of the rest masses and electric charges of the particles seen in $4D$ is an indirect observation of the existence of the $5D$ and also according to Ponce De Leon these variations of rest masses and electric charges can be regarded as new physical phenomena unambiguously associated with the experimental existence of Extra Dimensions and according to Ponce De Leon this can provide a wealth of new physics.[47].We adopt here the $5D$ General Relativity Ansatz given by Ponce De Leon according to the following equation(eq 12 and 14 pg 4 in [3],eq 5 pg 5 in [18] without Conformal Factors $\Omega(y) = 1$)

$$dS^2 = g_{\alpha\beta}(x^\rho, y)dx^\alpha dx^\beta - \Phi^2(x^\rho, y)dy^2 \tag{22}$$

Note that this Ansatz is exactly the same presented by Basini-Capozziello and can be made equal to the one presented by Shiromizu-Maeda-Sasaki making the term dS^2 equal in all of them. The variation of the rest-mass due to the presence of the $5D$ Extra Dimension is given by([3] eq 20,eq 13 pg 6 in [18] without Conformal Factors $\Omega(y) = 1$, [7] eq 1 and eq 21 pg 5 in [4]):

[41] see top of pg 2 in [3]

[42] note that Ponce De Leon also points out the fat that $11D$ Supergravity and $10D$ Superstrings also evolved from the Klein Compactification Mechanism

[43] see pg 2 in [4].Ponce De Leon argues that the Cylindrical Condition is not needed and also argues that we may live in a Universe of Large Extra Dimensions,so the Compactification Mechanism is not needed too.see also pg 2 in [18].Large Extra Dimensions are introduced in BraneWorld and STM theories with different motivations.we keep the point of view of STM

[44] see pg 2 in [3]

[45] see pg 2 in [3]

[46] see pg 2 in [4] and pg 2 in [18]

[47] see again pg 2 in [3],pg 2 in [4]

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2 (\frac{dy}{ds})^2}} \tag{23}$$

providing the following form for eq 20 pg 5 in [4][48]

$$u^4 = \frac{dx^4}{ds} = -\Phi \left(\frac{dy}{ds} \right). \tag{24}$$

The variation of the electric charge due to the presence of the $5D$ Extra Dimension is given by(eq 19 pg 5 in [4])[49]:

$$q = \frac{M_{(5)} \Phi u^4}{\sqrt{1 - (u^4)^2}} = \pm \frac{M_5 \Phi^2 \frac{dy}{ds}}{\sqrt{1 - \Phi^2 (\frac{dy}{ds})^2}} \tag{25}$$

The equation of Hamilton-Jacobi for the Action S defined by $S = S(x^\mu, y)$ is given by the following expression(eq 11 pg 6 in [18] without Conformal Factors $\Omega(y) = 1$)[50]:

$$g^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} \right) \left(\frac{\partial S}{\partial x^\nu} \right) - \frac{1}{\Phi^2} \left(\frac{\partial S}{\partial y} \right)^2 = M_{(5)}^2 \dashrightarrow g^{\mu\mu} \left(\frac{\partial S}{\partial x^\mu} \right) \left(\frac{\partial S}{\partial x^\mu} \right) - \frac{1}{\Phi^2} \left(\frac{\partial S}{\partial y} \right)^2 = M_{(5)}^2 \tag{26}$$

The rest-mass m_0 seen in $4D$ is given by(eq 12 pg 6 in [18]):

$$g^{\mu\nu} \left(\frac{\partial S}{\partial x^\mu} \right) \left(\frac{\partial S}{\partial x^\nu} \right) = m_0^2 \dashrightarrow g^{\mu\mu} \left(\frac{\partial S}{\partial x^\mu} \right) \left(\frac{\partial S}{\partial x^\mu} \right) = m_0^2 \tag{27}$$

Note that we can write the Hamilton-Jacobi equation as follows:

$$m_0^2 - \frac{1}{\Phi^2} \left(\frac{\partial S}{\partial y} \right)^2 = M_{(5)}^2. \tag{28}$$

$$m_0^2 = M_{(5)}^2 + \frac{1}{\Phi^2} \left(\frac{\partial S}{\partial y} \right)^2. \tag{29}$$

$$m_0 = \sqrt{M_{(5)}^2 + \frac{1}{\Phi^2} \left(\frac{\partial S}{\partial y} \right)^2}. \tag{30}$$

Look now to this form of the Hamilton-Jacobi equation(eq 17 pg 5 in [4])[51]:

$$m_0^2 - \frac{q^2}{\Phi^2} = M_{(5)}^2, \tag{31}$$

$$m_0^2 = M_{(5)}^2 + \frac{q^2}{\Phi^2}, \tag{32}$$

[48] without Electromagnetic Potential and a Spacelike Metric see bottom of pg 4 before section 2.2 in [4]

[49] the reader would ask why the + sign in an equation that originally have the − sign?.see eqs 55,58 and 60 in [3] and eq 18 in [4]

[50] Spacelike Metric and diagonalized metrics for the right terms below

[51] Spacelike Metric also

$$m_0 = \sqrt{M_{(5)}^2 + \frac{q^2}{\Phi^2}}, \tag{33}$$

From the equations above it can be seem that the rest-mass m_0 in $4D$ is obtained from partial derivatives of the $5D$ Action $S = S(x^\mu, y)$ with respect to the $4D$ Spacetime Coordinates while the electric charge q is obtained from the same Action but with partial derivatives related to the Extra Coordinate(see eq 18 pg 5 in [4])(see also eqs 55,58 and 60 in [3]).This is exactly the purpose of the Hamilton-Jacobi Equation:to extract masses and charges from the $5D$ Extra Dimensional Formalism

$$q = \pm\frac{\partial S}{\partial y} \dashrightarrow m_0 = \sqrt{g^{\mu\mu}}\frac{\partial S}{\partial x^\mu} \tag{34}$$

The equations of the rest mass m_0 and electric charge q written in function of the $5D$ Extra Dimension shows how masses and charges are generated by the Higher Dimensional Nature of the Universe.

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{35}$$

$$q = \pm\frac{M_5\Phi^2\frac{dy}{ds}}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} = \pm m_0\Phi^2\frac{dy}{ds} \tag{36}$$

Examining the Table of Elementary Particles given below[52]

Particle	spin ($\hbar$)	B	L	T	T_3	S	C	B*	charge (e)	m_0 (MeV)	antipart.
u	1/2	1/3	0	1/2	1/2	0	0	0	+2/3	5	$\bar{u}$
d	1/2	1/3	0	1/2	−1/2	0	0	0	−1/3	9	$\bar{d}$
s	1/2	1/3	0	0	0	−1	0	0	−1/3	175	$\bar{s}$
c	1/2	1/3	0	0	0	0	1	0	+2/3	1350	$\bar{c}$
b	1/2	1/3	0	0	0	0	0	−1	−1/3	4500	$\bar{b}$
t	1/2	1/3	0	0	0	0	0	0	+2/3	173000	$\bar{t}$
e^-	1/2	0	1	0	0	0	0	0	−1	0.511	e^+
μ^-	1/2	0	1	0	0	0	0	0	−1	105.658	μ^+
τ^-	1/2	0	1	0	0	0	0	0	−1	1777.1	τ^+
ν_e	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_e$
ν_μ	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_\mu$
ν_τ	1/2	0	1	0	0	0	0	0	0	0(?)	$\bar{\nu}_\tau$
γ	1	0	0	0	0	0	0	0	0	0	γ
gluon	1	0	0	0	0	0	0	0	0	0	$\overline{\text{gluon}}$
W^+	1	0	0	0	0	0	0	0	+1	80220	W^-
Z	1	0	0	0	0	0	0	0	0	91187	Z
graviton	2	0	0	0	0	0	0	0	0	0	graviton

Examine first the group of the Quarks *udscbt*.All these particles possesses a defined rest-mass m_0 seen in $4D$ and a defined electric charge q.Suppose that in $5D$ all these Quarks are the same Quark with the same $5D$ rest-mass M_5 and the Dimensional Reduction from $5D$ to $4D$ or the Hamilton-Jacobi equation

[52]extracted from the Formulary Of Physics by J.C.A. Wevers available on Internet

"projects" these "different" rest-masses m_0 seen in $4D$ as "images" of the same $5D$ rest-mass M_5 being the differences between each Quark generated by the respective Spacetime Coupling term $\sqrt{1-\Phi^2(\frac{dy}{ds})^2}$ assigned for each Quark.Hence for example the Quark u and the Quark t have the same $5D$ rest-mass $M_5 = 1$ but different Geometries from the $5D$ to $4D$ Dimensional Reduction generates two different Spacetime Couplings for each Quark $\sqrt{1-\Phi_u^2(\frac{dy[u]}{ds})^2}$ and $\sqrt{1-\Phi_t^2(\frac{dy[t]}{ds})^2}$ "projecting" in $4D$ two different rest-masses $m_0 = 5$ for the Quark u and $m_0 = 173000$ for the Quark t as "images" of the same $5D$ Quark mass $M_5 = 1$.This point of view could perhaps leads to a major revolution in Particle Physics[53]

Examining now the electric charges q or better the relation $\frac{q}{m_0}$

$$\frac{q}{m_0} = \pm\frac{1}{\sqrt{g^{\mu\mu}}}\frac{\partial x^\mu}{\partial y} = \pm\Phi^2\frac{dy}{ds} \tag{37}$$

Electric charges q are functions of the $4D$ rest-masses m_0.Note in the given Table of Elementary Particles that all the particles that possesses charges q also possesses masses m_0.There are no particles with electric charge q and rest mass $m_0 = 0$.This is one of the most important consequences of the Hamilton-Jacobi equation in the Ponce De Leon Formalism.For our Quarks u and t different Mass-to-Charge Couplings $\pm\Phi^2\frac{dy}{ds}$ one for the Quark u $\pm\Phi_u^2\frac{dy[u]}{ds}$ and another for the Quark t $\pm\Phi_t^2\frac{dy[t]}{ds}$ associated to the rest masses $m_0(u) = 5$ and $m_0(t) = 173000$ will generate the same electric charge $q(u) = +\frac{2}{3}$ and $q(t) = +\frac{2}{3}$ for both Quarks.Their respective antiparticles $\bar{\mathrm{u}}$ and $\bar{\mathrm{t}}$ possesses the same rest-masses but electric charges of different signs $q(\bar{\mathrm{u}}) = -\frac{2}{3}$ and $q(\bar{\mathrm{t}}) = -\frac{2}{3}$.The explanation why antiparticles have the same rest-masses of particles but different signs for electric charges will be given in Section 3 but we can say right now that the difference is being generated by the Mass-to-Charge Couplings $\pm\Phi^2\frac{dy}{ds}$.

In Section 4 we study the structure of a Higher Dimensional BraneWorld Star with a Tidal Charge defined by the Metric of Dadhich,Maartens,Papadopolous and Rezania(eq 11 pg 2 and eqs 12 and 14 pg 3 in [12]) with the analysis made by Germani-Maartens according to ([27]) and Kotrlova-Stuchlik-Torok according to ([25])in order to determine the physical nature of the Tidal Charge and to obtain numerical values needed to the calculations of Section 5.We will see that when the radius R of a given BraneWorld Star is greater than its own Schwarzschild Radius r_S the sign of the Tidal Charge is negative because(and in agreement with Kotrlova-Stuchlik-Torok) the great majority of the BraneWorld Stars possesses negative values for the Tidal Charge(see pg 2 of [25]) but when in a star collapse the Schwarzschild Radius is reached the value of the Tidal Charge reduces to zero and becomes positive as far as the collapse goes by with a radius R smaller than the Schwarzschild Radius.[54]Hence we can say that the Schwarzschild Radius inverts the sign of the Tidal Charge.

. The Spacetime Metric for an Higher Dimensional BraneWorld Star with a Tidal Charge is given by the following equation:(see eq 7 pg 5 in [25],eq 7 pg 5 in [10],eqs 1,2 pg 2 in [22],eqs 1,2 pg 2 in [21],eq 33 pg 4 in [27]) We adopted here the definitions of [25]

$$ds^2 = A(r)\mathrm{c}^2dt^2 - A^{-1}(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2, \tag{38}$$

$$A(r) = 1 - \frac{2r_{\mathrm{G}}}{r} + \frac{Q}{r^2}; \qquad r_{\mathrm{G}} = \frac{\mathrm{G}M}{\mathrm{c}^2}, \tag{39}$$

The equation above resembles the Reissner-Nordstrom Metric however the "Charge" defined by Q is

[53]see pg 60 in [6] and bottom of pg 2 and pg 3 in [7]

[54]we do not consider the Tidal Charge of a Singularity here

not an Electromagnetic Field[55] but instead it represents the projection in $4D$ Spacetime of the non-local $5D$ Bulk Weyl Tensor(see pg 5 before eq 6 and after eq 8 in [25],pg 4 after eq 3 in [10])

The parameter r_G defined for the Sun is given by:(see eq 7.57 pg 187 in [9])

$$r_{G\odot} = \frac{GM_\odot}{c^2} = 1.48 \times 10^3 \text{ m} \tag{40}$$

The Schwarzschild Radius r_S is defined by:(see end of pg 191 and top of 192 in [9])

$$r_S = \frac{2GM}{c^2} \tag{41}$$

For the Sun the Schwarzschild Radius is given by:

$$r_{S\odot} = \frac{2GM_\odot}{c^2} = 2.96 \times 10^3 \text{ m} \tag{42}$$

We assume here uniform(or constant) density stars(see pg 7 between eqs 19,20 in [10],pg 5 between eqs 6,7 in [25],pg 3 Section III and pg 4 between eqs 29,30 in [27])

The Tidal Charge can be written in function of the star mass M[56],radius R,density ρ and tension[57] λ. It can also be written in function of the Schwarzschild Radius r_S and parameter r_G

The equations for the Tidal Charge Q are given by:(see eq 9 combined with eq 8 pg 5 in [25],pg 7 between eqs 19 and 20 in [10],eq 35 pg 4 in [27])

$$Q = -\frac{3\mathrm{G}M}{\mathrm{c}^2} R\left(\frac{\rho}{\lambda}\right), \curvearrowright Q = -3r_G R\left(\frac{\rho}{\lambda}\right), \tag{43}$$

While the mass M,radius R and density ρ of a star are known we must calculate the tension λ.The equations are given by:(see eq 13 combined with eq 8 pg 5 in [25],eq 21 pg 7 in [10],eq 29 pg 4 in [27])[58]

$$\lambda \geq \left(\frac{\mathrm{G}M/\mathrm{c}^2}{R - 2\mathrm{G}M/\mathrm{c}^2}\right)\rho, \curvearrowright \lambda \geq \left(\frac{r_G}{R - r_S}\right)\rho, \tag{44}$$

$$\lambda = n\left(\frac{r_G}{R - r_S}\right)\rho, \curvearrowright n \geq 1 \tag{45}$$

$$\frac{\rho}{\lambda} = \frac{1}{n\left(\frac{r_G}{R - r_S}\right)} = \frac{1}{n}\frac{R - r_S}{r_G} = N\frac{R - r_S}{r_G} \curvearrowright N = \frac{1}{n} \curvearrowright N \leq 1 \tag{46}$$

From the expressions above we can now obtain an expression to calculate the Tidal Charge Q in function of the star radius R,Schwarzschild Radius r_S and parameter r_G

[55]Germani-Maartens outlines the fact that there is no EM Field,no Electrical Charge in this model:Their explanation is the fact that non-local Bulk Weyl Tensor from the $5D$ Spacetime leads to an Energy-Momentum Tensor projected in the $4D$ Spacetime that posesses the same mathematical structure of an Electric Field but without any Electrical Charge being present:see pg 4 in [27]

[56]Germany-Maartens outlines the fact that the Tidal Charge is generated by Weyl stresses and these are projections of the $5D$ Spacetime Bulk Weyl Tensor that responds non-locally to the gravitational fields of the $4D$ Spacetime and "backreacts".Weyl stresses will occur both in the interior and the exterior of the star.However only the exterior is considered.see pg 3 before Section III in [27]

[57]Germany-Maartens outines that we recover ordinary $4D$ General Relativity when $\frac{1}{\lambda} = 0$.In this case the Tidal Charge vanishes and we recover the ordinary Schwarzschild Solution.This tension λ is generated by the Bulk Weyl stresses.see pg 3 after eq 28 in [27]

[58]note that in the last two references the uniform density of the star is again outlined

$$Q = -3r_G R \left(\frac{\rho}{\lambda}\right) = -3r_G R \left(\frac{1}{n}\frac{R - r_S}{r_G}\right) = -3r_G R \left(N\frac{R - r_S}{r_G}\right), \tag{47}$$

The parameter r_G can be dropped from the equation leaving ourselves with only the star radius R and the Schwarzschild Radius r_S

The final expression for the Tidal Charge Q will then be:

$$Q = -3r_G R \left(N\frac{R - r_S}{r_G}\right), \curvearrowright Q = -3NR\left(R - r_S\right), \tag{48}$$

Note that for a BraneWorld Star with a radius R greater than the Schwarzschild Radius r_S the sign of the Tidal Charge Q is always negative in agreement with Kotrlova-Stuchlik-Torok (pg 2 of [25]).When in a collapse the star radius R contracts and when it reaches the Schwarzschild Radius r_S the Tidal Charge vanishes and becomes positive while the collapse proceed with a radius R smaller than the Schwarzschild radius r_S[59].We must bear in mind the following important thing:

- -The Schwarzschild Radius r_S is the point of inversion of the Tidal Charge Q sign in a gravitational collapse.

$$Q = -3NR\left(R - r_S\right) \curvearrowright R - r_S > 0 \curvearrowright R > r_S \curvearrowright Q < 0 \dashrightarrow Negative \tag{49}$$

$$Q = -3NR\left(R - r_S\right) \curvearrowright R - r_S = 0 \curvearrowright R = r_S \curvearrowright Q = 0 \dashrightarrow Zero \tag{50}$$

$$Q = -3NR\left(R - r_S\right) \curvearrowright R - r_S < 0 \curvearrowright R < r_S \curvearrowright Q > 0 \dashrightarrow Positive \tag{51}$$

In Section 4 we will exhaustively compute two possible numericsl values for the Tidal Charge of the Sun needed for the calculations of Section 5 but we present here the values found:

$$Q_\odot = -3,206219108 \times 10^{16} m^2 \tag{52}$$

$$Q_\odot = -5,06390231 \times 10^{18} m^2 \tag{53}$$

In Section 5 we analize the differences between the Gravitational Bending of Light in the Schwarzschild, Reissner-Nordstron and BraneWorld Black Hole Metrics following the approaches of Briet-Hobill according to ([11]),Kar-Sinha according to ([5]),Gergely-Darazs-Keresztes-Dwornik according to ([21]) and ([22]),Aliev-Talazan according to ([24]) and Bohmer-Harko-Lobo according to ([10]).
In this Section we consider the Sun a pointlike star of mass $M_\odot$ and Tidal Charge $Q_\odot$ and r is the distance between the photon beam and the Sun(our r is the r_0 and the Sun replaces the Black Hole as the point connected to the photon beam by r_0 both in fig 1 pg 8 of [11]) The purpose of this Section is to demonstrate that the Tidal Charge Q makes the BraneWorld Black Hole Metric the best candidate to demonstrate the Higher Dimensional Nature of the Universe because the terms $\frac{G}{c^2}$ and $\frac{G^2}{C^4}$ that will affect both the Schwarzschild and Reissner-Nordrstom Metrics do not affect the BraneWorld Black Hole Metric.Here is the reason why we will compare the Gravitational Bending of Light obtained from the Metric of Dadhich,Maartens,Papadopolous and Rezania(see eq 7 pg 5 in [25],eq 7 pg 5 in [10],eqs 1,2 pg 2 in [22],eqs

[59]again we do not consider the Singularity here

1,2 pg 2 in [21],eq 33 pg 4 in [27]) against both the Schwarzschild Metric(see eq 7.29 pg 177 in [9],eq $II.11$ pg 5 in [19]) and Reissner-Nordstrom Metric(see eqs 7.110,7.111 pg 209 in [9],eq $V.7$ pg 9 in [19]).
The absence of the terms $\frac{G}{c^2} \simeq 10^{-27}$ and $\frac{G}{c^4} \simeq 10^{-43}$ will make the Extra Dimension effects more noticeable in the BraneWorld Meric than in the Schwarzschild or Reissner-Nordstrom Metrics where these terms appear to reduce the effects of the Extra Dimension in a way that it is almost impossible to spot them.
We will analyze the following three situations for Gravitational Bending of Light in both Schwarzschild, Reissner-Nordstrom and BraneWorld Black Hole Metrics:

- 1)-photon beam passing the Sun[60] at a distance $r = 150.000km$ $r = 1,5 \times 10^8 m$
- 2)-photon beam passing the Sun[61] at a distance $r = 1.000.000km$ $r = 1 \times 10^9 m$
- 3)-photon beam passing the Sun[62] at a distance $r = 10.000.000km$[63] $r = 1 \times 10^{10} m$

Taking in mind that a $5D$ General Relativity Ansatz can be written as:(eq 12 and 14 pg 4 in [3],eq 5 pg 5 in [18] without Conformal Factors $\Omega(y) = 1$)

$$dS^2 = g_{\alpha\beta}(x^\rho, y)dx^\alpha dx^\beta - \Phi^2(x^\rho, y)dy^2 = ds^2 - \Phi^2(x^\rho, y)dy^2 \quad (54)$$

or as:
(see eq 12 pg 3 in [14] or eq 2 pg 2 in [15](see also eq 3.2 pg 8 in [13])

$$dS^2 = d\chi^2 + q_{\mu\nu}dx^\mu dx^\nu \,. \quad (55)$$

$$dS^2 = (n_\mu n_\nu + q_{\mu\nu})\, dx^\mu dx^\nu = (n_{\mu\nu} + q_{\mu\nu})\, dx^\mu dx^\nu = d\chi^2 + q_{\mu\nu}dx^\mu dx^\nu \; = d\chi^2 + ds^2,, \quad (56)$$

$$^{(5)}g_{\mu\nu} = q_{\mu\nu} + n_{\mu\nu} \quad (57)$$

(see pg 2 in [14] or pg 2 in [15])(see also pg 8 in [13])

$$n_\mu dx^\mu = d\chi \quad (58)$$

We consider in the forthcoming analysis only the $4D$ Spacetime part of the Ansatz ds^2 which means to say the metric of the Brane with however the induced effects(e.g Tidal Charge Q)from the $5D$ Spacetime:the Bulk.We also consider only first order terms(second order terms will be analyzed in Section 5).

The following parameter ϑ is very useful when computing Gravitational Bendings of Light in $4D$ or in $5D$(see eq 7.57 pg 187 in [9])

$$\vartheta = \frac{4GM}{c^2} \quad (59)$$

$$\vartheta_\odot = \frac{4GM_\odot}{c^2} = 5.92 \times 10^3 \text{ m} \quad (60)$$

[60] pointlike Sun
[61] pointlike Sun
[62] pointlike Sun
[63] this is the limit for the capability of detection of Gravitational Bending of Light angles by the European Space Agency Satellite GAIA see the shift of 5×10^{-7} pg 4 in [28]

The Schwarzschild Metric given below(see eq 7.29 pg 177 in [9],eq $II.11$ pg 5 in [19])

$$ds^2 = \left[1 - \frac{2GM}{c^2 r}\right] c^2 dt^2 - \left[1 - \frac{2GM}{c^2 r}\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{61}$$

possesses a coefficient of Gravitational Bending of Light given by:(eq 20 pg 9 in [11])

$$\delta\phi \simeq \frac{4GM}{c^2 r}. \tag{62}$$

In this case $\delta\phi = \frac{\vartheta}{r}$.Note that in the Schwarzschild case the term $\frac{G}{c^2} \simeq 10^{-27}$ will contribute to reduce our capability to measure the Gravitational Bending of Light making difficult the detection of an Extra Dimension.[64]

Computing the Gravitational Bending of Light in the Schwarzschild Metric for the three cases of photon beams passing the Sun we should expect for:

- 1)$r = 1,5 \times 10^8 m$

$$\delta\phi \simeq \frac{4GM_\odot}{c^2 r} = \frac{5,92 \times 10^3 m}{1,5 \times 10^8 m} = 3,9466 \times 10^{-5}. \tag{63}$$

- 2)$r = 1 \times 10^9 m$

$$\delta\phi \simeq \frac{4GM_\odot}{c^2 r} = \frac{5,92 \times 10^3 m}{1 \times 10^9 m} = 5,92 \times 10^{-6}. \tag{64}$$

- 3)$r = 1 \times 10^{10} m$

$$\delta\phi \simeq \frac{4GM_\odot}{c^2 r} = \frac{5,92 \times 10^3 m}{1 \times 10^{10} m} = 5,92 \times 10^{-7}. \tag{65}$$

These values are extremely small however it is possible to detect all of them.The last value is our limit of detection capability.[65]

The Reissner-Nordstrom Metric given below:(see eqs 7.110,7.111 pg 209 in [9],eq $V.7$ pg 9 in [19])

$$ds^2 = \left[1 - \frac{2GM}{c^2 r} + \frac{Gq^2}{c^4 r^2}\right] c^2 dt^2 - \left[1 - \frac{2GM}{c^2 r} + \frac{Gq^2}{c^4 r^2}\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{66}$$

possesses a coefficient of Gravitational Bending of Light given by:(eq 27 pg 10 in [11])

$$\delta\phi \simeq \frac{4GM}{c^2 r} - \frac{3}{4}\frac{Gq^2}{c^4 r^2}\pi. \tag{67}$$

As in the Schwarzschild Metric the term $\frac{G}{c^2} \simeq 10^{-27}$ also appears but now the case is even worst due to the term $\frac{G}{c^4} \simeq 10^{-43}$ associated to the Electric Charge making the detection of this term practically impossible.[66]

[64]some authors work with the units $c = G = 1$ but in this case we need to work with the real values

[65]consider again the European Space Agency Satellite GAIA see the shift of 5×10^{-7} pg 4 in [28]

[66]see pg 2 of [11] for the real possibility of electrically charged Black Holes.In our case we consider the Sun possessing a small Electric Charge q however the value of 10^{-43} is 10^{36} times smaller than our maximum capability of detection of 5×10^{-7} pg 4 in [28]

The values of the Gravitational Bending of Light in the Reissner-Nordstrom Metric are very similar to the ones obtained for the Schwarzschild Metric because the first term appears in both Metrics however a very small difference lies in the term $\frac{G}{c^4} \simeq 10^{-43}$.

We will concentrate ourselves in the following term:

$$\epsilon\phi = -\frac{3}{4}\frac{G}{c^4 r^2}\pi. \tag{68}$$

with:

$$G = 6,67 \times 10^{-11} Newton \times m^2/kg^2 \tag{69}$$

$$c^4 = 8,1 \times 10^{33} m^4/s^4 \tag{70}$$

$$\pi = 3,1415926536 \tag{71}$$

giving:

$$\epsilon\phi = -\frac{3}{4}\frac{G}{c^4 r^2}\pi. \curvearrowright \epsilon\phi = -2,3561944902\frac{G}{c^4 r^2} \curvearrowright \epsilon\phi = -2,3561944902 \times \frac{8,234567901 \times 10^{-45} m^2}{r^2} \tag{72}$$

$$\epsilon\phi = -\frac{3}{4}\frac{G}{c^4 r^2}\pi. \curvearrowright \epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{r^2} \tag{73}$$

Note that the case now becomes even more worst than in the Schwarzschild Metric due to the division by r^2 and not by r

Computing the Gravitational Bending of Light in the Reissner-Nordstrom Metric term $\epsilon\phi$ for the three cases of photon beams passing the Sun we should expect for:[67]

- 1)$r = 1,5 \times 10^8 m$

$$\epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{r^2} \curvearrowright \epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{2,25 \times 10^{16} m^2} \tag{74}$$

$$\epsilon\phi = -8,62321934 \times 10^{-61} \tag{75}$$

- 2)$r = 1 \times 10^9 m$

$$\epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{r^2} \curvearrowright \epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{1 \times 10^{18} m^2} \tag{76}$$

$$\epsilon\phi = -1,94022435180 \times 10^{-62} \tag{77}$$

- 3)$r = 1 \times 10^{10} m$

[67] the value of 10^{-61} is 10^{54} times smaller than our maximum capability of detection of 5×10^{-7} pg 4 in [28] so we dont even need to worry about the other values

$$\epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{r^2} \curvearrowright \epsilon\phi = -\frac{1,94022435180 \times 10^{-44} m^2}{1 \times 10^{20} m^2} \tag{78}$$

$$\epsilon\phi = -1,94022435180 \times 10^{-64} \tag{79}$$

All these values are too small to be detected independent of any residual Electric Charge so we can say that the Reissner-Nordstrom Metric have a Gravitational Bending of Light equal to the one obtained for the Schwarzschild Metric

The BraneWorld Black Hole Metric given below:(see eq 7 pg 5 in [25],eq 7 pg 5 in [10],eqs 1,2 pg 2 in [22],eqs 1,2 pg 2 in [21],eq 33 pg 4 in [27])

$$ds^2 = [1 - \frac{2GM}{c^2 r} + \frac{Q}{r^2}]dt^2 - [1 - \frac{2GM}{c^2 r} + \frac{Q}{r^2}]^{-1} dr^2 - r^2 d\Omega^2 \tag{80}$$

possesses some coefficients of Gravitational Bending of Light given by the formulas below:

- 1)Kar-Sinha Equation:(see eq 7 pg 4 in [5])

$$\delta\phi \simeq \frac{4GM}{c^2 r} + \frac{3\pi \mid Q \mid}{4r^2} \tag{81}$$

- 2)Gergely-Darazs-Keresztes-Dwornik Equation(see eq 25 pg 7 in [21],eq 24 pg 6 in [22])

$$\delta\phi \simeq \frac{4GM}{c^2 r} - \frac{3\pi Q}{4r^2} \tag{82}$$

- 3)Aliev-Talazan Equation [68](see eq 30 pg 11 in [24])

$$\delta\phi \simeq \frac{4GM}{c^2 r} - \frac{3\pi Q}{4r^2} \tag{83}$$

[68]if the rotating coefficient becomes zero.

- 4)Bohmer-Harko-Lobo Equation:(see eq 27 pg 8 in [10])[69]

$$\delta\varphi_{LD} = \delta\varphi_{LD}^{(GR)}\left(1 - \frac{2Q}{r^2}\right), \tag{84}$$

$$\delta\varphi_{LD}^{(GR)} = 4GM/c^2 r \tag{85}$$

As in the Schwarzschild Metric the term $\frac{G}{c^2} \simeq 10^{-27}$ also appears but now the case is even better because the term $\frac{G}{c^4} \simeq 10^{-43}$ do not appear. Hence the first term in the BraneWorld Metric match exactly the first term in both Schwarzschild or Reissner-Nordstrom Metrics but a significant difference appears in the second term and significant enough to be detected experimentally by Artificial Satellites(more on this in Section 6). We will examine first the formulas of the items 1 to 3[70] given above.Although item 1 uses a formula with a modulus of the Tidal Charge Q while items 2 to 3 uses a negative term for the Tidal Charge Q all these formulas are actually equivalent because the sign of the Tidal Charge Q is always negative in agreement with Kotrlova-Stuchlik-Torok (pg 2 of [25]) We will consider the term given below:

$$\psi = \frac{3\pi Q}{4r^2} \tag{86}$$

With the following value of the Tidal Charge Q obtained for the Sun

$$Q_\odot = -3,206219108 \times 10^{16} m^2 \tag{87}$$

We should expect for:

$$\psi = \frac{3\pi Q}{4r^2} \curvearrowright \psi = \frac{-3,206219108 \times 2,3561944902 \times 10^{16} m^2}{r^2} \tag{88}$$

$$\psi = \frac{3\pi Q}{4r^2} \curvearrowright \psi = \frac{-7,55447579664 \times 10^{16} m^2}{r^2} \tag{89}$$

Computing the Gravitational Bending of Light in the BraneWorld Metric term ψ for the three cases of photon beams passing the Sun we should expect for:[71]

- 1)$r = 1,5 \times 10^8 m$

$$\psi = \frac{3\pi Q}{4r^2} \curvearrowright \psi = \frac{-7,55447579664 \times 10^{16} m^2}{2,25 \times 10^{16} m^2} \tag{90}$$

$$\psi = \frac{3\pi Q}{4r^2} \curvearrowright \psi = -3,357447985 \tag{91}$$

- 2)$r = 1 \times 10^9 m$

[69] this method will be analyzed lately in Section 5

[70] details about the reason why these formulas are different will be given in Section 5

[71] the value of 10^{-4} is nearly 10^3 times greater than our maximum capability of detection of 5×10^{-7} from the European Space Agency Satellite GAIA see pg 4 in [28]

$$\psi = \frac{3\pi Q}{4r^2} \curvearrowright \psi = \frac{-7,55447579664 \times 10^{16} m^2}{1 \times 10^{18} m^2} \tag{92}$$

$$\psi = -7,55447579664 \times 10^{-2} \tag{93}$$

- $3) r = 1 \times 10^{10} m$

$$\psi = \frac{3\pi Q}{4r^2} \curvearrowright \psi = \frac{-7,55447579664 \times 10^{16} m^2}{1 \times 10^{20} m^2} \tag{94}$$

$$\psi = -7,55447579664 \times 10^{-4} \tag{95}$$

These values obtained dividing the Sun Tidal Charge Q by the square of the distance r in this case r^2 are better results than the ones obtained with the Metrics of Schwarzschild or Reissner-Nordstrom making the BraneWorld Black Hole Metric of Dadhich,Maartens,Papadopolous and Rezania the "emergent winner" and best candidate to prove the Higher Dimensionality of the Universe by Gravitational Bending of Light measured from Artificial Satellites because these values are in the range of our detection capability.

Section 6 is the most important section and as a matter of fact the main purpose of this work.It describes how an experiment using Artificial Satellites(eg European Space Agency Satellite GAIA) could be used to prove the fact that we live in a Universe of Higher Dimensional Nature by measuring again the Gravitational Bending of Light around the Sun but this time not from a faint Beam of Light from a distant star with a low power of resolution(eg the famous Sun Eclipse of 1919) but from a well defined and with a higher power of resolution Laser Beam generated by Artificial Satellites with higher precision equipment.We use the idea developed in [7] pgs 20 to 23 however adapted to the European Space Agency Satellite GAIA scheduled to be launched in 2012.GAIA is able to measure a Gravitational Bending of Light of 5×10^{-7} (see pg 4 in [28]) by far more than enough to detect the presence of the Extra Dimensions in our Universe.All the Metrics seen until here are useful mathematical tools but if we are going to use real conditions of experiment from a real star(eg our Sun) we cannot forget the fact that all the stars have movement of rotation and this means to say that all the stars have Angular Momentum and the coefficient of Angular Momentum affect the results of the measures of the Gravitational Bending of Light.Hence we must examine the rotational version of the BraneWorld Black Hole Metric of Dadhich,Maartens,Papadopolous and Rezania obtained by Aliev-Gümrükçüoğlu and Kotrlova-Stuchlik (eqs 2 to 4 pg 4 of [26],eqs 4 and 5 pg 5 in [24],eqs 34,35 pg 11 and pg 10 between eqs 26 and 27 in [29])[72] with the Gravitational Bending Of Light equations considering also the Angular Momentum obtained by Aliev-Talazan(see eq 30 pg 11 in [24]).Both equations are given below:[73][74]

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Sigma}\left[adt - (r^2+a^2)\,d\phi\right]^2 , \tag{96}$$

with the metric functions:

$$\Delta \;=\; r^2 + a^2 - 2mr + Q, , \qquad \Sigma = r^2 + a^2\cos^2\theta \tag{97}$$

[72] in this case the Metric is given with the signature $-,+,+,+$
[73] second order terms will be considered in Section 6
[74] we consider as in the introduction to Section 5 only the Brane induced part of the Metric ds^2

where $m = \frac{GM}{c^2}$ is the Mass(see pg 10 after eq $VI.1$ in [19]), a is the rotation parameter, or the Angular Momentum per Unit Mass, $a = J/M$, and Q is the Tidal Charge of the Black Hole.

$$\delta\phi = \frac{4GM}{c^2 r}\left(1 - \frac{a}{r}\right) - \frac{\pi Q}{4r^2}\left(3 - \frac{4a}{r}\right) \tag{98}$$

As we can see the rotational version of the BraneWorld Metric is very similar to the non-rotating and as a matter of fact if the coefficient of rotation is extremely low or close to zero both will give approximately the same results.Hence we will determine the coefficient of rotation(eg Angular Momentum)for our Sun.

The Momentum of Inertia for our Sun $I_\odot$ is given by the well-known formula below:[75]

$$I_\odot = (2/5) * M_\odot * R_\odot^2 \tag{99}$$

with the Sun Mass $M_\odot$ and Radius $R_\odot$ being given by the following values:

$$M_\odot = 1,9891 \times 10^{30} kg \tag{100}$$

$$R_\odot \cong 1,3 \times 10^9 m \tag{101}$$

Hence we should expect for:

$$I_\odot = (2/5) * M_\odot * R_\odot^2 = \frac{2}{5} \times 1,9891 \times 10^{30} kg \times 1,69 \times 10^{18} m^2 \tag{102}$$

$$I_\odot = (2/5) * M_\odot * R_\odot^2 = 1,3446316 \times 10^{48} kg \times m^2 \tag{103}$$

The Angular Momentum of the Sun $L_\odot$ is given by the following formula:

$$L_\odot = I_\odot \omega_\odot \tag{104}$$

where $\omega_\odot$ is the Angular Velocity of the Sun and we know that the Sun rotates over itself every 26 days.Then we should expect for:

$$\omega_\odot = \frac{2\pi}{26 days} = 2.8 \times 10^{-6} rad/s \tag{105}$$

Then the total Angular Momentum of the Sun $L_\odot$ is :

$$L_\odot = I_\odot \omega_\odot = 1,3446316 \times 10^{48} kg \times m^2 \times 2.8 \times 10^{-6} rad/s = 3,76496848 \times 10^{42} kg \times m^2 \times rad/s \tag{106}$$

Hence the Angular Momentum per Unit Mass for the Sun , $a_\odot = J_\odot / M_\odot$ is:[76]

$$a_\odot = \frac{3,76496848 \times 10^{42} kg \times m^2 \times rad/s}{1,9891 \times 10^{30} kg} \tag{107}$$

$$a_\odot = 1,8928 \times 10^{12} \times m^2 \times rad/s \tag{108}$$

[75] we assume a Sun rotation over the z-axis
[76] $J_\odot = L_\odot$

Now we are ready to compute the Gravitational Bending of Light for the Sun as a BraneWorld Star with the Equation of Aliev-Talazan(see eq 30 pg 11 in [24])[77]

$$\delta\phi_{\odot} = \frac{4GM_{\odot}}{c^2 r}\left(1 - \frac{a_{\odot}}{r}\right) - \frac{\pi Q_{\odot}}{4r^2}\left(3 - \frac{4a_{\odot}}{r}\right) \tag{109}$$

using the value given below for the Tidal Charge Q obtained for the Sun:

$$Q_{\odot} = -3,206219108 \times 10^{16} m^2 \tag{110}$$

$$\delta\phi_{\odot} = \frac{4GM_{\odot}}{c^2 r} - \frac{4GM_{\odot}a_{\odot}}{c^2 r^2} - \frac{3\pi Q_{\odot}}{4r^2} + \frac{\pi Q_{\odot} a_{\odot}}{r^3} \tag{111}$$

Since we already computed in the previous Section the values of the first and third term we concentrate ourselves in the second and four terms thereby giving:

$$\mu\odot = \frac{4GM_{\odot}a_{\odot}}{c^2 r^2} = \frac{1,1205376 \times 10^{16} m^3 \times rad/s}{r^2} \tag{112}$$

$$\nu\odot = \frac{\pi Q_{\odot} a_{\odot}}{r^3} = \frac{-1,90654823 \times 10^{29} m^4 \times rad/s}{r^3} \tag{113}$$

The term $\mu\odot$ is "Schwarzschild like" and the term $\nu\odot$ is "BraneWorld like"

Computing the Gravitational Bending of Light for the terms $\mu\odot$ and $\nu\odot$ using the Rotating BraneWorld Metric with the three situations given in the previous Section we should expect for:

- 1)$r = 1,5 \times 10^8 m$

$$\mu\odot = \frac{4GM_{\odot}a_{\odot}}{c^2 r^2} = \frac{1,1205376 \times 10^{16} m^3 \times rad/s}{2,25 \times 10^{16} m^2} \tag{114}$$

$$\nu\odot = \frac{\pi Q_{\odot} a_{\odot}}{r^3} = \frac{-1,90654823 \times 10^{29} m^4 \times rad/s}{3,375 \times 10^{24} m^3} \tag{115}$$

$$\mu\odot = \frac{4GM_{\odot}a_{\odot}}{c^2 r^2} = 4,9801671 \times 10^{-1} m \times rad/s \tag{116}$$

$$\nu\odot = \frac{\pi Q_{\odot} a_{\odot}}{r^3} = -5,694031 \times 10^{4} m \times rad/s \tag{117}$$

- 2)$r = 1 \times 10^9 m$

$$\mu\odot = \frac{4GM_{\odot}a_{\odot}}{c^2 r^2} = \frac{1,1205376 \times 10^{16} m^3 \times rad/s}{1 \times 10^{18} m^2} \tag{118}$$

$$\nu\odot = \frac{\pi Q_{\odot} a_{\odot}}{r^3} = \frac{-1,90654823 \times 10^{29} m^4 \times rad/s}{1 \times 10^{27} m^3} \tag{119}$$

[77] again here we consider first order terms.second order terms will be examined in details in Section 6

$$\mu\odot = \frac{4GM_\odot a_\odot}{c^2r^2} = 1,1205376 \times 10^{-2} m \times rad/s \tag{120}$$

$$\nu\odot = \frac{\pi Q_\odot a_\odot}{r^3} = -1,90654823 \times 10^{2} m \times rad/s \tag{121}$$

- $3) r = 1 \times 10^{10} m$

$$\mu\odot = \frac{4GM_\odot a_\odot}{c^2r^2} = \frac{1,1205376 \times 10^{16} m^3 \times rad/s}{1 \times 10^{20} m^2} \tag{122}$$

$$\nu\odot = \frac{\pi Q_\odot a_\odot}{r^3} = \frac{-1,90654823 \times 10^{29} m^4 \times rad/s}{1 \times 10^{30} m^3} \tag{123}$$

$$\mu\odot = \frac{4GM_\odot a_\odot}{c^2r^2} = 1,1205376 \times 10^{-4} m \times rad/s \tag{124}$$

$$\nu\odot = \frac{\pi Q_\odot a_\odot}{r^3} = -1,90654823 \times 10^{-1} m \times rad/s \tag{125}$$

Even the third case above give an excellent result:The term $\mu\odot$ is about 10^3 times bigger than 10^{-7} and the term $\nu\odot$ is about 10^6 times bigger than 10^{-7}.Hence we can say that these values are well within the Gravitational Bending of Light detection capabilities of the European Space Agency Satellite GAIA which is about 5×10^{-7} (see pg 4 in [28]).
In [7] abstract and pgs 20 to 23 it was proposed the use of Artificial Satellites with Laser Beams to measure the Gravitational Bending of Light in order to detect the Extra Terms generated by the Higher Dimensional Spacetime.The idea was to send a Laser Beam to the neighborhoods of the Sun that will act as a Gravitational Lens.If the rotating BraneWorld Metric of Dadhich,Maartens,Papadopolous and Rezania obtained by Kotrlova-Stuchlik really describe a real Star then the Tidal Charge generated by the Weyl Tensor of the $5D$ Higher Dimensional Spacetime will appear and will be detected.The Laser Beam is more suitable for this task than the Light of a faint distant Star and the Satellites in Outer Space do not suffer the interferences due to the Earth Atmosphere.The European Space Agency Satellite GAIA scheduled to be launched in 2012 can detect the Extra Terms in the Gravitational Bending of Light and can proof that as a matter of fact we live in a Universe of Higher Dimensional Nature.

2 Dimensional Reduction from a $5D$ Spacetime to a $4D$ Spacetime in both Induced Kaluza-Klein and Randall-Sundrum Formalisms.

The idea of an unified theory describing all the fundamental interactions in physics under the same standard was one of the main issues of the XX Century physics starting from the early efforts of Einstein,Kaluza-Klein and many others(see pg 1 in [2])until the recent and more sophisticated approaches.

However and among the large number of ideas up to now proposed and classified as "unified theories" almost all of them resulted unsuccessful due to the following reasons presented below:(see pg 2 in [2])

- 1)-Technical difficulties connected to the lack of an unitary mathematical description of all interactions
- 2)-The huge number of all new parameters introduced "ad hoc" [78] to "build up" the unified theory
- 3)-The fact that most of them cannot be observed neither at laboratory nor at astrophysical conditions
- 4)-The very wide(and questionable since this is not testable) number of Extra Dimensions requested by some of these approaches

According to Basini-Capozziello(see pg 2 in [2]) due to the reasons described by the framework above it seems that the "Goal" of an unified theory is(and will still be for a while) an useful(and aesthetic) paradigm but(unfortunately) by far to be achieved if the trend is continuing to try to unify interactions(which means to say to make something simple) by adding and adding new ingredients:new particles,new Extra Dimensions and new parameters.

A more classical approach should be to consider the very essential physical quantities(eg masses and charges) and try to achieve unification with no need and without the introduction of "ad-hoc" new ingredients [79] avoiding what Isaac Newton described as "Hypotheses Non Fingo"

Still according to Basini-Capozziello (see pg 2 in [2],see pg 10 in [1])the $5D$ space is the minimum dimensional space scaling up really able to contain and explain all the physics laws and by Dimensional Reduction from $5D$ to $4D$ it gives rises to the physical quantities that characterizes the dynamics of ordinary particles such as mass,charge and spin.(see pg 2 in [2],see pg 10 in [1])

In order to get a better picture of what the $5D$ Extra Dimension really is(see pg 9 in [1]) we can say that the $5D$ Extra Dimension have a real physical meaning associated to the generation of the mass of ordinary particles seen in a $4D$ Spacetime.

We do not perceive $4D$ the time dimension as a spacelike dimension and the situation is analogue for the $5D$ Extra Dimension

According to the Campbell-Maagard theorem it is always possible to define a $4D$ Riemannian Manifold embedded inside a $5D$ Riemannian Manifold.The components of the spacetime metric tensor are defined

[78] see the word "ad-hoc" in pg 2 of [2] and see also the footnote 3 of pg 2 in [2]

[79] again see the word "ad-hoc" in pg 2 of [2] and see also the footnote 3 of pg 2 in [2]

by the 3+1 and Extra Coordinate and the $5D$ generates in the $4D$ spacetime the mass charges and spins of ordinary $4D$ particles.(see pg 13 in [2],see pg 14 in [1])

According to Overduin-Wesson all the physics is allowed to depend on the $3+1$ and Extra Coordinate.(see pg 12 in [8])

Still with Overduin-Wesson all physical quantities specially the ones associated with the spacetime metric tensor depends on the Extra Coordinate and is this dependence that allows us to obtain masses and charges in $4D$ from the $5D$ Higher Dimensional Field Equations.(see pg 29 in [8] after Paragraph 6)

We will avoid further long and tedious explanations on Dimensional Reduction and will right now examine the main equations of the Basini-Capozziello-Overduin-Wesson formalism however good explanations on how Dimensional Reduction works can be found in [7] pg 12 to 15 until eq 63 and in [6] pg 15 to 17 and pg 40 to 41 before eq 332.

The $5D$ Spacetime Ansatz is given in matrix form by:(see eq 57 pg 14 in [1],see eq 43 pg 14 in [2],see eq 47 pg 30 in [8])

$$(g_{AB}) = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -\Phi^2 \end{pmatrix} \quad . \tag{126}$$

The $5D$ Spacetime Ansatz using the standard form of a Space-Time Metric of General Relativity is given by:(see eq 56 pg 14 in [1],see eq 42 pg 14 in [2],see eq 109 pg 15 in [6],see eq 5 pg 5 in [7])

$$dS^2 = g_{AB}dx^A dx^B = g_{\alpha\beta}dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{127}$$

In order to better figure out what will come further we introduce some useful relations between the Riemann and Ricci Tensors

- Contraction of a Contravariant Script in a Riemann Tensor:(see eq 3.76 pg 85 in [9])

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu} \ . \tag{128}$$

- Relation between the Riemann and Ricci Tensors:(see eq 3.90 pg 88 in [9])

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \ . \tag{129}$$

- The Ricci Tensor is always symmetric :(see eq 3.91 pg 88 in [9])

$$R_{\mu\nu} = R_{\nu\mu} \ , \tag{130}$$

- The Ricci Scalar is defined by:(see eq 3.92 pg 88 in [9])

$$R = R^{\mu}{}_{\mu} = g^{\mu\mu} R_{\mu\mu} \ . \tag{131}$$

The relation between the $5D$ and the respective $4D$ counterparts of both Riemann and Ricci Tensors is given by:(eqs 58 and 59 pg 14 and 15 in [1],eqs 44 and 45 pg 14 in [2],eqs 111 and 112 pg 16 in [6],eqs 7 and 8 pg 5 in [7],eqs 48 and 49 pg 31 and 32 in [8])

$$^{5}R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} - \frac{1}{2\Phi^{2}}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{132}$$

$$^{5}R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{\Phi_{,a;b}}{\Phi}g_{\mu\nu} - \frac{1}{2\Phi^{2}}g_{\mu\nu}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{133}$$

$$^{5}R = R - \frac{\Phi_{,a;b}}{\Phi}g^{\alpha\beta} - \frac{1}{2\Phi^{2}}g^{\alpha\beta}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu}g_{\alpha\lambda,4}g_{\beta\mu,4} - \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{134}$$

In order to simplify our point of view we will consider a complete diagonalized metric[80] $\alpha = \beta = \mu = \nu$.Then the equations are given by:(eqs 113 to 116 pg 16 in [6],eqs 9 to 12 pg 5 and 6 in [7]

$$^{5}R_{\alpha\beta} = R_{\alpha\beta} - \frac{\Phi_{,a;b}}{\Phi} - \frac{1}{2\Phi^{2}}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{135}$$

$$^{5}R_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{\Phi_{,a;b}}{\Phi}g_{\mu\nu} - \frac{1}{2\Phi^{2}}g_{\mu\nu}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{136}$$

$$^{5}R = R - \frac{\Phi_{,a;b}}{\Phi}g^{\alpha\beta} - \frac{1}{2\Phi^{2}}g^{\alpha\beta}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{137}$$

$$^{5}R = R - \frac{^{4}\Box\Phi}{\Phi} - \frac{1}{2\Phi^{2}}g^{\alpha\beta}(\frac{\Phi_{,4}g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + \frac{g^{\mu\nu}g_{\mu\nu,4}g_{\alpha\beta,4}}{2}) \tag{138}$$

$$^{5}R = R - \frac{^{4}\Box\Phi}{\Phi} \tag{139}$$

From above if we are in a Flat Spacetime of Special Relativity or in the neighborhoods of Earth where space is considered Flat all the derivatives of the Spacetime Metric Tensor vanishes.We live in a region of the Universe where the Scalar Field is $\Phi \simeq 1$ with low derivatives.Then the $5D$ Riemann,Ricci Tensors and Ricci Scalars all of them reduces from the original $5D$ form to its $4D$ Counterparts.This is the reason why we cannot tell if we live in a $5D$ or a $4D$ Spacetime because we live in a Flat Spacetime.This is the Essence of Dimensional Reduction:(see again Dimensional Reduction in [7] pg 12 to 15 until eq 63 and in [6] pg 15 to 17 and pg 40 to 41 before eq 332.)

[80] Our point of view would still be valid for non-diagonalized metrics however this would imply in more and un-necessary algebraic expresions

The Weyl Tensor is defined by:(see eq 7 pg 2 in [14],see eq $A.1$ pg 5 in [14],see eq 3.97 pg 89 in [9],eq 3.10 pg 9 in [13])

$$ {}^{(5)}R_{\mu\alpha\nu\beta} = \frac{2}{3}(g_{\mu[\nu}{}^{(5)}R_{\beta]\alpha} - g_{\alpha[\nu}{}^{(5)}R_{\beta]\mu}) - \frac{1}{6}g_{\mu[\nu}g_{\beta]\alpha}{}^{(5)}R + {}^{(5)}C_{\mu\alpha\nu\beta}, \tag{140} $$

$$ {}^{(5)}R_{\alpha\beta\mu\nu} = \frac{2}{3}(g_{\alpha[\mu}{}^{(5)}R_{\nu]\beta} - g_{\beta[\mu}{}^{(5)}R_{\nu]\alpha}) - \frac{1}{6}g_{\alpha[\mu}g_{\nu]\beta}{}^{(5)}R + {}^{(5)}C_{\alpha\beta\mu\nu}, \tag{141} $$

$$ {}^{(5)}R_{\alpha\beta} = \frac{2}{3}g^{\mu\nu}(g_{\alpha[\mu}{}^{(5)}R_{\nu]\beta} - g_{\beta[\mu}{}^{(5)}R_{\nu]\alpha}) - \frac{1}{6}g^{\mu\nu}g_{\alpha[\mu}g_{\nu]\beta}{}^{(5)}R + {}^{(5)}C_{\alpha\beta}, \tag{142} $$

For a complete diagonalized metric $\alpha = \beta = \mu = \nu$ we would get the following results:

$$ {}^{(5)}R_{\alpha\beta\mu\nu} = \frac{2}{3}(g_{\alpha\mu}{}^{(5)}R_{\nu\beta} - g_{\alpha\nu}{}^{(5)}R_{\mu\beta} - g_{\beta\mu}{}^{(5)}R_{\nu\alpha} + g_{\beta\nu}{}^{(5)}R_{\mu\alpha}) - \frac{1}{6}(g_{\alpha\mu}g_{\nu\beta} - g_{\alpha\nu}g_{\mu\beta}){}^{(5)}R + {}^{(5)}C_{\alpha\beta\mu\nu}, \tag{143} $$

$$ g_{\alpha\mu}{}^{(5)}R_{\nu\beta} - g_{\alpha\nu}{}^{(5)}R_{\mu\beta} = 0 \curvearrowright {}^{(5)}R_{\nu\beta} = {}^{(5)}R_{\mu\beta}g^{\mu}_{\nu} \curvearrowright g_{\alpha\nu} = g_{\alpha\mu}g^{\mu}_{\nu} \tag{144} $$

$$ g_{\nu\beta}{}^{(5)}R_{\mu\alpha} - g_{\beta\mu}{}^{(5)}R_{\nu\alpha} = 0 \curvearrowright {}^{(5)}R_{\nu\alpha} = {}^{(5)}R_{\mu\alpha}g^{\mu}_{\nu} \curvearrowright g_{\beta\nu} = g_{\beta\mu}g^{\mu}_{\nu} \tag{145} $$

$$ \frac{1}{6}(g_{\alpha\mu}g_{\nu\beta} - g_{\alpha\nu}g_{\mu\beta}){}^{(5)}R = 0 \tag{146} $$

$$ {}^{(5)}R_{\alpha\beta\mu\nu} = {}^{(5)}C_{\alpha\beta\mu\nu}, \tag{147} $$

Above is being presented one of the most important results of this Section.The Weyl Tensor responsible for the Tidal Charge appears in the Basini-Capozziello-Overduin-Wesson formalism for the $5D$ Kaluza Klein Induced Matter BraneWorld Model.Now we will examine how the "same" Weyl Tensor appears in the Shiromizu-Maeda-Sasaki formalism for the Randall-Sundrum BraneWorld Model in order to conclude that both gives similar results regardless and independently of the mathematical formalism being used because if both formalisms although "apparently different" describes the "same" $5D$ Extra Dimension and if the "same" Weyl Tensor is being "generated" by the "same" $5D$ in both formalisms then all the subsequent results concerning the Tidal Charge must be equivalent.

The $5D$ Extra Dimensional Ansatz used by Shromizu-Maeda-Sasaki for the Randall Sundrum BraneWorld Model can be described by the following equations(see [14] eq 12 pg 3,[13] eq 3.2 pg 8):

$$dS^2 = (n_\mu n_\nu + q_{\mu\nu})\, dx^\mu dx^\nu = d\chi^2 + q_{\mu\nu} dx^\mu dx^\nu \;\; = g_{\alpha\beta} dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{148}$$

$$dS^2 = (n_{\mu\nu} + q_{\mu\nu})\, dx^\mu dx^\nu = d\chi^2 + q_{\mu\nu} dx^\mu dx^\nu \;\; = g_{\alpha\beta} dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{149}$$

$$(n_{\mu\nu} + q_{\mu\nu})\, dx^\mu dx^\nu = q_{\mu\nu} dx^\mu dx^\nu + n_{\mu\nu} dx^\mu dx^\nu = d\chi^2 + q_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta} dx^\alpha dx^\beta - \Phi^2 dy^2 \tag{150}$$

$$ds^2 = q_{\mu\nu} dx^\mu dx^\nu = g_{\alpha\beta} dx^\alpha dx^\beta \tag{151}$$

In the equations above $q_{\mu\nu}$ and $g_{\alpha\beta}$ are our familiar $4D$ Spacetime Metric Tensor χ (in [14]) or y (in [13]) are the $5D$ Extra Dimension and n_μ or n_ν are the vectors normal to the $4D$ Spacetime defined as: ([14] bottom of pg 2,[13] between eq 3.1 and 3.2 pg 8)

$$n_\mu dx^\mu = d\chi = i\Phi dy \tag{152}$$

Combining two normal vectors n_μ and n_ν we get $n_{\mu\nu}$ that behaves for the $5D$ Spacetime exactly the same way as $q_{\mu\nu}$ and $g_{\alpha\beta}$ behaves for the $4D$ Spacetime. Then we can say that $n_{\mu\nu}$ is a "new Spacetime Metric Tensor" for the $5D$ Spacetime that plays the same role of $q_{\mu\nu}$ and $g_{\alpha\beta}$ for the $4D$ Spacetime.
Below are the relations between $n_{\mu\nu}$ and the Scalar field Φ from the equations of $5D$ in the Shiromizu-Maeda-Sasaki formalism coupled to the equations of $5D$ in the Basini-Capozziello-Overduin-Wesson formalism. We need these relations of equivalence because the $5D$ Ansatz dS^2 can be written as eq 56 pg 14 in [1] or eq 109 pg 15 in [6] according to Basini-Capozziello-Overduin-Wesson or the same $5D$ Ansatz can be written as eq 12 pg 3 in [14] according to Shiromizu-Maeda-Sasaki and we can always say that one Ansatz is equal to the other.

$$n_\mu = \frac{d\chi}{dx^\mu} = i\Phi \frac{dy}{dx^\mu} = q_\mu \frac{d\chi}{ds} \curvearrowright q_\mu = \frac{ds}{dx^\mu} \tag{153}$$

$$n^\mu = \frac{dx^\mu}{d\chi} = \frac{1}{i\Phi}\frac{dx^\mu}{dy} = q^\mu \frac{ds}{d\chi} \curvearrowright q^\mu = \frac{dx^\mu}{ds} \tag{154}$$

$$n_\mu dx^\mu n_\nu dx^\nu = d\chi^2 = n_{\mu\nu} dx^\mu dx^\nu = -\Phi^2 dy^2 \tag{155}$$

$$n_\mu n_\nu = -\frac{\Phi^2 dy^2}{dx^\mu dx^\nu} = \frac{d\chi^2}{dx^\mu dx^\nu} = n_{\mu\nu} \tag{156}$$

$$n^\mu n^\nu = -\frac{dx^\mu dx^\nu}{\Phi^2 dy^2} = \frac{dx^\mu dx^\nu}{d\chi^2} = n^{\mu\nu} \tag{157}$$

The $5D$ "new Spacetime Metric Tensor" $n_{\mu\nu}$ behaves Covariantly and Contravariantly likes the familiar $4D$ Spacetime Metric Tensors $q_{\mu\nu}$ and $g_{\alpha\beta}$ although $n_{\mu\nu}$ is not exactly equal to $q_{\mu\nu}$ and $g_{\alpha\beta}$.

$$n^{\mu}n_{\nu} = \frac{dx^{\mu}}{d\chi}\frac{d\chi}{dx^{\nu}} = \frac{dx^{\mu}}{dx^{\nu}} = n^{\mu}_{\nu} = q^{\mu}_{\nu} \tag{158}$$

$$n^{\mu} = n^{\mu\nu}n_{\nu} \curvearrowright n_{\mu} = n_{\mu\nu}n^{\nu} \curvearrowright n_{\mu\nu} \neq q_{\mu\nu} \curvearrowright n^{\mu\nu} \neq q^{\mu\nu} \tag{159}$$

The relation between the $5D$ "new Spacetime Metric Tensor" $n_{\mu\nu}$ and the ordinary $4D$ Spacetime Metric Tensors $q_{\mu\nu}$ or $g_{\alpha\beta}$ is given by

$$n^{\mu\nu} = q^{\mu\nu}(\frac{ds}{d\chi})^2 \curvearrowright n_{\mu\nu} = q_{\mu\nu}(\frac{d\chi}{ds})^2 \tag{160}$$

In order to advance further with the $5D$ Shiromizu-Maeda-Sasaki formalism we need to define the Extrinsic Curvature as:(eq 3.3 pg 8 in [13], pg 2 between eqs 2 and 3 in [14])

$$K_{\mu\nu} = q_{\mu}{}^{\alpha}q_{\nu}{}^{\beta}\nabla_{\alpha}n_{\beta} = q_{\mu}{}^{\alpha}q_{\nu}{}^{\beta}\nabla_{\alpha}(\frac{d\chi}{dx^{\beta}}) \tag{161}$$

$$K_{\alpha\beta} = \nabla_{\alpha}n_{\beta} \tag{162}$$

$$K_{\mu\nu} = q_{\mu}{}^{\alpha}q_{\nu}{}^{\beta}K_{\alpha\beta} \tag{163}$$

$$K_{AB} = \frac{1}{2}\pounds_{\mathbf{n}}\, g_{AB} = g_{A}{}^{C}\nabla_{C}n_{B}\,, \tag{164}$$

The Extrinsic Curvature (for example $K_{\nu\mu}$[81]) is taken over the Covariant Derivative of n_{μ} but we know that $n_{\mu} = n_{\mu\nu}n^{\nu}$ and $n^{\nu} = \frac{dx^{\nu}}{d\chi}$.Then we can see that the Extrinsic Curvature is not zero when the derivatives with respect to the $5D$ Extra Dimension do not vanish. If $n^{\nu} = \frac{dx^{\nu}}{d\chi} = 0$ then the Covariant Derivative will be zero and consequently the Extrinsic Curvature.We have a non-null Extrinsic Curvature when the derivatives of the $4D$ Spacetime Coordinates with respect to the $5D$ Extra Coordinate do not vanish.
The Covariant Derivative is defined as(see eq 3.7 pg 64 in [9]):

$$\nabla_{\mu}n_{\nu} = \partial_{\mu}n_{\nu} - \Gamma^{\lambda}_{\mu\nu}n_{\lambda} = \partial_{\mu}(\frac{d\chi}{dx^{\nu}}) - \Gamma^{\lambda}_{\mu\nu}(\frac{d\chi}{dx^{\lambda}}) \tag{165}$$

According to Shiromizu-Maeda Sasaki the Gauss Equation (eq 1 pg 2 in [14] and eq 3.5 pg 9 in [13]) gives the Relations between the $5D$ to $4D$ Riemann and Ricci Tensors. If the Extrinsic Curvature is zero then the $5D$ Riemann and Ricci Tensors reduces to its $4D$ counterparts in a process of Dimensional Reduction from $5D$ to $4D$ that resembles the one described by Basini-Capozziello-Overduin-Wesson.

We live in a region of Spacetime where the derivatives $\frac{dx^{\nu}}{d\chi}$ are close to zero but not entirely zero then we have an Extrinsic Curvature that although not null it is extremely difficult to be detected(at least on Earth). Hence the $5D$ Riemann and Ricci Tensors are not entirely equal to its $4D$ counterparts but the difference is by far too small to be noticed.

[81]we changed the scripts and this was intentional

This is exactly what happens with the Scalar Field Φ described in the Basini-Capozziello-Overduin-Wesson formalism where the D'Alembertian Operator applied over the Scalar Field Φ which means to say ${}^{(4)}\Box\Phi$ gives a result almost close to zero but not entirely zero generating again a very small difference between the $5D$ Riemann and Ricci Tensors and its $4D$ counterparts and like in the difference of Shiromizu-Maeda-Sasaki case the difference in the Basini-Capozziello-Overduin-Wesson case is also too small to be detected and then we cannot say if we live in a $5D$ or in a $4D$ Spacetime. Note that Basini-Capozziello-Overduin-Wesson uses derivatives of the $g_{\alpha\beta}$ the $4D$ Spacetime Metric Tensor or the Scalar Field Φ both taken with respect to the Extra Coordinate (and in an almost flat spacetime like Earth neighborhoods the derivatives have pretty small values almost close to zero too) while the Shiromizu-Maeda-Sasaki uses derivatives of the $4D$ Spacetime Coordinates with respect to the $5D$ Extra Dimension.

The Extrinsic Curvature in Shiromizu-Maeda-Sasaki plays the same role of the derivatives(or the D'Alembertian) of the Scalar Field Φ or the same role of the derivatives of $g_{\alpha\beta}$ the $4D$ Spacetime Metric Tensor these last two both in the Basini-Capozziello-Overduin-Wesson case where these derivatives are also taken with respect to the Extra Coordinate.

Looking again to the Gauss equation (eq 1 pg 2 in [14] and eq 3.5 pg 9 in [13])

$$ {}^{(4)}R^{\alpha}{}_{\beta\gamma\delta} = {}^{(5)}R^{\mu}{}_{\nu\rho\sigma} q_{\mu}{}^{\alpha} q_{\beta}{}^{\nu} q_{\gamma}{}^{\rho} q_{\delta}{}^{\sigma} + K^{\alpha}{}_{\gamma} K_{\beta\delta} - K^{\alpha}{}_{\delta} K_{\beta\gamma} \,, \tag{166} $$

$$ {}^{(4)}R^{\alpha}{}_{\beta\gamma\delta} = {}^{(5)}R^{\alpha}{}_{\beta\gamma\delta} + K^{\alpha}{}_{\gamma} K_{\beta\delta} - K^{\alpha}{}_{\delta} K_{\beta\gamma} \,, \tag{167} $$

It can be seen from above that if the Extrinsic Curvatures have small values close to zero then the $5D$ Riemann and Ricci Tensors present values close to its $4D$ counterparts.Compare this situation with what would happen to eqs 132 to 139 in this work for the case of an almost flat spacetime in the neighborhoods of Earth with low values for the derivatives of $g_{\alpha\beta}$ or Φ.

Another useful equation is the Codacci equation that defines the $5D$ Ricci Tensor in function of Covariant Derivatives of the Extrinsic Curvature.As already seen if the derivatives of the $4D$ Spacetime Coordinates with respect to the $5D$ Extra Dimension vanishes then the Extrinsic Curvature vanishes too and hence the $5D$ Ricci Tensor: (see eq 3.6 pg 9 in [13],eq 2 pg 2 in [14])

$$ D_{\nu} K_{\mu}{}^{\nu} - D_{\mu} K = {}^{(5)}R_{\rho\sigma} n^{\sigma} q_{\mu}{}^{\rho} \,, \tag{168} $$

$$ D_{\nu} K_{\mu}{}^{\nu} - D_{\mu} K = {}^{(5)}R_{\mu\sigma} n^{\sigma} \,, \tag{169} $$

$$ K = K^{\mu}_{\mu} = K_{\mu\mu} g^{\mu\mu} \tag{170} $$

$$ \nabla_B K^B_A - \nabla_A K = {}^5R_{BC}\, g^B_A n^C \,, \tag{171} $$

$$ \nabla_B K^B_A - \nabla_A K = {}^5R_{AC}\, n^C \,, \tag{172} $$

$$ K = K^A_A \tag{173} $$

From the Gauss equation we can obtain the "backreaction" term in the relation between the $5D$ to $4D$ Riemann and Ricci Tensors that generates in our $4D$ Spacetime the Weyl Tensor responsible for the BraneWorld Tidal Charge.(see eq 3 pg 2 in [14])

$$^{(4)}R_{\mu\nu} = {}^{(5)}R_{\rho\sigma}q_{\mu}{}^{\rho}q_{\nu}{}^{\sigma} - {}^{(5)}R^{\alpha}{}_{\beta\gamma\delta}n_{\alpha}q_{\mu}{}^{\beta}n^{\gamma}q_{\nu}{}^{\delta} + KK_{\mu\nu} - K_{\mu}{}^{\alpha}K_{\nu\alpha} \,. \tag{174}$$

$$^{(4)}R_{\mu\nu} = {}^{(5)}R_{\mu\nu} - {}^{(5)}R^{\alpha}{}_{\mu\gamma\nu}n_{\alpha}n^{\gamma} + KK_{\mu\nu} - K_{\mu}{}^{\alpha}K_{\nu\alpha} \,. \tag{175}$$

This "backreaction" term is given in $5D$ by the following equation.(see eq 5 pg 2 in [14])

$$\tilde{E}_{\mu\nu} \equiv {}^{(5)}R^{\alpha}{}_{\beta\rho\sigma}n_{\alpha}n^{\rho}q_{\mu}{}^{\beta}q_{\nu}{}^{\sigma} \,. \tag{176}$$

$$\tilde{E}_{\mu\nu} \equiv {}^{(5)}R^{\alpha}{}_{\mu\rho\nu}n_{\alpha}n^{\rho} \,. \tag{177}$$

From the definition of the Weyl Tensor we get the term that generates the "Tidal Electric Charge" in $4D$ as follows:(see eq 9 pg 2 in [14])

$$E_{\mu\nu} \equiv {}^{(5)}C^{\alpha}{}_{\beta\rho\sigma}n_{\alpha}n^{\rho}q_{\mu}{}^{\beta}q_{\nu}{}^{\sigma}. \tag{178}$$

$$E_{\mu\nu} \equiv {}^{(5)}C^{\alpha}{}_{\mu\rho\nu}n_{\alpha}n^{\rho}. \tag{179}$$

Using [13] we will analyze the results obtained from above.

- 1)-The Gauss equation is given by eq 3.5 pg 9 in [13]

$$R_{ABCD} = {}^{5}R_{EFGH}\, g^{E}_{A}g^{F}_{B}g^{G}_{C}g^{H}_{D} + 2K_{A[C}K_{D]B} \,, \tag{180}$$

$$R_{ABCD} = {}^{5}R_{ABCD} + 2K_{A[C}K_{D]B} \,, \tag{181}$$

- 2)-But we prefer the following version of the Gauss equation

$$R_{ABCD} = {}^{5}R_{EFGH}\, g^{E}_{A}g^{F}_{B}g^{G}_{C}g^{H}_{D} + K_{AC}K_{DB} - K_{AD}K_{CB} \tag{182}$$

$$R_{ABCD} = {}^{5}R_{ABCD} + K_{AC}K_{DB} - K_{AD}K_{CB} \tag{183}$$

- 3)-Multiplied the equations above by the Contravariant $4D$ diagonalized Spacetime Metric Tensor g^{AA} we get the following expressions:

$$g^{AA}R_{ABCD} = g^{AA5}R_{ABCD} + g^{AA}K_{AC}K_{DB} - g^{AA}K_{AD}K_{CB} \tag{184}$$

$$R^{A}_{BCD} = {}^{5}R^{A}_{BCD} + K^{A}_{C}K_{DB} - K^{A}_{D}K_{CB} \tag{185}$$

Compare eq 185 with eq 167 also given in this work.[82]

[82]QED:Quod Erad Demonstratum

- 4)-Introducing now the Maartens $5D$ Curvature Identities defined by eqs 3.7 to 3.9 pg 9 in [13]

$$^5R_{EFGH}g^E_Ag^F_Bg^G_Cn^H = 2\nabla_{[A}K_{B]C} \tag{186}$$

$$^5R_{EFGH}g^E_An^Fg^G_Bn^H = -\pounds_nK_{AB} + K_{AC}K^C_B \tag{187}$$

$$^5R_{CD}\,g^C_Ag^D_B = R_{AB} - \pounds_{\mathbf{n}}K_{AB} - KK_{AB} + 2K_{AC}K^C_B\,. \tag{188}$$

$$^5R_{ABCH}n^H = 2\nabla_{[A}K_{B]C} \tag{189}$$

$$^5R_{AFBH}n^Fn^H = -\pounds_nK_{AB} + K_{AC}K^C_B \tag{190}$$

$$^5R_{AB} = R_{AB} - \pounds_{\mathbf{n}}K_{AB} - KK_{AB} + 2K_{AC}K^C_B\,. \tag{191}$$

- 5)-Isolating the Lie Derivative term from eq 3.8 and inserting it into eq 3.9 we get the following results:

$$\pounds_nK_{AB} = -^5R_{AFBH}n^Fn^H + K_{AC}K^C_B \tag{192}$$

$$-\pounds_nK_{AB} =^5 R_{AFBH}n^Fn^H - K_{AC}K^C_B \tag{193}$$

$$^5R_{AB} = R_{AB} +^5 R_{AFBH}n^Fn^H - K_{AC}K^C_B - KK_{AB} + 2K_{AC}K^C_B\,. \tag{194}$$

$$^5R_{AB} = R_{AB} +^5 R_{AFBH}n^Fn^H - KK_{AB} + K_{AC}K^C_B\,. \tag{195}$$

$$R_{AB} =^5 R_{AB} -^5 R_{AFBH}n^Fn^H + KK_{AB} - K_{AC}K^C_B \tag{196}$$

Compare eq 196 above with eq 175 both in this work.
Working with the $5D$ "backreaction" term $^5R_{AFBH}n^Fn^H$ using the 5D curvature Weyl Tensor(eq 3.10 pg 9 in [13]):

$$^5R_{ABCD} = {}^5C_{ACBD} + \frac{2}{3}\left\{g_{A[C}{}^5R_{D]B} - g_{B[C}{}^5R_{D]A}\right\} - \frac{1}{6}g_{A[C}g_{D]B}{}^5R \tag{197}$$

We get the term that generates the "Tidal Electric Charge" in $4D$(see eq 3.12 pg 9 in [13],see eq $A.2$ pg 6 in [14]):

$$E_{\mu\nu} =^5 C_{ACBD}n^Cn^Dg^A_\mu g^B_\nu \tag{198}$$

$$E_{\mu\nu} =^5 C_{\mu C\nu D}n^Cn^D \tag{199}$$

Placing again together eqs 199 and 147 of this work both shown below:

$$E_{\mu\nu} =^5 C_{\mu C\nu D} n^C n^D \tag{200}$$

$$^{(5)}R_{\alpha\beta\mu\nu} = ^{(5)}C_{\alpha\beta\mu\nu}, \tag{201}$$

We can see that while one was obtained by a diagonalized metric in the Basini-Capozziello-Overduin-Wesson formalism applied to the Induced Kaluza-Klein BraneWorld the other was obtained by the Shiromizu-Maeda-Sasaki formalism applied to the Randall-Sundrum BraneWorld.Both uses a Weyl Tensor and if the derivatives of the $4D$ Spacetime Coordinates with respect to the $5D$ Extra Dimension do not vanishes[83] then an Electric Tidal Charge will appear in both formalisms because although the formalisms are "apparently" different[84] actually both formalisms describes the "same" $5D$ Extra Dimension

Defining $\kappa_4 = \frac{8\pi G_4}{c^4}$ and $\kappa_5 = \frac{8\pi G_5}{c^4}$ where G_4 and G_5 are the Gravitational Constants defined in $4D$ and $5D$ Spacetimes with the relation between G_4 and G_5 given by [6] pg 16 between eqs 116 and 117 we will analyze the equations given below:(see pg 2 after Section III in [12] after eq 8 and see eq 10.)

$$\mathcal{U} = -\left(\frac{\kappa_4}{\kappa_5}\right)^4 E_{\mu\nu}u^\mu u^\nu \curvearrowright \mathcal{U} = \left(\frac{\kappa_4}{\kappa_5}\right)^4 \frac{Q}{r^4} \curvearrowright \mathcal{U} = -\left(\frac{1}{\int d\chi}\right)^4 E_{\mu\nu}u^\mu u^\nu \curvearrowright \mathcal{U} = \left(\frac{1}{\int d\chi}\right)^4 \frac{Q}{r^4} \tag{202}$$

$$\mathcal{U} = -\left(\frac{1}{\int dy}\right)^4 E_{\mu\nu}u^\mu u^\nu \curvearrowright \mathcal{U} = \left(\frac{1}{\int dy}\right)^4 \frac{Q}{r^4} \tag{203}$$

$$E_{\mu\nu}u^\mu u^\nu = -\frac{Q}{r^4} \tag{204}$$

The Tidal Charge appears in the $4D$ Spacetime induced by the $5D$ Weyl Tensor.According to eqs 12 to 14 pg 3 in [12] this affects the ordinary Schwarzschild Metric generating the
Metric of Dadhich,Maartens,Papadopolous and Rezania as shown below:

$$ds^2 = [1 - \frac{2GM}{c^2 r} + \frac{Q}{r^2}]dt^2 - \frac{1}{1 - \frac{2GM}{c^2 r} + \frac{Q}{r^2}}dr^2 - r^2 d\Omega^2 \tag{205}$$

We will see in Sections 5 and 6 that this metric is the best candidate to demonstrate the Higher Dimensional Nature of the Universe.

[83] Overduin-Wesson-physics is allowed to depend on the $3+1$ and Extra Coordinate.(see pg 12 in [8])-all physical quantities specially the ones associated with the spacetime metric tensor depends on the Extra Coordinate and is this dependence that allows us to obtain masses and charges in $4D$ from the $5D$ Higher Dimensional Field Equations.(see pg 29 in [8] after Paragraph 6)

[84] keeping in mind the Basini-Capozziello paradigm of the "Hypotheses Non Fingo" of Isaac Newton if we remove conformal factors and compactification terms and even more "exotic" mathematical terms that only "complicate" the things we get a sample form of the $5D$ Ansatz dS^2.And we can say that both formalisms are the same formalism however described by different mathematical tools

3 Rest-Masses and Electric Charges seen in a $4D$ Spacetime but being generated by a $5D$ Spacetime due to the Geometrical Nature of the Hamilton-Jacobi Equation.

Although Dimensional Reduction from $5D$ to $4D$ can explain why we cannot detect the presence of the $5D$ Spacetime in the ordinary conditions of the $4D$ Flat Spacetime of Special Relativity the Geodesics Equations tells nothing about the masses and the charges of the particles seen in $4D$(See pg 2 and 3 in [4],See pg 2,3 and 4 in [18]).The masses and charges of the particles seen in $4D$ are also generated by the $5D$ Spacetime in a very attractive way.We can have a small group of particles in the $5D$ Spacetime each one having the same rest mass M_5 but due to different Spacetime Couplings between the $5D$ and $4D$ Spacetimes two particles having the same rest-mass in $5D$ will appear with different rest-masses m_0 in the $4D$ Spacetime.The Spacetime Coupling projects for each $5D$ particle a different image in $4D$. The same is also true for electric charges(See pg 3 in [4]).This is a very interesting point of view: for example we have 6 Quark each one having a different rest mass m_0 seen in $4D$ but it might be possible that all the Quarks in $5D$ have the same rest mass M_5 and due to different Spacetime Couplings the same $5D$ Quark appears in the $4D$ Spacetime with different images each one being a different projection of the $5D$ Spacetime into the $4D$ Spacetime one . The masses and charges generated in the $4D$ Spacetime as a geometrical projection from the $5D$ Spacetime are explained by the Hamilton-Jacobi equation(See pg 4 in [18],See pg 3 in [4]) .In this Section we follow the procedures and the approach of Ponce De Leon.

According to Ponce de Leon there are three possibilities for the projection of a $5D$ Spacetime into a $4D$ Spacetime giving three possible values for the $5D$ rest mass M_5(See eqs 3 to 5 pg 3 in [6][85]):

- Timelike $5D$ Geodesics:

$$dS^2 > 0 \curvearrowright dS^2 = ds^2 - \Phi^2 dy^2 \curvearrowright ds^2 > \Phi^2 dy^2 \curvearrowright 1 > \Phi^2 (dy/ds)^2 \curvearrowright M_5 > 0 \tag{206}$$

- Null-like $5D$ Geodesics:

$$dS^2 = 0 \curvearrowright dS^2 = ds^2 - \Phi^2 dy^2 \curvearrowright ds^2 = \Phi^2 dy^2 \curvearrowright 1 = \Phi^2 (dy/ds)^2 \curvearrowright M_5 = 0 \tag{207}$$

- Spacelike $5D$ Geodesics:

$$dS^2 < 0 \curvearrowright dS^2 = ds^2 - \Phi^2 dy^2 \curvearrowright ds^2 < \Phi^2 dy^2 \curvearrowright 1 < \Phi^2 (dy/ds)^2 \curvearrowright M_5 < 0 \tag{208}$$

[85] without conformal factors

- Case 1)- particles in a Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ with a $5D$ rest-mass $M_5 > 0$ giving a $4D$ rest-mass $m_0 > 0$

All the relations between M_5, m_0,dS^2 and ds^2 are given by the following equation(See eq 22 pg 5 in [3])[86]:

$$\frac{dS}{M_5} = \frac{ds}{m_0} \tag{209}$$

Now we will introduce the mathematical demonstration of the Hamilton-Jacobi Equation: Starting with the contravariant component of the $5D$ Momentum P^Q defined in function of M_5 as being $P^Q = M_5 U^Q$(See eq 15 pg 5 in [3],See eq 6 pg 5 in [18]) where $U^Q = (dx^q/dS, dy/dS)$ and $U^Q U_Q = 1$ because $U^Q = g^{QQ} U_Q$ and $U_Q = g_{QQ} U^Q$ giving $U^Q U_Q = g^{QQ} U_Q g_{QQ} U^Q = g^{QQ} g_{QQ} U_Q U^Q$ but we know that $g^{QQ} g_{QQ} = 1$ then $U^Q U_Q = 1$

Defining the contravariant and the covariant components of the Momentum in $5D$ and the product between the components we have(See eq 16 pg 5 in [3],See eq 7 pg 5 in [18]):

$$P^Q = M_5 U^Q \tag{210}$$

$$P_Q = M_5 U_Q \tag{211}$$

$$P(5) = P^Q P_Q = M_5 U^Q M_5 U_Q = M_5^2 U^Q U_Q = M_5^2 \tag{212}$$

The product between components of the 5D Momentum is given by:

$$P(5) = P^Q P_Q = M_5^2 \tag{213}$$

But we know that $Q = 0, 1, 2, 3, 4$ being the script 4 the Fifth Dimension

Also we know that dS^2 the $5D$ Spacetime Metric is not entirely seen by a $4D$ observer.The $4D$ observer can only access the $4D$ part of the trajectory(See abstract and pg 2 in [18],See pg 4 before eq 11 in [3]).Hence the $4D$ observer can only measure the $4D$ Momentum defined by its contravariant and covariant components as follows(See eq 17 pg 5 in [3],See eq 8 pg 5 in [18]):

$$p^q = m_0 U^q \tag{214}$$

$$p_q = m_0 U_q \tag{215}$$

with $p = P$ and being $q = 0, 1, 2, 3$ but also with:

$U^q = g^{qq} U_q$ and $U_q = g_{qq} U^q$ giving $U^q U_q = g^{qq} U_q g_{qq} U^q = g^{qq} g_{qq} U_q U^q$ and since $g^{qq} g_{qq} = 1$ then $U_q U^q = 1$ just like its $5D$ counterpart

then we should expect for:

$$p(4) = p^q p_q = m_0 U^q m_0 U_q = m_0^2 U^q U_q = m_0^2 \tag{216}$$

Hence the product between components of the $4D$ Momentum is given by:

$$p(4) = p^q p_q = m_0^2 \tag{217}$$

[86]we do not consider here conformal factors

Then the product between components of the $5D$ Momentum can be written as:

$$p(5) = P^Q P_Q = p^q p_q + P^4 P_4 = M_5^2 \tag{218}$$

with:

$$p(5) = p(4) + P^4 P_4 = M_5^2 \tag{219}$$

but we know that

$$p(4) = p^q p_q = m_0^2 \tag{220}$$

Then we should expect the following expression given below for the product between components of the $5D$ Momentum(See eq 19 pg 5 in [3],See eq 9 pg 5 in [18]):

$$p(5) = p^q p_q + P^4 P_4 = M_5^2 \curvearrowright p(5) = m_0^2 + P^4 P_4 = M_5^2 \tag{221}$$

Considering now the following $5D$ Spacetime Ansatz defined below as(See eq 14 pg 4 in [3],See eq 5 pg 5 in in [18])[87]:

$$dS^2 = g_{qr}(x^w, y)dx^q dx^r - \Phi^2(x^w, y)dy^2 \tag{222}$$

$$dS^2 = ds^2 - \Phi^2(x^w, y)dy^2 \tag{223}$$

$$ds^2 = g_{qr}(x^w, y)dx^q dx^r \tag{224}$$

Where w is the affine parameter and S is the $5D$ Action defined by(See pg 6 between eqs 10 and 11 in [18]):

$$S = S(x^w, y) \tag{225}$$

We can define the covariant components of the $5D$ or $4D$ Momentum in function of the $5D$ Action given above as follows:

$$p_q = P_q = -\frac{\partial S}{\partial x^q} \tag{226}$$

$$p_r = P_r = -\frac{\partial S}{\partial x^r} \tag{227}$$

$$p_4 = P_4 = -\frac{\partial S}{\partial y} \tag{228}$$

but we know that:

$$p^q = g^{qr} p_r \tag{229}$$

Rewriting the product between components of the $5D$ Momentum in function of the $5D$ Action we should expect for:

[87] without conformal factors

$$p(5) = p^q p_q + P^4 P_4 = M_5^2 = g^{qr} p_r p_q + P^4 P_4 = M_5^2 \tag{230}$$

Then we finally arrive at the Hamilton-Jacobi equation as defined by Ponce De Leon given below:

$$g^{qr} \frac{\partial S}{\partial x^q} \frac{\partial S}{\partial x^r} + P^4 P_4 = M_5^2 \tag{231}$$

but we also know that

$$P^4 = g^{44} P_4 \tag{232}$$

Hence the Hamilton-Jacobi equation now becomes:

$$g^{qr} \frac{\partial S}{\partial x^q} \frac{\partial S}{\partial x^r} + g^{44} P_4^2 = M_5^2 \tag{233}$$

but g_{44} from the $5D$ Spacetime Ansatz is given by $g_{44} = -\Phi^2$.Hence and since $g^{44} = -1/(\Phi^2)$ the Hamilton-Jacobi Equation as defined by Ponce De Leon is now(See eq 11 pg 6 in [18])[88][89]:

$$g^{qr} \frac{\partial S}{\partial x^q} \frac{\partial S}{\partial x^r} - \frac{1}{\Phi^2} P_4^2 = M_5^2 \tag{234}$$

$$g^{qr} \frac{\partial S}{\partial x^q} \frac{\partial S}{\partial x^r} - \frac{1}{\Phi^2} (\frac{\partial S}{\partial y})^2 = M_5^2 \tag{235}$$

being the $4D$ rest mass m_0 given by(See eq 12 pg 6 in [18]):

$$m_0^2 = g^{qr} \frac{\partial S}{\partial x^q} \frac{\partial S}{\partial x^r} \tag{236}$$

Then the Hamilton-Jaconi equation as defined by Ponce De Leon can now be written as(See eqs 17 and 18 pg 5 in [4]) [90][91][92][93]:

$$m_0^2 - \frac{1}{\Phi^2} (\frac{\partial S}{\partial y})^2 = M_5^2 \tag{237}$$

We already know that $\frac{dS}{M_5} = \frac{ds}{m_0}$.Hence we should expect for:

$$\frac{dS}{ds} = \frac{M_5}{m_0} \tag{238}$$

But $dS^2 = ds^2 - \Phi^2 dy^2$ giving $(dS/ds)^2 = 1 - \Phi^2 (dy/ds)^2 = (M_5/m_0)^2$

From the expressions above we can write the expressions given below:

$$M_5^2 = m_0^2 [1 - \Phi^2 (\frac{dy}{ds})^2] \tag{239}$$

$$m_0^2 = \frac{M_5^2}{1 - \Phi^2 (\frac{dy}{ds})^2} \tag{240}$$

[88]QED:Quod Erad Demonstratum

[89]without conformal factors and spacelike signature for the extra dimension

[90]QED:Quod Erad Demonstratum

[91]spacelike signature for the extra dimension

[92]note that this equation do not have conformal factors

[93]note also that the electric charge is defined as the extra component of the $5D$ Momentum.this agrees with pg 3 in [4]

And finally we arrive at the relation between the rest mass m_0 seen in a $4D$ Spacetime and the rest mass M_5 from the $5D$ Spacetime according to Ponce De Leon(See eq 20 pg 5 in [3],See eq 21 pg 5 in [4],See eq 13 pg 6 in [18]) [94][95][96]

$$m_0 = \frac{M_5}{\sqrt{1-\Phi^2(\frac{dy}{ds})^2}} \tag{241}$$

From the equation above it is now easy to see why two particles with the same rest mass M_5 in a $5D$ Spacetime (or two specimens of the same $5D$ particle) can appear in the $4D$ Spacetime with different rest masses m_0 looking apparently as different particles however the particles seen in $4D$ are different projections or different images of two identical $5D$ particles because each $5D$ particle and each $4D$ image moves with a different $5D$ Spacetime Ansatz dS^2 generating in the $4D$ Spacetime different terms of the form $\sqrt{1-\Phi^2(\frac{dy}{ds})^2}$ each term for each particle. The term $\sqrt{1-\Phi^2(\frac{dy}{ds})^2}$ is the Spacetime Coupling between the $5D$ rest mass M_5 and the $4D$ rest mass m_0.

Now its time to turn back to the Hamilton-Jacobi equation

$$g^{qr}\frac{\partial S}{\partial x^q}\frac{\partial S}{\partial x^r} - \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 = M_5^2 \tag{242}$$

$$m_0^2 - \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 = M_5^2 \tag{243}$$

where the electric charge q is defined as(See eq 18 pg 5 in [4])

$$q = P_4 = -\frac{\partial S}{\partial y} \tag{244}$$

From the equation above we can see that the electric charge seen in a $4D$ Spacetime is obtained purely by the derivative of the Hamilton-Jacobi Action S with respect to the extra dimension. In this case the $4D$ Spscetime electric charge q according to Ponce De Leon is generated by a pure geometric effect originated in the $5D$ Spacetime.

Rewriting the Hamilton-Jacobi equation according to Ponce De Leon as follows(See again eq 17 pg 5 in [4])[97]:

$$g^{qr}\frac{\partial S}{\partial x^q}\frac{\partial S}{\partial x^r} - \frac{q^2}{\Phi^2} = M_5^2 \tag{245}$$

$$m_0^2 - \frac{q^2}{\Phi^2} = M_5^2 \tag{246}$$

we can have a clear perspective about how the $5D$ Spacetime Action $S = S(x^w, y)$ generates in a $4D$ Spacetime the masses and charges of all the Elementary Particles observed.(See pg 6 between eqs 12 and 13 in [18]).

Combining together the Hamilton-Jacobi equation and the relation between the $5D$ rest mass M_5 and the $4D$ rest mass m_0 both as defined by Ponce De Leon we will find the following interesting result:

[94]QED:Quod Erad Demonstratum
[95]spacelike signature for the extra dimension
[96]without conformal factors
[97]QED:Quod Erad Demonstratum

$$m_0^2 - \frac{q^2}{\Phi^2} = M_5^2 \tag{247}$$

$$M_5^2 = m_0^2[1 - \Phi^2(\frac{dy}{ds})^2] \tag{248}$$

$$m_0^2 - \frac{q^2}{\Phi^2} = m_0^2[1 - \Phi^2(\frac{dy}{ds})^2] \tag{249}$$

dividing the expression above by m_0^2 we should expect for:

$$1 - \frac{q^2}{m_0^2\Phi^2} = 1 - \Phi^2(\frac{dy}{ds})^2 \tag{250}$$

$$\frac{q^2}{m_0^2\Phi^2} = \Phi^2(\frac{dy}{ds})^2 \tag{251}$$

$$q^2 = m_0^2\Phi^4(\frac{dy}{ds})^2 \tag{252}$$

$$q = \pm m_0\Phi^2\frac{dy}{ds} \tag{253}$$

but we know that

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{254}$$

Then we have the Ponce De Leon final expression for the electric charge seen in $4D$ Spacetime in function of the $5D$ rest mass M_5(See eq 19 pg 5 in [4])[98][99]

$$q = \pm\frac{M_5\Phi^2\frac{dy}{ds}}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{255}$$

This is another very interesting feature of the formalism developed by Ponce De Leon.Two identical particles in a given $5D$ Spacetime with the same rest-mass M_5 will appear not only with different rest masses m_0 in the $4D$ Spacetime looking apparently as different $4D$ particles or different $4D$ "images" of the same $5D$ particle with each "image" being defined by the each different $4D$ rest-mass m_0 and the differences between the $4D$ "images" are due to the different Spacetime Couplings for each $5D$ particle moving each particle in a different $5D$ Spacetime Ansatz dS^2 but also the electric charge q seen in $4D$ is a function of the $5D$ Spacetime.This means to say that two $5D$ Spacertime identical particles each one with the same rest mass M_5 will appear in the $4D$ Spacetime with different rest masses m_0 as a different $4D$ "images" of the same $5D$ particle but each "image" defined by the $4D$ rest mass m_0 possesses also an electric charge of positive or negative sign generated in $4D$ by the term $\Phi^2(dy/ds)$.Two $4D$ particles with the same $4D$ rest mass m_0 can have two possible values for the $4D$ electric charge:

[98]QED:Quod Erad Demonstratum

[99]we will explain why our result have the $\pm$ sign

- $+m_0\Phi^2\frac{dy}{ds}$
- $-m_0\Phi^2\frac{dy}{ds}$

The result above explains why an Elementary Particle seen in $4D$ with a rest-mass m_0 have an electric charge q of a given sign ($+$ in the case of the quarks u and c and $-$ in the case of the quarks s and b) and for every charged Elementary Particle in $4D$ there exists (also in $4D$) a corresponding Elementary Anti-Particle of equal rest mass m_0 and an electric charge q equal in modulus to the charge q of the corresponding Elementary Particle but opposite signs ($-$ in the case of the anti-quarks u^- and c^- and $+$ in the case of the anti-quarks s^- and b^-)

This leads ourselves to the following combinations:

- positive matter corresponds to negative anti-matter

$$Matter(+) = q(+) = +m_0\Phi^2\frac{dy}{ds} \tag{256}$$

$$AntiMatter(-) = q(-) = -m_0\Phi^2\frac{dy}{ds} \tag{257}$$

- negative matter corresponds to positive anti-matter

$$Matter(-) = q(-) = -m_0\Phi^2\frac{dy}{ds} \tag{258}$$

$$AntiMatter(+) = q(+) = +m_0\Phi^2\frac{dy}{ds} \tag{259}$$

The term $\pm\Phi^2\frac{dy}{ds}$ is known as the Mass to Charge Coupling.It plays between the $4D$ rest mass m_0 and the $4D$ electric charge q a role almost similar to the role played between the $5D$ rest mass M_5 and the $4D$ rest mass m_0 by the Spacetime Coupling.

The scenario described above between matter and anti-matter is by far well-known but however one fundamental question remains:

- What generates this scenario in the first place????

This scenario can be entirely demonstrated mathematically by the formalism developed by Ponce De Leon.

- Here we go:

According to Ponce De Leon our $4D$ Universe lies in the intersection point between two different $5D$ BraneWorld Universes and the intersection point is the $5D$ Extra Dimension y when $y = 0$ .One of these $5D$ BraneWorld Universes is the responsible for the Matter seen in our $4D$ Universe and the other $5D$ BraneWorld Universe is the responsible for the Antimatter seen in our $4D$ Universe.Below are the $5D$ Spacetime Ansatz of two different BraneWorld Universes defined in function of the Extra Dimension y and an affine parameter w as follows(See eq 55 pg 10 in [3]):

$$dS^2 = g_{qr}(x^w, +y)dx^q dx^r - \Phi^2(x^w, +y)dy^2 \curvearrowright y(+) \geq 0 \curvearrowright 5DBraneWorldMatterUniverse \tag{260}$$

$$dS^2 = g_{qr}(x^w, -y)dx^q dx^r - \Phi^2(x^w, -y)dy^2 \curvearrowright y(-) \leq 0 \curvearrowright 5DBraneWorldAntiMatterUniverse \tag{261}$$

Each one of these $5D$ BraneWorld Universes possesses particles of $5D$ rest-mass M_5 and perhaps these $5D$ particles are similar in both Universes.However according to the Ponce De Leon relations between the $5D$ rest-mass M_5 and the $4D$ rest-mass m_0 and the $4D$ electric charge q we have an interesting feature: the $4D$ "image" of one of these $5D$ Universes correspond to the $4D$ matter particles seen in our Universe while the $4D$ "image" of the other $5D$ Universe correspond to the $4D$ anti-matter particles also seen in our Universe and what is more remarkable:all this agrees with the Hamilton-Jacobi equation.

Our Visible $4D$ Universe lies exactly in the point $y = 0$

Each one of these $5D$ BraneWorld Universe defines an Action for the Hamilton-Jacobi equation as shown below(See eq 58 pg 10 in [3]):[100]

$$S(+) = S(x^r, +y) \curvearrowright S(+) = S(x^q, +y) \tag{262}$$

Above is written the Action for the $5D$ BraneWorld Matter Universe

$$S(-) = S(x^r, -y) \curvearrowright S(-) = S(x^q, -y) \tag{263}$$

Above is written the Action for the $5D$ BraneWorld Anti Matter Universe

Using separation of variables for both Actions we have:

$$S(+) = S(x^r, +y) = A(x^r) + B(+y) \curvearrowright S(+) = S(x^q, +y) = A(x^q) + B(+y) \tag{264}$$

Above is the Action for the $5D$ BraneWorld Matter Universe with the $5D$ and $4D$ components separated.

$$S(-) = S(x^r, -y) = A(x^r) + B(-y) \curvearrowright S(-) = S(x^q, -y) = A(x^q) + B(-y) \tag{265}$$

Above is the Action for the $5D$ BraneWorld Anti-Matter Universe with the $5D$ and $4D$ components separated.

From above we can see that the $4D$ part of both Actions $A(x^q)$ or $A(x^r)$ are equal for both BraneWorld Universes.The difference lies in the $5D$ part of both Actions $B(+y)$ and $B(-y)$ responsible for the electric charge (remember that $q = -\frac{\partial S}{\partial y}$).

Considering for example the parts of the Action responsible for the $4D$ rest mass m_0 inside the Hamilton-Jacobi equation for the two $5D$ BraneWorld Universes defined below involving two particles:an electron and a positron lying the electron in the $5D$ BraneWorld Matter Universe $y(+) >= 0$ and the positron lying in the $5D$ BraneWorld Antimatter Universe $y(-) <= 0$ we have(See eq 55 pg 10 in [3]):

[100] the Action of the Hamilton-Jacobi equation is described as a sum.see for example eq 66 pg 11 in [3],or eq 32 pg 11 in [18]

$$dS^2 = g_{qr}(x^w,+y)dx^q dx^r - \Phi^2(x^w,+y)dy^2 \curvearrowright y(+) \geq 0 \curvearrowright Electron \curvearrowright q(+) < 0 \tag{266}$$

$$dS^2 = g_{qr}(x^w,-y)dx^q dx^r - \Phi^2(x^w,-y)dy^2 \curvearrowright y(-) \leq 0 \curvearrowright Positron \curvearrowright q(-) > 0 \tag{267}$$

$$P(+)_q = -1 \times \frac{\partial S(+)}{\partial x^q} = -\frac{\partial A(x^q)}{\partial x^q} \tag{268}$$

$$P(+)_r = -1 \times \frac{\partial S(+)}{\partial x^r} = -\frac{\partial A(x^r)}{\partial x^r} \tag{269}$$

$$P(-)_q = -1 \times \frac{\partial S(-)}{\partial x^q} = -\frac{\partial A(x^q)}{\partial x^q} \tag{270}$$

$$P(-)_r = -1 \times \frac{\partial S(-)}{\partial x^r} = -\frac{\partial A(x^r)}{\partial x^r} \tag{271}$$

$$m_0^2 = g^{qr}\frac{\partial S(+)}{\partial x^q}\frac{\partial S(+)}{\partial x^r} = g^{qr}\frac{\partial A(x^q)}{\partial x^q}\frac{\partial A(x^r)}{\partial x^r} \tag{272}$$

$$m_0^2 = g^{qr}\frac{\partial S(-)}{\partial x^q}\frac{\partial S(-)}{\partial x^r} = g^{qr}\frac{\partial A(x^q)}{\partial x^q}\frac{\partial A(x^r)}{\partial x^r} \tag{273}$$

The result above is very important:it shows that the $4D$ part of the Hamilton-Jacobi equation in both BraneWorld Universes is equal generating equal rest masses m_0. This explains for example why electron and positron have the same $4D$ rest-mass m_0

Looking now to the $5D$ part of the Actions responsible for the electric charge $q = -\frac{\partial S}{\partial y}$

$$P_4(+) = -\frac{\partial S(+)}{\partial y} = -\frac{\partial B(+y)}{\partial y} \curvearrowright y(+) \geq 0 \tag{274}$$

$$P_4(-) = -\frac{\partial S(-)}{\partial y} = -\frac{\partial B(-y)}{\partial y} \curvearrowright y(-) \leq 0 \tag{275}$$

We can clearly see that the part of the Action responsible for the charge in the $5D$ BraneWorld Matter Universe is equal in modulus but have an opposite sign when compared to the part of the Action responsible for the charge in the $5D$ BraneWorld Anti Matter Universe

Or in other words:

$$B(+y) = -B(-y) \curvearrowright B(-y) = -B(+y) \tag{276}$$

And this implies in

$$q_4(+) = -\frac{\partial B(+y)}{\partial y} = -\frac{\partial(-B(-y))}{\partial y} = \frac{\partial B(-y)}{\partial y} = -q_4(-) \tag{277}$$

$$q_4(-) = -\frac{\partial B(-y)}{\partial y} = -\frac{\partial(-B(+y))}{\partial y} = \frac{\partial B(+y)}{\partial y} = -q_4(+) \tag{278}$$

From above we can see that if $q(+)$ is the charge of the electron resulting in a $q(+) < 0$ then $q(-)$ will have the same modulus but opposite signs resulting in a $q(-) > 0$ for the positron

Again using the example of the electron

$q_4(+) < 0$ then according to the Ponce De Leon $5D$ to $4D$ mass-to-charge relation:

$$q_4(+) = -\frac{M_5 \Phi^2 \frac{dy}{ds}}{\sqrt{1 - \Phi^2 (\frac{dy}{ds})^2}} \tag{279}$$

$$q_4(+) = -m_0 \Phi^2 \frac{dy}{ds} \tag{280}$$

We know that both the electron and the positron have the same $5D$ rest mass M_5 and the same $4D$ rest mass m_0

But Φ is the Scalar Field and looking back to the definition of the dS^2 in both $5D$ BraneWorld Universes we have:

$$\Phi^2(x^w, +y) \curvearrowright y(+) >= 0 \tag{281}$$

Above is the square of Scalar Field for the $5D$ BraneWorld Matter Universe

$$\Phi^2(x^w, -y) \curvearrowright y(-) <= 0 \tag{282}$$

Above is the square of the Scalar Field for the $5D$ BraneWorld Anti Matter Universe

Again using separation of variables we have:[101]

$$\Phi(x^w, +y) = U(x^w) V(+y) \tag{283}$$

$$\Phi(x^w, -y) = U(x^w) V(-y) \tag{284}$$

From above and in a similar situation compared to the Action S for the Hamilton-Jacobi equation,the $4D$ part of each Scalar Field is equal in both $5D$ BraneWorld Universes and the difference between Scalar Fields in each BraneWorld Universe lies exclusively in the $5D$ part of each Scalar Field.Hence we can clearly see that

$$V(-y) = -V(+y) \curvearrowright V(+y) = -V(-y) \tag{285}$$

then we have:

$$\Phi^2(x^w, +y) = U^2(x^w) V^2(+y) \tag{286}$$

$$\Phi^2(x^w, -y) = U^2(x^w) V^2(-y) \tag{287}$$

implying directly in:

[101] while the Action of the Hamilton-Jacobi equation in separation of variables is a sum the Scalar Field in separation of variables is a product.see for example eq 132 pg 19 in [6]

$$V^2(-y) = (-V(+y))^2 \curvearrowright V^2(+y) = (-V(-y))^2 \quad (288)$$

$$V^2(-y) = V^2(+y) \quad (289)$$

$$\Phi^2(x^w, +y) = \Phi^2(x^w, -y) \quad (290)$$

The square of the Scalar Field for the electron and the positron are exactly equal in both $5D$ BraneWorld Universes.Examining again the $4D$ equation of the electron charge:

$$q_4(+) = -m_0 \Phi^2 \frac{dy}{ds} \quad (291)$$

From above Φ^2 and m_0 are the same for the electron and the positron.Then the difference that generates two different charges of equal modulus and opposite signs in the $4D$ Universe according to

$$q = \pm m_0 \Phi^2 \frac{dy}{ds} \quad (292)$$

or even better for our example[102]:

$$q_4(\pm) = \mp m_0 \Phi^2 \frac{dy}{ds} \quad (293)$$

must reside in the term $\frac{dy}{ds}$

Note that from the equation above we can extract the equations of the electric charges $\mp q$ of both the electron and the positron as shown below:

- electron:

$$q_4(+) = -m_0 \Phi^2 \frac{dy(+)}{ds} \quad (294)$$

- positron:

$$q_4(-) = m_0 \Phi^2 \frac{dy(-)}{ds} \quad (295)$$

The electron is located in a $5D$ Spacetime where $y(+) >= 0$ and then $\frac{dy(+)}{ds} >= 0$[103] while the positron is located in a $5D$ Spacetime where $y(-) =< 0$ and then $\frac{dy(-)}{ds} <= 0$[104].

Note that the term $\frac{dy}{ds}$ for the positron is exactly the same for the electron multiplied by -1 and vice versa.Then:

[102] note the difference between $\pm$ and $\mp$.we defined the electron lying in the $5D$ Matter BraneWorld Universe $y(+) >= 0$ with a $q(+) < 0$ and the positron lying in the $5D$ BraneWorld AntiMatter Universe $y(-) <= 0$ with a $q(-) > 0$

[103] assuming linear displacement in y

[104] assuming again linear displacement in y

$$\frac{dy(+)}{ds} = -\frac{dy(-)}{ds} \tag{296}$$

$$\frac{dy(-)}{ds} = -\frac{dy(+)}{ds} \tag{297}$$

Again using the equation of the electron:

$$q_4(+) = -m_0\Phi^2\frac{dy(+)}{ds} \tag{298}$$

$$q_4(+) = -m_0\Phi^2(-\frac{dy(-)}{ds}) \tag{299}$$

$$q_4(+) = m_0\Phi^2\frac{dy(-)}{ds} \tag{300}$$

Note that we inserted in the equation of the electron the term $\frac{dy(-)}{ds}$ of the positron and since $m_0\Phi^2$ are the same for both particles the equation above no longer represents the electron because the motion now occurs in the $5D$ Universe $y(-) <= 0$.Then:

$$q_4(-) = m_0\Phi^2\frac{dy(-)}{ds} \tag{301}$$

And this agrees with our previous equation for the positive charge of the positron.

- Case 2)- particles in a Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$ with a $5D$ rest-mass $M_5 = 0$ giving a $4D$ rest-mass $m_0 = 0$

We have seen so far the case of particles in a Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ with a $5D$ rest-mass $M_5 > 0$ giving a $4D$ rest-mass $m_0 > 0$. But what happens if the $5D$ Spacetime Ansatz dS^2 becomes Null-Like which means to say $dS^2 = 0$???

The first thing to take in mind is the fact that a Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ always require a $5D$ rest-mass M_5 different than 0 otherwise dS/M_5 with $dS > 0$ and $M_5 = 0$ would produce an invalid result.

Then we cannot have $5D$ particles in a Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ with a null $5D$ rest-mass $M_5 = 0$.

On the other hand if the $5D$ Spacetime Ansatz dS^2 becomes Null-Like $dS^2 = 0$ then the Ponce De Leon relation between the $5D$ rest-mass M_5 and the $4D$ rest-mass m_0 will require a zero $5D$ rest-mass M_5 otherwise since $dS^2 = 0$ then $ds^2 = \Phi^2 dy^2$ and $1 = \Phi^2(dy/ds)^2$. This will generate a zero Spacetime Coupling $\sqrt{1 - \Phi^2(dy/ds)^2} = 0$ and since according to Ponce De Leon $m_0 = M_5/\sqrt{1 - \Phi^2(dy/ds)^2}$ if $M_5 > 0$ and $\sqrt{1 - \Phi^2(dy/ds)^2} = 0$ we would get an invalid result for m_0

Then a Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$ always require a $5D$ rest-mass $M_5 = 0$
Rewriting the Hamilton-Jacobi equation according to Ponce de Leon for the case of a zero $5D$ rest mass $M_5 = 0$ as follows:

$$g^{qr}\frac{\partial S}{\partial x^q}\frac{\partial S}{\partial x^r} - \frac{q^2}{\Phi^2} = 0 \tag{302}$$

$$m_0^2 - \frac{q^2}{\Phi^2} = 0 \tag{303}$$

We will obtain the following result(See eq 24 pg 6 in [4]):

$$m_0^2 = \frac{q^2}{\Phi^2} \tag{304}$$

$$m_0 = \pm\frac{q}{\Phi} \tag{305}$$

$$q = \pm m_0\Phi \tag{306}$$

The two signs for the electric charge above are being generated by the term $1 = \Phi^2(dy/ds)^2$ or $1 = \pm\Phi(dy/ds)$

Or even better(See pg 6 after eq 24 in [4]):

$$\Phi(dy/ds) = \pm 1 \tag{307}$$

$$\frac{dy}{ds} = \pm\frac{1}{\Phi} \tag{308}$$

Note that like in the previous case the expression above encompasses the $5D$ BraneWorld Matter Universe for the electron with $y(+) >= 0$ and the $5D$ BraneWorld Antimatter Universe for the positron with $y(-) <= 0$.

But we know that according to Ponce De Leon $q = -\frac{\partial S}{\partial y}$.Then we can write the following expression for the $4D$ rest mass m_0 generated from a Null Like $5D$ Ansatz $dS^2 = 0$ as follows(See eq 27 pg 6 in [3]):

$$m_0 = \pm\frac{1}{\Phi}\frac{\partial S}{\partial y} \tag{309}$$

From above we have the following expressions for the $4D$ rest-mass m_0 in a Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$[105]

$$m_0 = \frac{1}{\Phi}\frac{\partial S}{\partial y} \curvearrowright y(+) >= 0 \tag{310}$$

$$m_0 = -\frac{1}{\Phi}\frac{\partial S}{\partial y} \curvearrowright y(-) <= 0 \tag{311}$$

And both provides always a positive m_0 which means to say that in a Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$ the rest-mass m_0 seen in $4D$ is obtained purely by the derivative of the Hsmilton-Jacobi Action with respect to the extra dimension as a pure geometrical effect originated in the $5D$ Spacetime and the $4D$ electric charge q is also generated by the same geometric effect originated in the $5D$

[105] note that the minus sign in the $y(-) <= 0$ cancels with the minus sign giving a positive m_0 in this case.

Again back to the Hamilton-Jacobi equation according to Ponce De Leon as follows:

$$g^{qr}\frac{\partial S}{\partial x^q}\frac{\partial S}{\partial x^r} - \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 = 0 \tag{312}$$

$$g^{qr}\frac{\partial S}{\partial x^q}\frac{\partial S}{\partial x^r} = \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 \tag{313}$$

We will obtain this interesting result: [106]

$$\sqrt{g^{qr}\frac{\partial S}{\partial x^q}\frac{\partial S}{\partial x^r}} = \pm\frac{1}{\Phi}(\frac{\partial S}{\partial y}) \tag{314}$$

For diagonalized metrics we have:[107]

$$g^{rr}\frac{\partial S}{\partial x^r}\frac{\partial S}{\partial x^r} - \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 = 0 \tag{315}$$

$$g^{rr}(\frac{\partial S}{\partial x^r})^2 - \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 = 0 \tag{316}$$

$$g^{rr}(\frac{\partial S}{\partial x^r})^2 = \frac{1}{\Phi^2}(\frac{\partial S}{\partial y})^2 \tag{317}$$

$$\sqrt{g^{rr}}(\frac{\partial S}{\partial x^r}) = \pm\frac{1}{\Phi}(\frac{\partial S}{\partial y}) \tag{318}$$

$$\sqrt{g^{rr}}(\frac{\partial y}{\partial x^r}) = \pm\frac{1}{\Phi} \tag{319}$$

$$\Phi = \pm\sqrt{g_{rr}}(\frac{\partial x^r}{\partial y}) \tag{320}$$

And at least we got for the Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$ in a diagonalized metric a set of valid expressions for the Scalar Field Φ.One of these expressions corresponds to the $5D$ BraneWorld Matter Universe:

$$\Phi = \sqrt{g_{rr}}(\frac{\partial x^r}{\partial y}) \curvearrowright y(+) >= 0 \tag{321}$$

While the other corresponds to the $5D$ BraneWorld Antimatter Universe:[108]

$$\Phi = -\sqrt{g_{rr}}(\frac{\partial x^r}{\partial y}) \curvearrowright y(-) <= 0 \tag{322}$$

In order to terminate this second case:we are now left with two different expressions for the Hamilton-Jacobi equation according to Ponce De Leon

[106]note the signs $\pm$
[107]the signs $\pm$ appears again
[108]again we assume a linear displacement with respect to y in order to use the minus sign of y to cancel the minus sign in the beginning of the expression giving a positive Scalar Field.The square of both expressions must match as we have seen before

$$m_0^2 - \frac{q^2}{\Phi^2} = M_5^2 \quad (323)$$

$$m_0^2 - \frac{q^2}{\Phi^2} = 0 \quad (324)$$

In the end of this section we provide a Table of Elementary Particles.Note that all the particles possessing an electric charge always possesses a rest mass.We can have particles of zero $4D$ rest mass $m_0 = 0$(eg photons) but these particles will always have a null electric charge $q = 0$.We cannot have a particle with zero 4D rest-mass and a non-null electric charge.This is one of the most important consequences of the Hamilton-Jacobi equation according to Ponce De Leon formalism.

- Case 3)- particles in a Spacelike $5D$ Spacetime Ansatz $dS^2 < 0$ with a $5D$ rest-mass $M_5 < 0$ giving a $4D$ rest-mass $m_0 > 0$

We already know that $dS/M_5 = ds/m_0$ then since the $4D$ rest-mass m_0 is always positive[109] we must "always" have a negative $5D$ rest mass $M_5 < 0$ since $dS < 0$ in order to make the term dS/M_5 "always" positive.Also note that the $4D$ Spacetime Ansatz ds^2 is "always" Timelike or Null-Like.

- Lastly we would like to discuss a fundamental question:Why does the electron annihilates with the positron?'Why each particle annihilates with its own antiparticle counterpart??

- Considering the case 1)- particles in a Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ with a $5D$ rest-mass $M_5 > 0$ giving a $4D$ rest-mass $m_0 > 0$:

We already know that the equations relating the $4D$ rest-mass m_0 and the $4D$ electric charge q according to Ponce de Leon for a particle and its antiparticle counterpart are given by:

$$q_4(+) = -m_0 \Phi^2 \frac{dy(+)}{ds} \curvearrowright q_4(+) < 0 \quad (325)$$

$$q_4(-) = m_0 \Phi^2 \frac{dy(-)}{ds} \curvearrowright q_4(-) > 0 \quad (326)$$

Imagine that one particle and its antiparticle counterpart collides:Both are travelling in two different Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ each one for each particle.Although we defined the antiparticles moving in the $5D$ BraneWorld Universe $y(-) <= 0$ remember that $y(-)^2 >= 0$ and consequently $dy^2 >= 0$ and the square of the Scalar Fields is the same.Both $5D$ Spacetime Ansatz seems to be equal however "they" are not.Both particles and antiparticles share the same $4D$ Spacetime Universe and the same $4D$ rest-mass m_0 because according to the nature of the Hamilton-Jacobi equation and the formalism of Ponce De Leon two equal $5D$ rest masses M_5 but however located in two different $5D$ BraneWorld Universes are being projected into the same $4D$ Spacetime.
Suppose that our electron collides with our positron:we have now the following situations:

[109]we do not consider here exotic matter

- Sum of the charges:Both particles possesses charges of equal modulus but opposite signs.In the collision both charges enters in contact with each other.Consequantly one charge will cancel the other.Then we should expect for:

$$q_4(+) + q_4(-) = 0 \tag{327}$$

- Sum of the masses:using the equation above we have:

$$q_4(+) + q_4(-) = -m_0\Phi^2\frac{dy(+)}{ds} + m_0\Phi^2\frac{dy(-)}{ds} = 0 \tag{328}$$

But we know that both particles share the same term $m_0\Phi^2$.Hence it seems to be legitimate to write:

$$q_4(+) + q_4(-) = m_0\Phi^2(-\frac{dy(+)}{ds} + \frac{dy(-)}{ds}) = 0 \tag{329}$$

But we also know that:

$$\frac{dy(+)}{ds} = -\frac{dy(-)}{ds} \tag{330}$$

$$\frac{dy(-)}{ds} = -\frac{dy(+)}{ds} \tag{331}$$

Then we would have two situations:

$$q_4(+) + q_4(-) = m_0\Phi^2(\frac{dy(-)}{ds} + \frac{dy(-)}{ds}) = 0 \tag{332}$$

$$q_4(+) + q_4(-) = m_0\Phi^2 \times 2 \times (\frac{dy(-)}{ds}) = 0 \tag{333}$$

$$q_4(+) + q_4(-) = m_0\Phi^2(-\frac{dy(+)}{ds} - \frac{dy(+)}{ds}) = 0 \tag{334}$$

$$q_4(+) + q_4(-) = m_0\Phi^2 \times 2 \times (-\frac{dy(+)}{ds}) = 0 \tag{335}$$

Note that in order to produce a total charge of the system electron-positron equal to zero the term $m_0\Phi^2 \times 2$ [110]must also becomes equal to zero.Hence the total mass of the system electron-positron according to the Hamilton-Jacobi equation will be zero.This leads us to an important conclusion:

- A zero $4D$ rest-mass requires a Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$.The total rest-mass of the electron-positron $m_0 = 0$ seen in $4D$ is the mass of the observed photon that will appear in the collision.Then in the collision the electron-positron system changes the geometry from two different and independent Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ to a single one and unified Null-Like $5D$ Spacetime Ansatz $dS^2 = 0$

[110]the sum of two equal $4D$ rest-masses m_0 one for the electron and the other for the positron

We will terminate this Section with two fundamental questions(and possible answers):

- 1)-Why we have in our $4D$ Universe two "kinds" of "matter" for non-zero rest-mass particles:(Matter and Antimatter) and not a third one???
- 2)-Why Matter prevails over Antimatter and not the inverse????

This picture of two different $5D$ BraneWorld Universes one for Matter and the another for Antimatter suggest us that perhaps the Big Bang was a "shock-wave",a collision between two different "plane waves" in $5D$ that generated $13,7$ billions of years ago what we know as the $4D$ Big Bang.(See abstract of [16] and abstract of and page 2 of [17].Note that in the last one it is mentioned explicitly the $4D$ Brane as the "plane" of the collision between two different $5D$ "plane waves" propagating in opposite directions along the Extra Dimension.)

A collision between two different $5D$ BraneWorld Universes is pictured below:(See eqs 23 and 24 pg 5 in [17])

$$d\mathcal{S}^2 = n^2(t+\lambda y)dt^2 - a^2(t+\lambda y)\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] - \Phi^2(t+\lambda y)dy^2, \tag{336}$$

$$d\mathcal{S}^2 = n^2(t-\lambda y)dt^2 - a^2(t-\lambda y)\left[\frac{dr^2}{(1-kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] - \Phi^2(t-\lambda y)dy^2, \tag{337}$$

The $5D$ BraneWorld Universe $y(+) > 0$ represents the Matter in our $4D$ Universe and the $5D$ BraneWorld Universe $y(-) < 0$ represents the Antimatter in our $4D$ Universe.According to Ponce de Leon they can be interpreted as plane-waves propagating in "opposite" directions along the fifth dimension, and colliding at $y = 0.(k = -1, 0, +1)$.

- If the Big Bang was a collision between two different $5D$ BraneWorlds $13,7$ billions of years ago then we can easily figure out that:
- Although both these $5D$ Universes possessed the same kind of $5D$ rest-mass M_5, one of these $5D$ Universes was more massive than the other.In this case the $5D$ Matter Universe M_5 that generates the $4D$ rest-masses m_0 for the electron and not for the positron.This can be the reason why Matter prevailed over Antimatter
- The reason why we have two "kinds" of matter seen in our $4D$ Universe is due to the fact that it was a collision between two $5D$ Universes of the same kind of $5D$ rest-mass and not a collision between three of four $5D$ Universes with different kinds of $5D$ rest-mass

Below there is presented a Chart of Elementary Particles.Note that all the Elementary Particles known always possesses a positive $4D$ rest mass m_0:Examining carefully the Chart using the Ponce De Leon equations of mass and charge:

$$m_0 = \frac{M_5}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{338}$$

$$q = \pm \frac{M_5 \Phi^2 \frac{dy}{ds}}{\sqrt{1 - \Phi^2(\frac{dy}{ds})^2}} \tag{339}$$

Particle	spin ($\hbar$)	B	L	T	T_3	S	C	B*	charge (e)	m_0 (MeV)	antipart.
u	1/2	1/3	0	1/2	1/2	0	0	0	+2/3	5	$\overline{\mathrm{u}}$
d	1/2	1/3	0	1/2	−1/2	0	0	0	−1/3	9	$\overline{\mathrm{d}}$
s	1/2	1/3	0	0	0	−1	0	0	−1/3	175	$\overline{\mathrm{s}}$
c	1/2	1/3	0	0	0	0	1	0	+2/3	1350	$\overline{\mathrm{c}}$
b	1/2	1/3	0	0	0	0	0	−1	−1/3	4500	$\overline{\mathrm{b}}$
t	1/2	1/3	0	0	0	0	0	0	+2/3	173000	$\overline{\mathrm{t}}$
e^-	1/2	0	1	0	0	0	0	0	−1	0.511	e^+
μ^-	1/2	0	1	0	0	0	0	0	−1	105.658	μ^+
τ^-	1/2	0	1	0	0	0	0	0	−1	1777.1	τ^+
ν_e	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_e$
ν_μ	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_\mu$
ν_τ	1/2	0	1	0	0	0	0	0	0	0(?)	$\overline{\nu}_\tau$
γ	1	0	0	0	0	0	0	0	0	0	γ
gluon	1	0	0	0	0	0	0	0	0	0	$\overline{\mathrm{gluon}}$
W^+	1	0	0	0	0	0	0	0	+1	80220	W^-
Z	1	0	0	0	0	0	0	0	0	91187	Z
graviton	2	0	0	0	0	0	0	0	0	0	graviton

We can easily see that:

- 1)-We can have a set of $5D$ Quarks all of them with the same given rest-mass M_5 in a given $5D$ Spacetime generating as $4D$ Spacetime "images" all the six $4D$ Quarks each one with its own $4D$ rest-mass m_0 because the same $5D$ rest-mass M_5 each one for each $5D$ Quark is being divided by different Spacetime Couplings each one for each $4D$ Quark
- 2)-The group of Leptons in $5D$ corresponds to two $5D$ set of particles.One for the Electron-Muon Group and the other for the Neutrino Group in a situation similar to the one described for Quarks.Both moves in Timelike $5D$ Spacetime Ansatz $dS^2 > 0$ however in the Ansatz for the Neutrino Group the derivative of the Hamilton-Jacobi Action with respect to the extra coordinate is zero .
- 3)-As pointed before all the charged particles possesses mass
- 4)-particle Z like the Neutrino Group is stationary in the $5D$ Spacetime

4 The Structure of a BraneWorld Star according to the Metric of Dadhich,Maartens,Papadopolous and Rezania

If the $5D$ Extra Dimension really exists then Bulk Stresses(non-local) effects from the $5D$ Spacetime Bulk Weyl Tensor projected onto the $4D$ Spacetime(the Brane) generates a Tidal Charge Q of Reissner-Nordstrom type but always with negative sign.(see pg 2 in [25]).Then we can say that the Tidal Charge Q is the projection of the effects of the $5D$ Spacetime Stresses into the $4D$ Spacetime one.(see abstract of [10]).
These Weyl Stresses arises from the projection into the $4D$ Spacetime(Brane) of the the $5D$ Spacetime Bulk Weyl Tensor responding non-locally to the gravitational field inside the Brane therefore making "backreactions" on the Brane itself.(see pg 3 before Section III in [27]).Still these non-local Stresses from the $5D$ Spacetime Bulk Weyl Tensor leads to an Energy Momentum Tensor inside the $4D$ Spacetime(Brane) that have the mathematical form of an Electric Field but actually there are no "Real" Electric Fields being present(see pg 4 before eq 33 in [27]).

The exterior[111] solution describing a Higher Dimensional BraneWorld Star with a Tidal Charge Q and Reissner-Nordstrom type Spacetime Metric is given by the following equation:(see eqs 7 and 8 pg 5 in [25],eq 7 pg 5 in [10],eqs 1,2 pg 2 in [22],eqs 1,2 pg 2 in [21],eq 33 pg 4 in [27]) We adopted here the definitions of [25]

$$ds^2 = A(r)\mathrm{c}^2 dt^2 - A^{-1}(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \tag{340}$$

$$A(r) = 1 - \frac{2r_{\mathrm{G}}}{r} + \frac{Q}{r^2}; \qquad r_{\mathrm{G}} = \frac{\mathrm{G}M}{\mathrm{c}^2}, \tag{341}$$

The parameter r_G will be very useful in the forthcoming calculations.The value of r_G for the Sun is given by:(see eq 7.57 pg 187 in [9])

$$r_{G\odot} = \frac{GM_\odot}{c^2} = 1.48 \times 10^3 \text{ m} \tag{342}$$

The Schwarzschild Radius r_S will also be very useful and is defined by:(see end of pg 191 and top of 192 in [9])

$$r_S = \frac{2GM}{c^2} \tag{343}$$

For the Sun the Schwarzschild Radius is given by:

$$r_{S\odot} = \frac{2GM_\odot}{c^2} = 2.96 \times 10^3 \text{ m} \tag{344}$$

The Tidal Charge Q can be written in function of the star mass M,radius R,density ρ and tension λ. It can also be written in function of the Schwarzschild Radius r_S and parameter r_G.Our analysis considers R,r_S and r_G to illustrate an important question:What happens to the Tidal Charge Q when in a Gravitational Collapse the BraneWorlds Star reaches the Schwarzschild Radius r_S?

The equations for the Tidal Charge Q according to the parameters defined above are given by:(see eq 9 combined with eq 8 pg 5 in [25],pg 7 between eqs 19 and 20 in [10],eq 35 pg 4 in [27])

$$Q = -\frac{3\mathrm{G}M}{\mathrm{c}^2} R \left(\frac{\rho}{\lambda}\right), \curvearrowright Q = -3r_G R\left(\frac{\rho}{\lambda}\right), \tag{345}$$

[111] we consider only exterior solutions.see pg 3 before Section III in [27]

The equation above is valid only for BraneWorld Stars of uniform (or constant)density.(see pg 7 between eqs 19,20 in [10],pg 5 between eqs 6,7 in [25],pg 1,pg 3 Section III and pg 4 between eqs 29,30 in [27])

While the mass M,radius R and density ρ of a star are known we must calculate the tension λ.The equations are given by:(see eq 13 combined with eq 8 pg 5 in [25],eq 21 pg 7 in [10],eq 29 pg 4 in [27])[112]

$$\lambda \geq \left(\frac{\mathrm{G}M/\mathrm{c}^2}{R - 2\mathrm{G}M/\mathrm{c}^2} \right) \rho \, , \curvearrowright \lambda \geq \left(\frac{r_G}{R - r_S} \right) \rho \, , \tag{346}$$

From the equations above we can obtain these two very important results for our forthcoming analysis:the parameters n and N.

$$\lambda = n \left(\frac{r_G}{R - r_S} \right) \rho \, , \curvearrowright n \geq 1 \tag{347}$$

$$\frac{1}{\lambda} = \frac{1}{n\rho \left(\frac{r_G}{R - r_S} \right)} = \frac{1}{n\rho} \frac{R - r_S}{r_G} = \frac{N}{\rho} \frac{R - r_S}{r_G} \curvearrowright N = \frac{1}{n} \curvearrowright N \leq 1 \tag{348}$$

$$\frac{\rho}{\lambda} = \frac{1}{n \left(\frac{r_G}{R - r_S} \right)} = \frac{1}{n} \frac{R - r_S}{r_G} = N \frac{R - r_S}{r_G} \curvearrowright N = \frac{1}{n} \curvearrowright N \leq 1 \tag{349}$$

Germany-Maartens outines that we recover ordinary $4D$ General Relativity when $\frac{1}{\lambda} = 0$.In this case the Tidal Charge Q vanishes and we recover the ordinary Schwarzschild Solution.This tension λ is generated by the $5D$ Spacetime Bulk Weyl stresses and these stresses are associated with the parameter N.When $N = 0$,$Q = 0$[113] then $\frac{1}{\lambda} = 0$ and we recover the ordinary $4D$ Schwarzschild solution.(see pg 3 after eq 28 in [27])

Now we can present our equations for the Tidal Charge Q defined using only the star radius R,Schwarzschild Radius r_S and the parameters r_G and N.

$$Q = -3r_G R \left(\frac{\rho}{\lambda} \right) = -3r_G R \left(\frac{1}{n} \frac{R - r_S}{r_G} \right) = -3r_G R \left(N \frac{R - r_S}{r_G} \right) , \tag{350}$$

If we drop the parameter r_G we will get an expression for the Tidal Charge Q using only the star radius R,Schwarzschild Radius r_S and the parameter N.

$$Q = -3r_G R \left(N \frac{R - r_S}{r_G} \right) , \curvearrowright Q = -3NR \left(R - r_S \right) , \tag{351}$$

This always implies in $R > 2\mathrm{G}M/\mathrm{c}^2$, or $R > r_S$ i.e., the Schwarzschild radius is still the relevant limit as in general relativity.(see pg 5 in [25]).

We can now proceed with our next move:to calculate the Tidal Charge $Q_\odot$ for our Sun and this result will be very useful in Section 5.

For our Sun we can write the following equations:

$$\lambda_\odot \geq \left(\frac{\mathrm{G}M_\odot/\mathrm{c}^2}{R_\odot - 2\mathrm{G}M_\odot/\mathrm{c}^2} \right) \rho_\odot \, , \curvearrowright \lambda_\odot \geq \left(\frac{r_{G_\odot}}{R_\odot - r_{S_\odot}} \right) \rho_\odot \, , \tag{352}$$

[112]note that in all the references the uniform density of the star is again outlined with the given equations

[113]the star mass M,radius R,density ρ will never be equal to zero and we do not consider Singularities here.Then in order to make $\frac{1}{\lambda} = 0$ we must really have $N = 0$

$$\lambda_\odot = n_\odot \left(\frac{r_{G_\odot}}{R_\odot - r_{S_\odot}}\right) \rho_\odot \, , \curvearrowright n_\odot \geq 1 \tag{353}$$

The mass of the Sun is given by:

$$M_\odot = 1,9891 \times 10^{30} kg \tag{354}$$

and the radius by:

$$R_\odot \cong 1,3 \times 10^9 m \tag{355}$$

Now we can calculate the Sun density as follows:

$$\rho_\odot = \frac{3M_\odot}{4\pi R_\odot^3} = \frac{3 \times 1,9891 \times 10^{30} kg}{4 \times 3,1415926536 \times 2,197 \times 10^{27} m^3} \tag{356}$$

$$\rho_\odot = \frac{3M_\odot}{4\pi R_\odot^3} = \frac{5,9673 \times 10^{30} kg}{2,76083162398368 \times 10^{28} m^3} \tag{357}$$

$$\rho_\odot = \frac{3M_\odot}{4\pi R_\odot^3} = 2,16141395518703 \times 10^2 kg/m^3 \tag{358}$$

Inserting the values of the Sun density $\rho_\odot$,radius $R_\odot$,Schwarzschild Radius $r_{S_\odot}$ and parameter $r_{G_\odot}$ we can compute the value of the Sun tension $\lambda_\odot$ however we still need to determine the values of the Sun parameters $n_\odot$ and $N_\odot$ according to the calculations shown below:

$$\lambda_\odot = n_\odot \times \frac{1.48 \times 10^3 m}{1,29999740 \times 10^9 m} \times 2,16141395518703 \times 10^2 kg/m^3 \tag{359}$$

$$\lambda_\odot = n_\odot \times 1,138463815389 \times 10^{-6} \times 2,16141395518703 \times 10^2 kg/m^3 \tag{360}$$

$$\lambda_\odot = n_\odot \times 2,46069157805 \times 10^{-4} kg/m^3 \tag{361}$$

$$\frac{\lambda_\odot}{n_\odot} = 2,46069157805 \times 10^{-4} kg/m^3 \tag{362}$$

$$\frac{\rho_\odot}{\lambda_\odot} = \frac{1}{n_\odot} \frac{R_\odot - r_{S_\odot}}{r_{G_\odot}} = N_\odot \frac{R_\odot - r_{S_\odot}}{r_{G_\odot}} \tag{363}$$

$$\frac{\rho_\odot}{\lambda_\odot} = \frac{1}{n_\odot} \frac{1,29999740 \times 10^9 m}{1.48 \times 10^3 \text{ m}} = N_\odot \frac{1,29999740 \times 10^9 m}{1.48 \times 10^3 \text{ m}} \tag{364}$$

$$\frac{\rho_\odot}{\lambda_\odot} = \frac{1}{n_\odot} \times 8,7837662 \times 10^5 = N_\odot \times 8,7837662 \times 10^5 \tag{365}$$

$$\rho_\odot = \frac{\lambda_\odot}{n_\odot} \times 8,7837662 \times 10^5 = \lambda_\odot N_\odot \times 8,7837662 \times 10^5 \tag{366}$$

From the computations above it seems that the Sun parameters $n_\odot$ and $N_\odot$ leads us to a "dead-end" way but however there exists a solution and we will present it below:

Assuming the our Sun is a constant density BraneWorld Star we can use the following equation for the tension λ(see eq 28 pg 9 in [10])[114]

$$\lambda \geq \frac{3GM}{c^2}\frac{\rho}{R\Delta_{LD}} . \curvearrowright \lambda \geq 3r_G\frac{\rho}{R\Delta_{LD}} \curvearrowright \Delta_{LD} \leq 1,7 \times 10^{-4} \tag{367}$$

According to Bohmer-Harko-Lobo the parameter $\Delta_{LD} \leq 1,7 \times 10^{-4}$ is an Universal quantity that gives the absolute deviation from standard $4D$ General Relativity.(see top of pg 9 and also pg 9 before Section 3.3 in [10]). Adapting the equation above for our Sun we should expect for:

$$\lambda_{\odot} \geq 3r_{G_{\odot}}\frac{\rho_{\odot}}{R_{\odot}\Delta_{LD}} . \tag{368}$$

We will consider for this equation the condition of equality given below:[115]

$$\lambda = \frac{3GM}{c^2}\frac{\rho}{R\Delta_{LD}} . \curvearrowright \lambda = 3r_G\frac{\rho}{R\Delta_{LD}} \curvearrowright \Delta_{LD} = 1,7 \times 10^{-4} \tag{369}$$

Equalizing both equations already obtained for the Sun tension $\lambda_{\odot}$:

$$\lambda_{\odot} = 3r_{G_{\odot}}\frac{\rho_{\odot}}{R_{\odot}\Delta_{LD}} . = n_{\odot}\left(\frac{r_{G_{\odot}}}{R_{\odot} - r_{S_{\odot}}}\right)\rho_{\odot} \tag{370}$$

Looking to the equality above we are now in position to compute the values of the parameters $n_{\odot}$ and $N_{\odot}$.
Therefore we have:

$$\lambda_{\odot} = 3 \times 1.48 \times 10^3 \text{ m} \times \frac{2,16141395518703 \times 10^2 kg/m^3}{1,3 \times 10^9 m \times 1,7 \times 10^{-4}} . \tag{371}$$

$$\lambda_{\odot} = 3 \times 1.48 \times 10^3 \text{ m} \times \frac{2,16141395518703 \times 10^2 kg/m^3}{2,47 \times 10^5 m} . \tag{372}$$

$$\lambda_{\odot} = 3 \times 1.48 \times 10^3 \text{ m} \times 8,7616985905680 \times 10^{-4} kg/m^{-3} \tag{373}$$

At last we arrived to the final value of the Sun tension $\lambda_{\odot}$

$$\lambda_{\odot} = 3,890194742 kg/m^3 = n_{\odot} \times 2,46069157805 \times 10^{-4} kg/m^3 \tag{374}$$

And finally we have the values for the parameters $n_{\odot}$ and $N_{\odot}$

$$n_{\odot} = \frac{3,890194742 kg/m^3}{2,46069157805 \times 10^{-4} kg/m^3} \tag{375}$$

$$n_{\odot} = 1,5809855202015 \times 10^4 \curvearrowright N_{\odot} = 6,3251686 \times 10^{-3} \tag{376}$$

Finally we will obtain the numerical value of the Sun Tidal Charge $Q_{\odot}$

$$Q_{\odot} = -3N_{\odot}R_{\odot}\left(R_{\odot} - r_{S\odot}\right), \tag{377}$$

[114] note that before the equation the need of a constant density star is again outlined
[115] because we already have the parameters $n_{\odot}$ and $N_{\odot}$ and more inequalities would lead to more parameters

$$Q_{\odot} = -3 \times N_{\odot} \times 1,3 \times 10^{9}m \times \left(1,3 \times 10^{9}m - 2,96 \times 10^{3}m\right), \tag{378}$$

$$Q_{\odot} = -3 \times N_{\odot} \times 1,3 \times 10^{9}m \times 1,29999740 \times 10^{9}m, \tag{379}$$

$$Q_{\odot} = -3 \times N_{\odot} \times 1,689662 \times 10^{18}m^{2} \curvearrowright Q_{\odot} = -3 \times 6,3251686 \times 10^{-3} \times 1,689662 \times 10^{18}m^{2} \tag{380}$$

This value will be very useful in the calculations of Section 5.[116] .

$$Q_{\odot} = -3,206219108 \times 10^{16}m^{2} \tag{381}$$

Looking again to the equations of the tension λ

$$\lambda \geq \left(\frac{\mathrm{G}M/\mathrm{c}^{2}}{R - 2\mathrm{G}M/\mathrm{c}^{2}}\right)\rho\,, \curvearrowright \lambda \geq \left(\frac{r_G}{R - r_S}\right)\rho\,, \curvearrowright \lambda \geq \frac{3GM}{c^2}\frac{\rho}{R\Delta_{LD}}\,. \curvearrowright \lambda \geq 3r_G\frac{\rho}{R\Delta_{LD}} \tag{382}$$

And focusing ourselves in the equations that uses the radius R,Schwarzschild Radius r_S and the parameter r_G we will find the interesting result given below:

$$\lambda \geq \left(\frac{r_G}{R - r_S}\right)\rho\,, \curvearrowright \lambda \geq 3r_G\frac{\rho}{R\Delta_{LD}} \curvearrowright \left(\frac{r_G}{R - r_S}\right) = 3r_G\frac{1}{R\Delta_{LD}} \tag{383}$$

Hence from the equation written above we can see that perhaps the parameter Δ_{LD} may not be the Universal quantity that gives the absolute deviation from standard $4D$ General Relativity because we have now Δ_{LD} defined in function of a star radius R and a Schwarzschild radius r_S.Then for stars of different masses and radius we will have different values of Δ_{LD} each star possessing its own value.

$$\Delta_{LD} = \frac{3}{R}\left(R - r_S\right) \tag{384}$$

Computing $\Delta_{LD_{\odot}}$ for our Sun we have:

$$\Delta_{LD_{\odot}} = \frac{3}{R_{\odot}}\left(R_{\odot} - r_{S_{\odot}}\right) = \frac{3}{1,3 \times 10^{9}m}\left(1,29999740 \times 10^{9}m\right) = 3\frac{1,29999740 \times 10^{9}m}{1,3 \times 10^{9}m} \tag{385}$$

$$\Delta_{LD_{\odot}} = 2,9994 = 2,9994 \times 10^{0} > 1,7 \times 10^{-4} \tag{386}$$

A value by far higher than the $1,7 \times 10^{-4}$ given in the top of pg 9 in [10]

Making all the calculations again for our Sun with the new value $\Delta_{LD_{\odot}}$ we will get the following results:

$$\lambda_{\odot} \geq 3r_{G_{\odot}}\frac{\rho_{\odot}}{R_{\odot}\Delta_{LD_{\odot}}}\,. \tag{387}$$

$$\lambda_{\odot} = 3r_{G_{\odot}}\frac{\rho_{\odot}}{R_{\odot}\Delta_{LD_{\odot}}}\,. = n_{\odot}\left(\frac{r_{G_{\odot}}}{R_{\odot} - r_{S_{\odot}}}\right)\rho_{\odot}\,, = n_{\odot} \times 2,4606 \times 10^{-4}kg/m^{3} \tag{388}$$

[116]the sign of the Sun Tidal Charge $Q_{\odot}$ is negative.more on negative signs in this Section

$$\lambda_{\odot} = 3 \times 1.48 \times 10^3 \text{ m} \times \frac{2,1614 \times 10^2 kg/m^3}{1,3 \times 10^9 m \times 2,9994}. \tag{389}$$

$$\lambda_{\odot} = 3 \times 1.48 \times 10^3 \text{ m} \times \frac{2,1614 \times 10^2 kg/m^3}{3,89922 \times 10^9 m}. \tag{390}$$

$$\lambda_{\odot} = 3 \times 1.48 \times 10^3 \text{ m} \times 5,5431599140 \times 10^{-8} \frac{kg/m^3}{m}. \tag{391}$$

$$\lambda_{\odot} = 2,461162513 \times 10^{-4} kg/m^3 = n_{\odot} \times 2,4606 \times 10^{-4} kg/m^3. \tag{392}$$

$$n_{\odot} = \frac{2,461162513 \times 10^{-4} kg/m^3}{2,46069157805 \times 10^{-4} kg/m^3} = 1,00100383 \tag{393}$$

$$N_{\odot} = \frac{1}{n_{\odot}} = 9,989971 \times 10^{-1} \tag{394}$$

Examining the results obtained above we can clearly see that the introduction of the new value $\Delta_{LD_{\odot}}$ adapted for the Sun affects deeply(and radically) the values of the Sun tension $\lambda_{\odot}$ and the values of the parameters $n_{\odot}$ and $N_{\odot}$.
This(of course and as expected) affects the final value of the Sun Tidal Charge $Q_{\odot}$:

$$Q_{\odot} = -3N_{\odot}R_{\odot}\left(R_{\odot} - r_{S\odot}\right), \tag{395}$$

$$Q_{\odot} = -3 \times N_{\odot} \times 1,3 \times 10^9 m \times 1,29999740 \times 10^9 m \tag{396}$$

$$Q_{\odot} = -3 \times N_{\odot} \times 1,689662 \times 10^{18} m^2 \curvearrowright Q_{\odot} = -3 \times 9,989971 \times 10^{-1} \times 1,689662 \times 10^{18} m^2 \tag{397}$$

The new value for the Sun Tidal Charge $Q_{\odot}$ will then be:

$$Q_{\odot} = -5,06390231 \times 10^{18} m^2 \tag{398}$$

We rewrite here the previous value to enhance the comparison:

$$Q_{\odot} = -3,206219108 \times 10^{16} m^2 \tag{399}$$

Lastly and in agreement with Kotrlova-Stuchlik-Torok the Tidal Charge Q for Non-Collapsed BraneWorld Stars always have negative sign.(see pg 2 in [25])
. The Non-Collapsed BraneWorld Stars with a stellar radius R greater than the Schwarzschild radius r_S ($R > r_S$) accounts for the major part of all the Galaxies.However when a BraneWorld Star starts the Gravitational Collapse it begins to contract its volume and hence its radius R in the direction of the Schwarzschild radius r_S.
When the radius R of a BraneWorld Star becomes equal to the Schwarzschild Radius r_S ($R = r_S$) the Tidal Charge becomes zero $Q = 0$ and when the contraction proceed beyond the Schwarzschild Radius towards the Singularity but before reaching it ($R < r_S$) the sign of the Tidal Charge Q becomes positive.

In order to terminate this Section the following important items should always be considered.

- The Schwarzschild Radius r_S is the point in the Gravitational Collapse of a BraneWorld Star when the Tidal Charge Q inverts the ordinary negative sign to get a positive sign of a BraneWorld Black Hole.

- Non-Collapsed BraneWorld Stars (eg Sun) have always Tidal Charges Q of negative sign while BraneWorld Black Holes[117] always have Tidal Charges Q of positive sign.

$$Q = -3NR\,(R - r_S) \curvearrowright R - r_S > 0 \curvearrowright R > r_S \curvearrowright Q < 0 \tag{400}$$

$$Q = -3NR\,(R - r_S) \curvearrowright R - r_S = 0 \curvearrowright R = r_S \curvearrowright Q = 0 \tag{401}$$

$$Q = -3NR\,(R - r_S) \curvearrowright R - r_S < 0 \curvearrowright R < r_S \curvearrowright Q > 0 \tag{402}$$

[117] a star with a radius smaller than the Schwarzschild Radius even before reaching the Singularity is already a Black Home.we do not consider Singularities here

5 Gravitational Bending Of Light in both BraneWorld Black Hole and Reissner-Nordstrom Spacetime Metrics

The procedure to find out exact deflection angles of Gravitational Bending of Light is by the expression of the azimuthal angle in terms of the radial distance.The total deflection angle is found by solving the following integral given below in terms of the $4D$ Spacetime Metric Coordinates.Note that the Metric Coefficient associated with the $5D$ Extra Dimension[118][119][120] do not appear here.(see pg 14 in [11]).Note that Briet-Hobill mentions a numerical evaluation of the integral by the Simpson Rule(see pg 14 in [11] after eq 41).

The generic expression for the $4D$ Spacetime Induced spherically symmetric line element ds^2 is given by(see eq 9 pg 7 in [11],eq 5 pg 3 in [5]).

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - C(r)r^2(d\theta^2 + \sin\theta d\phi^2). \tag{403}$$

$$ds^2 = A(r)c^2dt^2 - A(r)^{-1}dr^2 - C(r)r^2(d\theta^2 + \sin\theta d\phi^2). \tag{404}$$

$$B(r) = A(r)^{-1} \tag{405}$$

The generic expression for the integral of the deflection angle is given by:(eq 41 pg 14 in [11],eq 6 pg 3 in [5])

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{\sqrt{B(r)}}{r\sqrt{\left(\frac{r}{r_0}\right)^2\left(\frac{A(r_0)}{A(r)}\right) - 1}}dr - \pi \tag{406}$$

Note that Eiroa-Romero-Torres outlines in pg 15 of [23] the fact that this integral is of elliptic nature and they mentions appropriated software to evaluate these types of integrals(see pg 15 before eq 73 in [23]). Note also that Aliev-Talazan also presents the integral as eq 29 pg 10 in [24] and both also mentions the fact that in the most generic cases the integral can be evaluated only numerically(see pg 11 after eq 30 in [24]).

Rewriting the integral in a way that is more easy for further evaluation we get the following results:

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{\sqrt{B(r)}}{r\sqrt{\left(\frac{r}{r_0}\right)^2\left(\frac{A(r_0)}{A(r)}\right) - 1}}dr - \pi = 2\int_{r_0}^{\infty} \frac{\sqrt{A(r)^{-1}}}{r\sqrt{\left(\frac{r}{r_0}\right)^2\left(\frac{A(r_0)}{A(r)}\right) - 1}}dr - \pi \tag{407}$$

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{r\sqrt{A(r)}\sqrt{\left(\frac{r}{r_0}\right)^2\left(\frac{A(r_0)}{A(r)}\right) - 1}}dr - \pi = 2\int_{r_0}^{\infty} \frac{1}{r\sqrt{A(r)\left(\frac{r}{r_0}\right)^2\left(\frac{A(r_0)}{A(r)}\right) - A(r)}}dr - \pi \tag{408}$$

[118] the $5D$ Metric Coeficient is $-\Phi^2dy^2$ see eq 12 in [3].Note that for photons $\frac{dy}{ds} = 0$ because the photon do not possess mass in $5D$ so we have in this case $dS^2 = 0$ and also $ds^2 = 0$ see pg 6 before eq 26 in [3]

[119] photons are stationary particles in the $5D$ Spacetime.this means so say that photons do not move in the Extra Dimension

[120] we also assume here the Induced Metric in the $4D$ Spacetime part of the $5D$ Spacetime Metric.This means to say that we consider ds^2 with the Induced effects from the $5D$ eg Tidal Charge.We do not consider here the complete $5D$ Metric dS^2

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{r\sqrt{\left(\frac{r}{r_0}\right)^2 (A(r_0)) - A(r)}} dr - \pi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{r^2 \left(\frac{r}{r_0}\right)^2 (A(r_0)) - A(r)r^2}} dr - \pi \tag{409}$$

Note that Eiroa-Romero-Torres presents the integral for the Reissner-Nordstrom Spacetime written in a way that fits in the first integral above(see eq 10 pg 4 in [23]).
The final expression for the integral will then be:

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{\left(\frac{r^4}{r_0^2}\right) (A(r_0)) - A(r)r^2}} dr - \pi \tag{410}$$

In the integral above r_0 have a constant(and numerically defined) value and this generates also a constant value for $A(r_0)$.Then dividing $\frac{A(r_0)}{r_0^2} = \Xi$ produces also a constant value for Ξ.
Rewriting the integral we should expect for:

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{r^4\Xi - A(r)r^2}} dr - \pi \tag{411}$$

We will proceed now with the BraneWorld Black Hole Metric[121] in order to get a "polynomial" expression for the integral above:
The Metric will then be:(see eqs 7 and 8 pg 5 in [25])

$$ds^2 = A(r)\mathrm{c}^2 dt^2 - A^{-1}(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2, \tag{412}$$

$$A(r) = 1 - \frac{2r_{\mathrm{G}}}{r} + \frac{Q}{r^2}; \qquad r_{\mathrm{G}} = \frac{\mathrm{G}M}{\mathrm{c}^2}, \tag{413}$$

The term r_G possesses also a constant numerical value for stars of constant mass M

$$A(r) = [1 - \frac{2r_G}{r} + \frac{Q}{r^2}] \tag{414}$$

$$A(r_0) = [1 - \frac{2r_G}{r_0} + \frac{Q}{r_0^2}] \tag{415}$$

Note that $\frac{2r_G}{r_0} = \sigma$ and $\eta = \frac{Q}{r_0^2}$ also are constant and numerical values.The Tidal Charge Q is also a constant and numerical value.

$$A(r)r^2 = r^2[1 - \frac{2r_G}{r} + \frac{Q}{r^2}] = [r^2 - \frac{2r_G r^2}{r} + \frac{Qr^2}{r^2}] = [r^2 - 2r_G r + Q] \tag{416}$$

Since r_G is a constant and numerical value then $2r_G = \vartheta = r_S$ must also possesses a constant and numerical value.This leads us to the following expressions:[122]

$$A(r)r^2 = r\,[r - 2r_G] + Q \tag{417}$$

[121]because is the best candidate to determine the Higher Dimensional Nature of the Universe
[122]the expression with the Schwarzschild Radius will appear again in the Bohmer-Harko-Lobo work as eq 13 pg 6 in [10]

$$A(r)r^2 = r\left[r - \vartheta\right] + Q = r\left[r - r_S\right] + Q \tag{418}$$

$$A(r_0) = \left[1 - \sigma + \eta\right] \tag{419}$$

Inserting the expressions above in the integral we have:

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{\left(\frac{r^4}{r_0^2}\right)(A(r_0)) - A(r)r^2}}dr - \pi \tag{420}$$

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{\left(\frac{r^4}{r_0^2}\right)([1 - \delta + \eta]) - (r\left[r - \vartheta\right] + Q)}}dr - \pi \tag{421}$$

Remember that:

$$\frac{A(r_0)}{r_0^2} = \frac{[1 - \delta + \eta]}{r_0^2} = \Xi \tag{422}$$

And now finally we really have the polynomial expression for the deflection angle of the BraneWorld Black Hole Metric that can be more easily evaluated by Simpson Rule or elliptical integrals software packages.The integrals are:

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{(r^4\Xi) - (r\left[r - \vartheta\right] + Q)}}dr - \pi \tag{423}$$

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{r^4\Xi - r^2 + r\vartheta - Q}}dr - \pi \tag{424}$$

All we have to do now is to solve the improper integral of the square root of a $4th$ order polynomial in r with term zero in r^3 and constants Ξ,ϑ and Q.
The following forms of the integral are more suitable for integration by parts:

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{r^2\left(r^2\Xi - r + \vartheta\right) - Q}}dr - \pi \tag{425}$$

$$\delta\phi = 2\int_{r_0}^{\infty} \frac{1}{\sqrt{r^2\Xi\left(r^2 - \Upsilon r + \beth\right) - Q}}dr - \pi \tag{426}$$

with $\Upsilon = \frac{1}{\Xi}$ and $\beth = \frac{\vartheta}{\Xi}$. Note that according to Aliev-Talazan pg 11 in [24] the integrant can be expanded using small parameters of the weak field $\mu = \frac{GM}{c^2 r_0} = \frac{r_G}{r_0} =$ and $\eta = \frac{Q}{r_0^2}$.Also we can restrain ourselves to second order terms.

Examining now the complete second-order formulas of the Gravitational Bending of Light for the Schwarzschild or Reissner-Nordstrom Metrics given by Briet-Hobill in order to verify if second order terms accounts for a significant deviation in the Gravitational Bending of Light[123]:

[123] the first order terms were already evaluated in Section 1

Schwarzschild:(eq 20 pg 9 in [11])

$$\delta\phi \simeq \frac{4GM}{c^2 r} + \frac{G^2M^2}{c^4r^2}\left(\frac{15}{4}\pi - 4\right) \tag{427}$$

Reissner-Nordstrom:(eq 27 pg 10 in [11])

$$\delta\phi \simeq \frac{4GM}{c^2 r} + \frac{G^2M^2}{c^4r^2}\left(\frac{15}{4}\pi - 4\right) - \frac{3}{4}\frac{Gq^2}{c^4r^2}\pi. \tag{428}$$

We can clearly see that the second-order term which is common to both Metrics is :

$$\varpi = \frac{G^2M^2}{c^4r^2}\left(\frac{15}{4}\pi - 4\right) \tag{429}$$

For our Sun we have:

$$\varpi_{\odot} = \frac{G^2M_{\odot}^2}{c^4r^2}\left(\frac{15}{4}\pi - 4\right) \tag{430}$$

$$\varpi_{\odot} = \frac{2,1904 \times 10^6 m^2}{r^2} \times \left(\frac{15}{4}\pi - 4\right) \tag{431}$$

$$\varpi_{\odot} = \frac{2,1904 \times 10^6 m^2}{r^2} \times \left(1,1780972 \times 10^1 - 4\right) \tag{432}$$

$$\varpi_{\odot} = \frac{2,1904 \times 10^6 m^2}{r^2} \times 7,780972 = \frac{1,704344 \times 10^7 m^2}{r^2} \tag{433}$$

Again we will analyze the following three situations already given before in Section 1[124]:

- 1)-photon beam passing the Sun at a distance $r = 150.000km$ $r = 1,5 \times 10^8 m$
- 2)-photon beam passing the Sun at a distance $r = 1.000.000km$ $r = 1 \times 10^9 m$
- 3)-photon beam passing the Sun at a distance $r = 10.000.000km$ $r = 1 \times 10^{10} m$

- 1)$r = 1,5 \times 10^8 m$

$$\varpi_{\odot} = \frac{1,704344 \times 10^7 m^2}{2,25 \times 10^{16} m^2} = 7,574862 \times 10^{-10} \tag{434}$$

- 2)$r = 1 \times 10^9 m$

$$\varpi_{\odot} = \frac{1,704344 \times 10^7 m^2}{1 \times 10^{18} m^2} = 1,704344 \times 10^{-11} \tag{435}$$

- 3)$r = 1 \times 10^{10} m$

[124] pointlike Sun

$$\varpi_{\odot} = \frac{1,704344 \times 10^{7} m^{2}}{1 \times 10^{20} m^{2}} = 1,704344 \times 10^{-13} \tag{436}$$

From the results above(and as we expected) we can see that second(or higher) order terms in the Schwarzschild or Reissner-Nordstrom Metrics do not account too much to affect the Gravitational Bending of Light since the best result of $7,574862 \times 10^{-10}$ is nearly 10^{3} times smaller than our most accurate detection capability of 5×10^{-7} (see pg 4 in [28]). (European Space Agency Satellite GAIA)

Focusing ourselves now in the Gravitational Bending of Light from the BraneWorld Black Hole Metric we have the following expressions to compute it:[125]

- Kar-Sinha Equation:(see eq 7 pg 4 in [5])[126]

$$\delta\phi \simeq \frac{4GM}{c^{2}r} + \frac{G^{2}M^{2}}{c^{4}r^{2}}\left(\frac{15}{4}\pi - 4\right) + \frac{3\pi \mid Q \mid}{4r^{2}} + \frac{57\pi\left(\mid Q \mid\right)^{2}}{64r^{4}} - \frac{GM \mid Q \mid}{c^{2}r^{3}}\left(\frac{3\pi}{2} - 14\right) \tag{437}$$

- Gergely-Darazs-Keresztes-Dwornik and Aliev-Talazan Equation:(see eq 25 pg 7 in [21],eq 24 pg 6 in [22] and [127] eq 30 pg 11 in [24])

$$\delta\phi \simeq \frac{4GM}{c^{2}r} + \frac{G^{2}M^{2}}{c^{4}r^{2}}\left(\frac{15}{4}\pi - 4\right) - \frac{3\pi Q}{4r^{2}} + \frac{57\pi Q^{2}}{64r^{4}} + \frac{GMQ}{c^{2}r^{3}}\left(\frac{3\pi}{2} - 14\right) \tag{438}$$

- Bohmer-Harko-Lobo Equation:(see eq 27 pg 8 in [10])

$$\delta\varphi_{LD} = \delta\varphi_{LD}^{(GR)}\left(1 - \frac{2Q}{r^{2}}\right), \tag{439}$$

$$\delta\varphi_{LD}^{(GR)} = 4GM/c^{2}r \tag{440}$$

We will leave the Bohmer-Harko-Lobo Equation to the end of this Section.Now we will concentrate ourselves in the first two equations above.At first sight both appear to be different equations but actually we are facing here two versions of the same equation[128] because as seen in Section 4 ordinary BraneWorld Stars always possesses a negative Tidal Charge Q[129] and comparing some terms taken from both equations we can see that $-\frac{3\pi Q}{4r^{2}}$ when $Q < 0$ will numerically be equivalent to $+\frac{3\pi|Q|}{4r^{2}}$ because the substraction of a negative quantity is equal to the addiction of the modulus of the same quantity,$+\frac{57\pi Q^{2}}{64r^{4}}$ will be numerically equivalent to $+\frac{57\pi(|Q|)^{2}}{64r^{4}}$ because the square of a negative number is always equal to the square of the modulus of the same number and finally $+\frac{GMQ}{c^{2}r^{3}}\left(\frac{3\pi}{2} - 14\right)$ will be numerically equivalent to $-\frac{GM|Q|}{c^{2}r^{3}}\left(\frac{3\pi}{2} - 14\right)$ because the addiction of a negative quantity is equal to the subtraction of the modulus of the same quantity.

[125] note that the second order term common to the Schwarzschild and Reissner-Nordstrom Metrics also appears here

[126] the original equation do not provide second order terms. we included these second order terms

[127] if the rotating coefficient becomes zero.

[128] Why??:this can be explained if each set of authors used a different software package to compute elliptic integrals,or different versions of the same software or even different ways to introduce the parameters of the integral into the computer program

[129] for positive Tidal Charges both equations disagree with each other but fortunately normal stars have negative Tidal Charges so both equations remains valid

Rewriting the Gravitational Bending of Light Equation of Gergely-Darazs-Keresztes-Dwornik and Aliev-Talazan using the parameter r_G we should expect for:

$$\delta\phi \simeq \frac{4r_G}{r} + \frac{r_G^2}{r^2}\left(\frac{15}{4}\pi - 4\right) - \frac{3\pi Q}{4r^2} + \frac{57\pi Q^2}{64r^4} + \frac{r_G Q}{r^3}\left(\frac{3\pi}{2} - 14\right) \tag{441}$$

The first order terms were already computed in Section 1 and the second order term that is common to the BraneWorld,Schwarzschild and Reissner-Nordstrom Metrics we just finished to see that the term do not account too much for the Gravitational Bending of Light so we can also ignore this term.
This leaves us with the second order terms originated by the Tidal Charge Q.These terms are:

$$\digamma = \frac{57\pi Q^2}{64r^4} \tag{442}$$

$$\daleth = \frac{r_G Q}{r^3}\left(\frac{3\pi}{2} - 14\right) \tag{443}$$

For our Sun we would have:

$$\digamma_{\odot} = \frac{57\pi Q_{\odot}^2}{64r^4} \tag{444}$$

$$\daleth_{\odot} = \frac{r_{G_{\odot}} Q_{\odot}}{r^3}\left(\frac{3\pi}{2} - 14\right) \tag{445}$$

We will analyze again the three situations described before for these new terms $\digamma$ and $\daleth$ with the following values for $r_{G_{\odot}}$ and $Q_{\odot}$ given below:

$$Q_{\odot} = -3,206219108 \times 10^{16} m^2 \tag{446}$$

$$r_{G\odot} = \frac{GM_{\odot}}{c^2} = 1.48 \times 10^3 \text{ m} \tag{447}$$

This would give the expressions shown below:

$$\digamma_{\odot} = \frac{57\pi Q_{\odot}^2}{64r^4} = \frac{57 \times \pi \times 1,2798409 \times 10^{33} m^4}{64r^4} \tag{448}$$

$$\digamma_{\odot} = \frac{57\pi Q_{\odot}^2}{64r^4} = \frac{8,90625 \times \pi \times 1,2798409 \times 10^{32} m^4}{r^4} \tag{449}$$

$$\digamma_{\odot} = \frac{57\pi Q_{\odot}^2}{64r^4} = \frac{3,580970466 \times 10^{33} m^4}{r^4} \tag{450}$$

$$\daleth_{\odot} = \frac{r_{G_{\odot}} Q_{\odot}}{r^3}\left(\frac{3\pi}{2} - 14\right) \tag{451}$$

$$\daleth_{\odot} = \frac{r_{G_{\odot}} Q_{\odot}}{r^3} \times -9,2876110196 \tag{452}$$

$$\daleth_{\odot} = \frac{-4,74520427 \times 10^{19} m^3}{r^3} \times -9,2876110196 = \frac{4,4071611468 \times 10^{20} m^3}{r^3} \tag{453}$$

Hence for the known three situations already described we would get these following results for the second order terms $\digamma$ and $\daleth$ originated by the Tidal Charge Q.

- 1)$r = 1,5 \times 10^8 m$

$$\digamma_{\odot} = \frac{3,580970466 \times 10^{33} m^4}{5,0628 \times 10^{32} m^4} = 7,07310276 \tag{454}$$

$$\daleth_{\odot} = \frac{4,4071611468 \times 10^{20} m^3}{3,375 \times 10^{24} m^3} = 1,3058255 \times 10^{-4} \tag{455}$$

- 2)$r = 1 \times 10^9 m$

$$\digamma_{\odot} = \frac{3,580970466 \times 10^{33} m^4}{1 \times 10^{36} m^4} = 3,580970466 \times 10^{-3} \tag{456}$$

$$\daleth_{\odot} = \frac{4,4071611468 \times 10^{20} m^3}{1 \times 10^{27} m^3} = 4,4071611468 \times 10^{-7} \tag{457}$$

- 3)$r = 1 \times 10^{10} m$

$$\digamma_{\odot} = \frac{3,580970466 \times 10^{33} m^4}{1 \times 10^{40} m^4} = 3,580970466 \times 10^{-7} \tag{458}$$

$$\daleth_{\odot} = \frac{4,4071611468 \times 10^{20} m^3}{1 \times 10^{30} m^3} = 4,4071611468 \times 10^{-10} \tag{459}$$

The results above are of great importance because these results shows to ourselves that second-order terms do not account too much for the Gravitational Bending of Light in the Schwarzschild or Reissner-Nordstrom Metrics but however the second-order terms originated from the BraneWorld Black Hole Metric accounts for significant contributions to the Gravitational Bending of Light at least in the two first situations described above:Note that in the third situation corresponding to the worst case even the first second-order term is inside the detection capabilities of the European Space Agency Satellite GAIA. With the results already presented in Section 1 and the results presented in this Section we can take for granted the fact that the BraneWorld Black Hole Metric of Dadhich,Maartens,Papadopolous and Rezania is the best candidate to demonstrate the Higher Dimensional Nature of the Universe.

. We will now terminate this Section analyzing the Gravitational Bending of Light method developed by Bohmer-Harko-Lobo. The Metric will then be:(see eqs 7 and 8 pg 5 in [25])

$$ds^2 = A(r)c^2 dt^2 - A^{-1}(r) dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{460}$$

$$A(r) = 1 - \frac{r_S}{r} + \frac{Q}{r^2}; \qquad r_S = \frac{2GM}{c^2}, \tag{461}$$

In order to determine the trajectory of a massless particle in the metric above the Hamilton-Jacobi equation must be used:(see eq 9 pg 5 in [10]).

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} - m^2 c^2 = 0, \curvearrowright g^{ii} \left(\frac{\partial S}{\partial x^i} \right)^2 - m^2 c^2 = 0, \tag{462}$$

Choosing the motion in the plane defined by the angle $\theta = \frac{\pi}{2}$,$d\theta = 0$ and $\sin\theta = 1$(see pg 6 in [10]) we can write the Hamilton-Jacobi equation as follows:

$$g^{ii}\left(\frac{\partial S}{\partial x^i}\right)^2 - m^2c^2 = g^{00}\left(\frac{\partial S}{\partial x^0}\right)^2 + g^{11}\left(\frac{\partial S}{\partial x^1}\right)^2 + g^{33}\left(\frac{\partial S}{\partial x^3}\right)^2 - m^2c^2 = 0 \tag{463}$$

or even better(see eq 10 pg 6 in [10]):

$$\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1}\left(\frac{\partial S}{c\partial t}\right)^2 - \left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 - m^2c^2 = 0. \tag{464}$$

The standard procedure for solving the Hamilton-Jacobi equation requires S to be written in the following form with the energy E and Angular Momentum L being constants of motion(see eq 11 pg 6 in [10]):

$$S = -Et + L\varphi + S_r(r), \tag{465}$$

Then computing the partial derivatives for time t and angle φ:

$$\frac{\partial S}{c\partial t} = -\frac{E}{c} \tag{466}$$

$$\frac{\partial S}{\partial \varphi} = L \tag{467}$$

The Hamilton-Jaconi equation will now be:

$$\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1}\left(\frac{E}{c}\right)^2 - \left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{L^2}{r^2} - m^2c^2 = 0. \tag{468}$$

$$\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1}\left(\frac{E}{c}\right)^2 - \left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)\left(\frac{\partial S}{\partial r}\right)^2 = \frac{L^2}{r^2} + m^2c^2 \tag{469}$$

We need now to compute the partial derivative with respect to r

$$\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)\left(\frac{\partial S}{\partial r}\right)^2 = \left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1}\left(\frac{E}{c}\right)^2 - \left(\frac{L^2}{r^2} + m^2c^2\right) \tag{470}$$

$$\left(\frac{\partial S}{\partial r}\right)^2 = \left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-2}\left(\frac{E}{c}\right)^2 - \left(\frac{L^2}{r^2} + m^2c^2\right)\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1} \tag{471}$$

$$\left(\frac{\partial S}{\partial r}\right) = \sqrt{\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-2}\left(\frac{E}{c}\right)^2 - \left(\frac{L^2}{r^2} + m^2c^2\right)\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1}} \tag{472}$$

The final result for $S_r(r)$ will then be(se eq 12 pg 6 in [10]):

$$S_r = \int\sqrt{\frac{E^2}{c^2}\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-2} - \left(m^2c^2 + \frac{L^2}{r^2}\right)\left(1 - \frac{r_S}{r} + \frac{Q}{r^2}\right)^{-1}}\, dr. \tag{473}$$

Making the following substitution in the integrand we have:

$$\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)=\frac{r\left(r-r_s\right)+Q}{r^2} \tag{474}$$

Rewriting again the integral we should expect for:

$$S_r=\int\sqrt{\frac{E^2}{c^2}\left(\frac{r^2}{r\left(r-r_s\right)+Q}\right)^2-\left(m^2c^2+\frac{L^2}{r^2}\right)\left(\frac{r^2}{r\left(r-r_s\right)+Q}\right)}\,dr. \tag{475}$$

$$S_r=\int\sqrt{\frac{E^2}{c^2}\left(\frac{r^4}{\left[r\left(r-r_s\right)+Q\right]^2}\right)-\left(m^2c^2+\frac{L^2}{r^2}\right)\left(\frac{r^2}{r\left(r-r_s\right)+Q}\right)}\,dr. \tag{476}$$

$$S_r=\int\sqrt{\frac{E^2}{c^2}\left(\frac{r^4}{\left[r^2-r_sr+Q\right]^2}\right)-\left(m^2c^2+\frac{L^2}{r^2}\right)\left(\frac{r^2}{r^2-r_sr+Q}\right)}\,dr. \tag{477}$$

An integrand with these terms will appear in the work of Bohmer-Harko-Lobo as eq 23 pg 8 in [10]. The following transformations will be very useful(see eqs 13 and 14 pg 6 in [10]):

$$r\left(r-r_S\right)+Q=r'^2, \tag{478}$$

$$\frac{r}{r'}\approx 1+\frac{r_S}{2r'}+\frac{r_S^2}{8r'^2}-\frac{Q}{2r'^2}. \tag{479}$$

providing a star radius r much greater than the Schwarzschild Radius r_S we can write:

$$\frac{r}{r'}\approx 1+\frac{r_S}{2r'}-\frac{Q}{2r'^2}. \tag{480}$$

The propagation of light is described by the eikonal equation derived from the Hamilton-Jacobi equation(see eq 22 pg 8 in [10]):

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k}=0.\curvearrowright g^{ii}\left(\frac{\partial\psi}{\partial x^i}\right)^2=0 \tag{481}$$

Assuming that the light ray is again moving in the plane $\theta=\pi/2$ the equation of the eikonal ψ can be given as: $\psi=-\omega_0t+L\varphi+\psi_r\left(r\right)$, with ω_0 being the frequency of the light and L a constant. The radial part of the eikonal $\psi_r\left(r\right)$ is given by(see pg 8 in [10] before eq 23):

$$g^{ii}\left(\frac{\partial\psi}{\partial x^i}\right)^2=g^{00}\left(\frac{\partial\psi}{\partial x^0}\right)^2+g^{11}\left(\frac{\partial\psi}{\partial x^1}\right)^2+g^{33}\left(\frac{\partial\psi}{\partial x^3}\right)^2=0 \tag{482}$$

$$\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)^{-1}\left(\frac{\partial\psi}{c\partial t}\right)^2-\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)\left(\frac{\partial\psi}{\partial r}\right)^2-\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\varphi}\right)^2=0. \tag{483}$$

$$\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)^{-1}\left(\frac{\omega_0}{c}\right)^2-\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)\left(\frac{\partial\psi}{\partial r}\right)^2-\frac{L^2}{r^2}=0. \tag{484}$$

$$\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)^{-1}\left(\frac{\omega_0}{c}\right)^2-\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)\left(\frac{\partial\psi}{\partial r}\right)^2=\frac{L^2}{r^2}. \tag{485}$$

$$\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)\left(\frac{\partial\psi}{\partial r}\right)^2=\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)^{-1}\left(\frac{\omega_0}{c}\right)^2-\frac{L^2}{r^2} \tag{486}$$

$$\left(\frac{\partial\psi}{\partial r}\right)^2 = \left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)^{-2}\left(\frac{\omega_0}{c}\right)^2 - \frac{L^2}{r^2}\left(1-\frac{r_S}{r}+\frac{Q}{r^2}\right)^{-1} \tag{487}$$

using the substitutions in the integrand already made before we have:

$$\left(\frac{\partial\psi}{\partial r}\right)^2 = \left(\frac{r^4}{[r^2-r_s r+Q]^2}\right)\left(\frac{\omega_0}{c}\right)^2 - \frac{L^2}{r^2}\left(\frac{r^2}{r^2-r_s r+Q}\right) \tag{488}$$

$$\left(\frac{\partial\psi}{\partial r}\right)^2 = \left(\frac{r^4}{[r^2-r_s r+Q]^2}\right)\left(\frac{\omega_0}{c}\right)^2 - \left(\frac{L^2}{r^2-r_s r+Q}\right) \tag{489}$$

placing the term $\left(\frac{\omega_0}{c}\right)$ in evidence and using the term $l = \frac{cL}{\omega_0}$

$$\left(\frac{\partial\psi}{\partial r}\right)^2 = \left(\frac{\omega_0}{c}\right)^2\left[\left(\frac{r^4}{[r^2-r_s r+Q]^2}\right) - \left(\frac{l^2}{r^2-r_s r+Q}\right)\right] \tag{490}$$

And finally we arrive at the main result from Bohmer-Harko-Lobo given below(see eq 23 pg 8 in [10]):

$$\psi_r(r) = \frac{\omega_0}{c}\int\sqrt{\frac{r^4}{(r^2-r_S r+Q)^2} - \frac{l^2}{r^2-r_S r+Q}}\,dr, \tag{491}$$

The equation above explains how Bohmer-Harko-Lobo arrived at the final result.We will now summarize their conclusions as the final explanations of [10].
Applying the transformations given by eqs 13 and 14 pg 6 in [10][130] we will arrive at this result(see eq 24 pg 8 in [10]):

$$\psi_r(r) = \frac{\omega_0}{c}\int\sqrt{1+\frac{2r_S}{r}-\frac{l^2+2Q}{r^2}}\,dr. \tag{492}$$

The integrand above can be expanded in powers of r_S/r to obtain(see eq 25 pg 8 in [10]):

$$\psi_r = \psi_r^{(0)} + \frac{\omega_0 r_S}{c}\int\frac{dr}{\sqrt{r^2-(l^2+2Q)}} = \psi_r^{(0)} + \frac{\omega_0 r_S}{c}\cosh^{-1}\frac{r}{\sqrt{l^2+2Q}}, \tag{493}$$

Still according to Bohmer-Harko-Lobo in pg 8 of [10] between eqs 25 and 26 the term $\psi_r^{(0)}$ is equivalent to the classical straight ray, with $r = l/\cos\varphi$. The total change in ψ_r during the propagation of the light from a very distant point R to the point $r = l$ and then going back again to R is $\Delta\psi_r = \Delta\psi_r^{(0)} + (2\omega_0 r_S/c)\cosh^{-1}\left(r/\sqrt{l^2+2Q}\right)$.Note that now we have $(2\omega_0 r_S/c)$ and not $\frac{\omega_0 r_S}{c}$ because the ray is going back and forth to R so it makes the travel two times(see again pg 8 between eqs 25 and 26 in [10]).

The change in the polar angle is obtained by differentiating $\Delta\psi_r$ with respect to L(see eq 26 pg 8 in [10]):

$$\Delta\varphi_{LD} = -\frac{\partial\Delta\psi_r}{\partial L} = -\frac{\partial\Delta\psi_r^{(0)}}{\partial L} + \frac{2r_S}{l}\left(1+\frac{2Q}{l^2}\right)^{-1}\left(1-\frac{l^2+2Q}{r^2}\right)^{-1/2}. \tag{494}$$

Differentiation to L means to say differentiate to $l = \frac{cL}{\omega_0}$ and also $r = l/\cos\varphi = \frac{cL}{\omega_0}/\cos\varphi$

[130] and with some tedious algebra

If and according to Bohmer-Harko-Lobo R is a very distant point then is valid the limit $R \to \infty$ and taking into account that if the light ray passes on a straight line that corresponds to $\Delta\varphi = \pi$, they found that the angle $\delta\varphi_{LD} = \Delta\varphi_{LD} - \pi$ between the two asymptotes of the light ray differs from π by the angle(see eq 27 pg 8 in [10]):

$$\delta\varphi_{LD} = \frac{2r_S}{l} - \frac{4Qr_S}{l^3} = \delta\varphi_{LD}^{(GR)} \left(1 - \frac{2Q}{l^2}\right), \tag{495}$$

with $\delta\varphi_{LD}^{(GR)} = 4GM/c^2 l$.
The equation above is the final result from Bohmer-Harko-Lobo for the Gravitational Bending of Light associated to the Metric of Dadhich,Maartens,Papadopolous and Rezania.Note that while other authors used the improper integral in the beginning of this Section Bohmer-Harko-Lobo used the Hamilton-Jacobi equation.This is the reason why the results from all other authors converge when rotating terms becomes zero making the result of Aliev-Talazan becomes equal to the one of the Kar-Sinha or Gergely-Darazs-Keresztes-Dwornik while the result from Bohmer-Harko-Lobo is different.Note also that Bohmer-Harko-Lobo took the Non-Rotating Metric.The same analysis using again the Hamilton-Jacobi equation could be extended to include the Rotating BraneWorld Black Hole Metric of Dadhich,Maartens,Papadopolous and Rezania as defined by Kotrlova-Stuchlik.

In October 2009 another work by Bohmer-Harko-Lobo together with De Risi appeared(see [30]).Looking to eq 73 pg 8 of [30] they used first order terms and arrived at exactly the same result obtained by Gergely-Darazs-Keresztes-Dwornik and Aliev-Talazan (see eq 25 pg 7 in [21],eq 24 pg 6 in [22] and [131] eq 30 pg 11 in [24]) but the approach of Bohmer-Harko-Lobo in [10] deserves further studies and should be extended to other Spacetime Metrics.

[131] again if the rotating coefficient becomes zero.

6 The Rotating BraneWorld Black Hole Metric of Dadhich,Maartens,Papado and Rezania

As seen in previous Sections of this work if the Universe have more than $4D$ Dimensions then the non-local effects from the $5D$ Extra Dimension generates a non-local imprint seen in the $4D$ Dimension with the mathematical form of a Coulomb-type effect(a Tidal Charge Q) although no Electrical Charges are present(see pg 4 before eq 33 in [27],pg 3 before Section III in [27] or pg 11 and 12 between eqs 35 and 36 in [29]).This effect from the Extra Dimensional Bulk Space generates in $4D$ Spacetime a contracted form of the $5D$ Spacetime Bulk Weyl Tensor as shown below:(see eq 9 pg 2 in [14])(see also eq $A2$ pg 6 in [14],eq 7 pg 2 in [15],eq 3.12 pg 9 in [13],eq 6 pg 2 in [12], eq 47 pg 7 in [17],eq 21 pg 6 in [16],eq 28 pg 10 in [29],eq 2 pg 4 in [24] and eq 1 pg 4 in [26])

$$E_{\mu\nu} \equiv {}^{(5)}C^{\alpha}{}_{\beta\rho\sigma} n_{\alpha} n^{\rho} q_{\mu}{}^{\beta} q_{\nu}{}^{\sigma}. \tag{496}$$

$$E_{ij} = {}^{(5)}C_{\alpha\beta\gamma\delta}\, n^{\alpha}\, n^{\gamma} q_i^{\beta} q_j^{\delta} \tag{497}$$

And computing the values for this contracted form of the $5D$ Spacetime Bulk Weyl Tensor the Tidal Charge Q can be easily seen as shown below:(see eq 36 pg 12 in [29]).

$$\begin{aligned} E_t{}^t &= -E_\phi{}^\phi = -\frac{Q}{\Sigma^3}\left(\Sigma - 2\left(r^2+a^2\right)\right) ,\\ E_r{}^r &= -E_\theta{}^\theta = \frac{Q}{\Sigma^2} ,\\ E_\phi{}^t &= -(r^2+a^2)\sin^2\theta\, E_t{}^\phi = -\frac{2Qa}{\Sigma^3}\left(r^2+a^2\right)\sin^2\theta , \end{aligned} \tag{498}$$

The expressions above although generated by the contraction of the $5D$ Spacetime Bulk Weyl Tensor have a mathematical form that resembles the Stress-Energy-Momentum Tensor of a $4D$ Spacetime Charged Rotating Black Hole defined by the Kerr-Newman Metric(see eq 37 pg 12 in [29])

$$\begin{aligned} T_t{}^t &= -T_\phi{}^\phi = \frac{q^2}{8\pi\Sigma^3}\left(\Sigma - 2\left(r^2+a^2\right)\right) ,\\ T_r{}^r &= -T_\theta{}^\theta = -\frac{q^2}{8\pi\Sigma^2} ,\\ T_\phi{}^t &= -(r^2+a^2)\sin^2\theta\, T_t{}^\phi = \frac{q^2 a}{4\pi\Sigma^3}\left(r^2+a^2\right)\sin^2\theta , \end{aligned} \tag{499}$$

Then we can see that the presence of the Tidal Charge Q affects the ordinary and uncharged Kerr Metric and hence we must include the Tidal Charge Q in the equation of the uncharged Kerr Metric in order to reproduce the realistic non-local physical effects of the $5D$ Bulk Extra Dimensional Space projected onto our Brane $4D$ Dimensional ordinary space.[132].

The Metric that correctly describes a Rotating BraneWorld Black Hole is known as the Rotating Metric of Dadhich,Maartens,Papadopolous and Rezania.It was defined by Aliev-Gümrükçüoğlu and Kotrlova-Stuchlik as:(see eq 34 and 35 pg 11 in [29] and eq 2 pg 4 in [26])

[132] if the $5D$ Extra Dimension really exists

$$ds^2 = -\left(1 - \frac{2\omega r - Q}{\Sigma}\right) dt^2 - \frac{2a(2\omega r - Q)}{\Sigma} \sin^2\theta \, dt \, d\phi$$
$$+\frac{\Sigma}{\Delta} dr^2 + \Sigma \, d\theta^2 + \left(r^2 + a^2 + \frac{2\omega r - Q}{\Sigma} \, a^2 \sin^2\theta\right) \sin^2\theta \, d\phi^2, \tag{500}$$

$$\Delta = r^2 + a^2 - 2\omega r + Q, \Sigma = r^2 + a^2 \cos^2\theta. \tag{501}$$

It was also defined by Aliev-Talazan as:(see eqs 4 and 5 pg 5 in [24])

$$ds^2 = -\frac{\Delta}{\Sigma} \left(dt - a \sin^2\theta \, d\phi\right)^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Sigma} \left[adt - (r^2 + a^2) \, d\phi\right]^2 , \tag{502}$$

$$\Delta \;\; = \;\; r^2 + a^2 - 2\omega r + Q, , \qquad \Sigma = r^2 + a^2 \cos^2\theta \tag{503}$$

In the Metrics above $\omega = \frac{GM}{c^2}$ is the mass,(see pg 10 after eq $VI.1$ in [19]) a is the rotation parameter, or the angular momentum per unit mass, $a = J/M$, and Q is the Tidal Charge of the black hole.

Also in the previous Sections we have seen so far that the Rotating Metric of Dadhich,Maartens,Papadopolous and Rezania is the best candidate to demonstrate the Higher Dimensional structure of the Universe because unlike the Schwarzschild and Reissner-Nordstrom Metrics first or second order terms do not affect the Gravitational Bending of Light even the second order terms connected to the Tidal Charge Q from the BraneWorld Metric affects significantly the Gravitational Bending of Light and also all the realistic stars have Angular Momentum so the Metrics of Schwarzschild and Reissner-Nordstrom cannot be used in a real fashion and even the term in the Rotating Metric of Dadhich,Maartens,Papadopolous and Rezania of first order associated to the Tidal Charge Q connected to the Angular Momentum of a star have a significant account in the Gravitational Bending of Light Equation of Aliev-Talazan and this account can be spotted by Artificial Satellites(eq:European Space Agency Satellite GAIA)

We will briefly describe in this Section how Aliev-Talazan arrived at the result $\delta\phi$ because it is important to understand how this result was obtained if we really intend to determine experimentally and what is more important:if we want to get success in the determination of the Real Dimensional Nature of the Universe by Gravitational Bending of Light.

Starting again with the Hamilton-Jacobi equation as follows but considering photons:(see eq 69 pg 20 in [29])

$$g^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = 0, \tag{504}$$

Using the following Action:(see eq 70 pg 20 in [29])

$$S = -Et + L\phi + S_r(r) + S_\theta(\theta) \, , \tag{505}$$

But however considering:(see eq 73 pg 21 in [29])

$$\frac{dx^i}{d\lambda} = h^{ij}\,\frac{\partial S}{\partial x^j}\;, \tag{506}$$

and rewriting the Hamilton-Jacobi equation as:(see eq 14 pg 8 in [24])

$$\frac{\partial S}{\partial \lambda} + \frac{1}{2}\,g^{\mu\nu}\,\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} = 0\,, \tag{507}$$

We would get the following geodesic equations of motion extracted from the Hamilton-Jacobi equation:(see eqs 18 to 23 pgs 8 to 9 in [24])

$$\Sigma\frac{dt}{d\lambda} = a(L - aE\sin^2\theta) + \frac{r^2+a^2}{\Delta}\left[(r^2+a^2)E - aL\right]\;, \tag{508}$$

$$\Sigma\frac{d\phi}{d\lambda} = \left(\frac{L}{\sin^2\theta} - aE\right) + \frac{a}{\Delta}\left[(r^2+a^2)E - aL\right]\;, \tag{509}$$

$$\Sigma\frac{dr}{d\lambda} = \sqrt{\mathcal{R}}\,, \tag{510}$$

$$\Sigma\frac{d\theta}{d\lambda} = \sqrt{\Theta}\,, \tag{511}$$

where the functions $\mathcal{R}(r)$ and $\Theta(\theta)$ are given by

$$\mathcal{R} = \left[(r^2+a^2)E - aL\right]^2 - \Delta\left[\mathcal{K} + (L-aE)^2\right]\;, \tag{512}$$

$$\Theta = \mathcal{K} + \cos^2\theta\left(a^2E^2 - \frac{L^2}{\sin^2\theta}\right)\;, \tag{513}$$

In the expressions above (K) is a constant of separation

Considering only a geodesic in the equatorial plane $\theta = \pi/2$(see pg 8 between eqs 14 and 15 in [24],pg 5 between eqs 9 and 10 in [26] and pg 21 before eq 73 in [29]) and a constant of separation $(K) = 0$ the geodesics equations would then be given by:

$$\Sigma\frac{dt}{d\lambda} = a(L - aE) + \frac{r^2+a^2}{\Delta}\left[(r^2+a^2)E - aL\right]\;, \tag{514}$$

$$\Sigma\frac{d\phi}{d\lambda} = (L - aE) + \frac{a}{\Delta}\left[(r^2+a^2)E - aL\right]\;, \tag{515}$$

$$\Sigma\frac{dr}{d\lambda} = \sqrt{\mathcal{R}}\,, \tag{516}$$

$$\Sigma\frac{d\theta}{d\lambda} = 0\,, \tag{517}$$

$$\mathcal{R} = \left[(r^2+a^2)E - aL\right]^2 - \Delta\left[(L-aE)^2\right]\;, \tag{518}$$

Note that the geodesics given above are written in function of the Energy E and Angular Momentum L then we have two quantities to worry about.Would be nice to have only one quantity relating E and L in order to simplify the further analysis we must still carry on.This new quantity is the Impact Parameter u defined by:(see pg 10 between eqs 26 and 27 in [24])

$$u = \frac{L}{E} \curvearrowright E = \frac{L}{u} \tag{519}$$

Making the following algebraic substitutions:

$$(L - aE) = (L - a\frac{L}{u}) = L(1 - \frac{a}{u}) \tag{520}$$

$$\left[(r^2 + a^2)E - aL\right] = \left[(r^2 + a^2)\frac{L}{u} - aL\right] = L\left[(r^2 + a^2)\frac{1}{u} - a\right] = L\left[\frac{r^2 + a^2}{u} - a\right] \tag{521}$$

The geodesics equation can be written now only in function of the Angular Momentum L

$$\Sigma\frac{dt}{d\lambda} = aL(1 - \frac{a}{u}) + \frac{r^2 + a^2}{\Delta}L\left[\frac{r^2 + a^2}{u} - a\right], \tag{522}$$

$$\Sigma\frac{d\phi}{d\lambda} = L(1 - \frac{a}{u}) + \frac{a}{\Delta}L\left[\frac{r^2 + a^2}{u} - a\right], \tag{523}$$

$$\Sigma\frac{dr}{d\lambda} = \sqrt{\mathcal{R}} = L\sqrt{\mathcal{P}}, \tag{524}$$

$$\mathcal{R} = \left[L\left[\frac{r^2 + a^2}{u} - a\right]\right]^2 - \Delta(L(1 - \frac{a}{u}))^2, \tag{525}$$

$$\mathcal{R} = L^2\left[\frac{r^2 + a^2}{u} - a\right]^2 - L^2\Delta(1 - \frac{a}{u})^2, \tag{526}$$

$$\mathcal{P} = \left[\frac{r^2 + a^2}{u} - a\right]^2 - \Delta(1 - \frac{a}{u})^2, \tag{527}$$

And now what is more important:Note that when computing $\frac{d\phi}{dr}$ or $\frac{dr}{d\phi}$ and we will need to use these expressions the Angular Momentum L disappears leaving only in the equations the Impact Parameter u.

$$\frac{d\phi}{dr} = \frac{(1 - \frac{a}{u}) + \frac{a}{\Delta}\left[\frac{r^2+a^2}{u} - a\right]}{\sqrt{\left[\frac{r^2+a^2}{u} - a\right]^2 - \Delta(1 - \frac{a}{u})^2}} \tag{528}$$

$$\frac{dr}{d\phi} = \frac{\sqrt{\left[\frac{r^2+a^2}{u} - a\right]^2 - \Delta(1 - \frac{a}{u})^2}}{(1 - \frac{a}{u}) + \frac{a}{\Delta}\left[\frac{r^2+a^2}{u} - a\right]} \tag{529}$$

We rewrote the geodesics equations in function of the Angular Momentum L and we verified that it can be dropped from the expressions $\frac{d\phi}{dr}$ and $\frac{dr}{d\phi}$.Since Energy E is related to the Angular Momentum L by the Impact Parameter u we will rewrite the geodesics equations in function of the Energy E to see what happens:

$$u = \frac{L}{E} \curvearrowright L = uE \tag{530}$$

Making the following algebraic substitutions:

$$(L - aE) = (uE - aE) = E(u - a) \tag{531}$$

$$\left[(r^2 + a^2)E - aL\right] = \left[(r^2 + a^2)E - auE\right] = E\left[(r^2 + a^2) - au\right] \tag{532}$$

The geodesics equations now can be given by the following expressions:

$$\Sigma\frac{dt}{d\lambda} = aE(u - a) + \frac{r^2 + a^2}{\Delta}E\left[(r^2 + a^2) - au\right] , \tag{533}$$

$$\Sigma\frac{d\phi}{d\lambda} = E(u - a) + \frac{a}{\Delta}E\left[(r^2 + a^2) - au\right] , \tag{534}$$

$$\Sigma\frac{dr}{d\lambda} = \sqrt{\mathcal{R}} = E\sqrt{\mathcal{P}}, \tag{535}$$

$$\mathcal{R} = \left[E\left[(r^2 + a^2) - au\right]\right]^2 - \Delta\left[(E(u - a))^2\right] , \tag{536}$$

$$\mathcal{R} = E^2\left[r^2 + a^2 - au\right]^2 - \Delta\ E^2\left[u - a\right]^2 , \tag{537}$$

$$\mathcal{P} = \left[r^2 + a^2 - au\right]^2 - \Delta\ \left[u - a\right]^2 , \tag{538}$$

Note that rewriting the geodesics equations in function of the Energy E and recomputing $\frac{d\phi}{dr}$ and $\frac{dr}{d\phi}$ the same thing that happened with the Angular Momentum L happens again:The Energy E can be dropped from the equations leaving ourselves with the following expressions written using the Impact Parameter u:

$$\frac{d\phi}{dr} = \frac{(u - a) + \frac{a}{\Delta}\left[(r^2 + a^2) - au\right]}{\sqrt{\left[r^2 + a^2 - au\right]^2 - \Delta\ \left[u - a\right]^2}} \tag{539}$$

$$\frac{dr}{d\phi} = \frac{\sqrt{\left[r^2 + a^2 - au\right]^2 - \Delta\ \left[u - a\right]^2}}{(u - a) + \frac{a}{\Delta}\left[(r^2 + a^2) - au\right]} \tag{540}$$

Note also that the expressions $\frac{d\phi}{dr}$ and $\frac{dr}{d\phi}$ obtained in apparently different cases are actually equivalent expressions.As a matter of fact:(see eq 26 pg 10 in [24] only for $\mathfrak{f}(u, r)$).

$$\frac{d\phi}{dr} = \frac{(1-\frac{a}{u}) + \frac{a}{\Delta}\left[\frac{r^2+a^2}{u} - a\right]}{\sqrt{\left[\frac{r^2+a^2}{u} - a\right]^2 - \Delta(1-\frac{a}{u})^2}} = \mathfrak{f}(u,r) \tag{541}$$

$$\frac{d\phi}{dr} = \frac{(u-a) + \frac{a}{\Delta}\left[(r^2+a^2) - au\right]}{\sqrt{\left[r^2+a^2-au\right]^2 - \Delta\ \left[u-a\right]^2}} = \mathfrak{f}(u,r) \tag{542}$$

And it can be easily verified in the first expression for $\mathfrak{f}(u,r)$ that:

$$\sqrt{\left[\frac{r^2+a^2}{u} - a\right]^2 - \Delta(1-\frac{a}{u})^2} = \frac{1}{u}\sqrt{\left[r^2+a^2-au\right]^2 - \Delta\ \left[u-a\right]^2} \tag{543}$$

Multiplying the numerator of the first expression for $\mathfrak{f}(u,r)$ by u

$$u\left\{(1-\frac{a}{u}) + \frac{a}{\Delta}\left[\frac{r^2+a^2}{u} - a\right]\right\} = (u-a) + \frac{a}{\Delta}\left[(r^2+a^2) - au\right] \tag{544}$$

So we managed to transform the first expression for the $\mathfrak{f}(u,r)$ in the second one.This function $\mathfrak{f}(u,r)$ is the function that will be integrated by the improper elliptic integral given by eq 29 pg 10 in [24] however with further treatment on the Impact Parameter u.

Aliev-Talazan defined the Impact Parameter u using the following expressions:(see eq 24 and 25 pg 9 in [24])

$$u_{\perp} = r^2 \sin\theta_0 \left(\frac{d\phi}{dt}\right)_{r\to\infty} = \frac{L}{E\sin\theta_0}\,, \tag{545}$$

$$u_{\parallel} = h\sin\theta_0 = r^2\left(\frac{d\theta}{dt}\right)_{r\to\infty} = \frac{\mathcal{K}}{E^2} + \left(a^2 - {u_{\perp}}^2\right)\cos^2\theta_0\,, \tag{546}$$

In the expressions above the vertical angle $\theta_0 = \pi/2 - \psi_0$ contains in its definition the inclination angle ψ_0 between the light ray and the equatorial plane and h is the height of the light ray on the equatorial plane.

The most simplest situation to be evaluated by experiments using Artificial Satellites and Laser Beams is the case where there are no inclination angle $\psi_0 = 0$ and height $h = 0$.In this case we are working in the Equatorial Plane and the Impact Parameter is given as below:(see again pg 10 between eqs 26 and 27 in [24])

$$u_{\perp} = r^2\left(\frac{d\phi}{dt}\right)_{r\to\infty} = \frac{L}{E}\,, \tag{547}$$

$$u_{\parallel} = h = 0,, \tag{548}$$

In order to calculate the Gravitational Bending of Light in the Equatorial Plane of the Rotating Braneworld Metric as the simplest case with $\theta = \pi/2,\ \psi_0 = 0,\ u_{\parallel} = 0$(see pg 10 before eq 26 in [24]) we must combine eqs 19 and 20 pg 9 in [24] in order to obtain the trajectory of the light ray described by the equation:(see eq 26 pg 10 in [24])

$$\frac{d\phi}{dr} = \mathfrak{f}(u,r)\,, \tag{549}$$

We already got our expressions for $\mathfrak{f}(u,r)$.The Aliev-Talazan expression for $\mathfrak{f}(u,r)$ is given below:(see eq 27 pg 10 in [24]).

$$\mathfrak{f}(u,r) = \frac{(\Delta - a^2)u + a(2\omega r - Q)}{\Delta\sqrt{(r^2+a^2-au)^2 - \Delta(u-a)^2}}\,. \tag{550}$$

Placing one of our expressions for $\mathfrak{f}(u,r)$ together with the Aliev-Talazan expression for $\mathfrak{f}(u,r)$

$$\frac{d\phi}{dr} = \frac{(u-a) + \frac{a}{\Delta}\left[(r^2+a^2) - au\right]}{\sqrt{\left[r^2+a^2-au\right]^2 - \Delta\ \left[u-a\right]^2}} = \mathfrak{f}(u,r) \tag{551}$$

Look carefully to the square root in the denominators of both expressions:

After some lengthly algebra our expression can be transformed in the Aliev-Talazan expression.Remeber that:

$$\Delta \;=\; r^2 + a^2 - 2\omega r + Q \tag{552}$$

$$r^2 + a^2 \;=\; \Delta + 2\omega r - Q \tag{553}$$

Only the starting point:

$$\frac{d\phi}{dr} = \frac{\Delta(u-a) + a\left[(r^2+a^2) - au\right]}{\Delta\sqrt{\left[r^2+a^2-au\right]^2 - \Delta\ \left[u-a\right]^2}} = \mathfrak{f}(u,r) \tag{554}$$

In a real situation we must write the impact parameter u in in function of the distance r_0 of closest approach to the BraneWorld Star. This can be done by making $dr/d\phi = 0$.(see pg 10 in [24]).
In our case:

$$\frac{dr}{d\phi}_{r \to\ r_0} = \frac{\sqrt{\left[\frac{r^2+a^2}{u} - a\right]^2 - \Delta(1-\frac{a}{u})^2}}{(1-\frac{a}{u}) + \frac{a}{\Delta}\left[\frac{r^2+a^2}{u} - a\right]} = 0 \tag{555}$$

or:

$$\frac{dr}{d\phi}_{r \to\ r_0} = \frac{\sqrt{\left[r^2+a^2-au\right]^2 - \Delta\ \left[u-a\right]^2}}{(u-a) + \frac{a}{\Delta}\left[(r^2+a^2) - au\right]} = 0 \tag{556}$$

Our second expression gives the following result for the relation between r_0 and u:

$$u = \frac{r_0^2 + a^2 + a\sqrt{\Delta_0}}{\sqrt{\Delta_0} + a} \tag{557}$$

$$u = \frac{\Delta_0 + 2\omega r_0 - Q + a\sqrt{\Delta_0}}{\sqrt{\Delta_0} + a} \tag{558}$$

It can be easily obtained making:

$$\sqrt{[r^2 + a^2 - au]^2 - \Delta\ [u-a]^2} = 0 \tag{559}$$

$$\left[r^2 + a^2 - au\right]^2 - \Delta\ [u-a]^2 = 0 \tag{560}$$

$$\left[r^2 + a^2 - au\right]^2 = \Delta\ [u-a]^2 \tag{561}$$

$$\left[r^2 + a^2 - au\right] = \sqrt{\Delta}\,[u-a] \tag{562}$$

Our expression will look familiar if we combine together eqs 26 and 27 of Aliev-Talazan in [24] as follows:

$$\frac{d\phi}{dr} = \mathfrak{f}(u,r) = \frac{(\Delta - a^2)u + a(2\omega r - Q)}{\Delta\sqrt{(r^2 + a^2 - au)^2 - \Delta(u-a)^2}} \tag{563}$$

Taking in mind that

$$\frac{dr}{d\phi}_{r \to\ r_0} = \frac{\Delta\sqrt{(r^2 + a^2 - au)^2 - \Delta(u-a)^2}}{(\Delta - a^2)u + a(2\omega r - Q)} = 0 \tag{564}$$

The Aliev-Talazan expression for the relation between r_0 and u is given by(see eq 28 pg 10 in [24])

$$u = \frac{a(Q - 2\omega r_0) \pm {r_0}^2\sqrt{\Delta_0}}{{r_0}^2 - 2\omega r_0 + Q}\ . \tag{565}$$

The $+$ sign refers to the light ray in the same direction of the BraneWorld Star rotation , while the $-$ sign corresponds to the case of a light ray in the opposite direction of the BraneWorld star rotation.

Aliev-Talazan consider the $+$ sign.

The total shift in the azimuthal angle ϕ begins when the light ray starts from infinity reaches the minimal impact distance point r_0 and then goes to infinity again.Hence it makes the travel two times[133](see pg 10 between eqs 28 and 29 in [24]).

The azimuthal angle ϕ is given by:

$$\phi = \int \mathfrak{f}(r,u)dr \tag{566}$$

Replacing each expression for the relation between r_0 and u inside each respective $\mathfrak{f}(r,u)$ in order to get the expressions for both $\mathfrak{f}(r,r_0)$ and taking in mind that the trajectory of light at infinity is a straight line, we find that the Gravitational Bending of Light angle from the straight line is given by:(see eq 29 pg 10 in [24])

$$\delta\phi = 2\int_{r_0}^{\infty} \mathfrak{f}(r,r_0)dr\ \ - \pi\,. \tag{567}$$

[133]as in the case of Bohmer-Harko-Lobo in the end of Section 5

This is of course an elliptic integral and can be evaluated only numerically(see comment on pg 11 after eq 30 in [24]).

Evaluating the integral analytically by expanding the integrand in small parameters of $\epsilon = \omega/r_0$, $\eta = Q/r_0^2$ and $\delta = a/r_0$. considering only second order terms we find the analytical expression for the deflection angle(see eq 30 pg 11 in [24]).

$$\delta\phi = \frac{4GM}{c^2 r_0}\left(1 - \frac{a}{r_0}\right) + \frac{G^2M^2}{4c^4r_0^2}(15\pi - 16) - \frac{\pi Q}{4r_0^2}\left(3 - \frac{4a}{r_0}\right) + \frac{GMQ}{2cr_0^3}(3\pi - 28) + \frac{57\pi}{64}\frac{Q^2}{r_0^4}\ . \tag{568}$$

Since the first order terms were already analyzed in Section 1 and according to the equation above the Angular Momentum seems to affect only first order terms[134] we will terminate this Section making a quick and simplified analysis of the second order terms in this equation but we exclude π and numbers and we will work only with the powers of 10 im order to go faster to get:

$$\iota = \frac{GMQ}{cr_0^3} \tag{569}$$

$$\varphi = \frac{Q^2}{r_0^4} \tag{570}$$

$$\iota = \frac{10^{-11} \times 10^{30} \times 10^{16}}{10^8 \times r_0^3} = \frac{10^{35}}{10^8 \times r_0^3} \tag{571}$$

$$\varphi = \frac{10^{32}}{r_0^4} \tag{572}$$

We will consider again like we did before the three following situations:

- 1)-photon beam passing the Sun[135] at a distance $r = 150.000km$ $r = 1,5 \times 10^8 m$
- 2)-photon beam passing the Sun[136] at a distance $r = 1.000.000km$ $r = 1 \times 10^9 m$
- 3)-photon beam passing the Sun[137] at a distance $r = 10.000.000km$ $r = 1 \times 10^{10} m$
- 1)-$r = 1,5 \times 10^8 m$

$$\iota = \frac{10^{35}}{10^8 \times r_0^3} = \frac{10^{35}}{10^8 \times 10^{24}} = \frac{10^{35}}{10^{32}} = 10^3 \tag{573}$$

$$\varphi = \frac{10^{32}}{r_0^4} = \frac{10^{32}}{10^{32}} = 1 \tag{574}$$

[134] remember that this integral was obtained numerically by computer software
[135] pointlike Sun
[136] pointlike Sun
[137] pointlike Sun

- 2)-$r = 1 \times 10^9 m$

$$\iota = \frac{10^{35}}{10^8 \times r_0^3} = \frac{10^{35}}{10^8 \times 10^{27}} = \frac{10^{35}}{10^{35}} = 1 \tag{575}$$

$$\varphi = \frac{10^{32}}{r_0^4} = \frac{10^{32}}{10^{36}} = 10^{-4} \tag{576}$$

- 3)-$r = 1 \times 10^{10} m$

$$\iota = \frac{10^{35}}{10^8 \times r_0^3} = \frac{10^{35}}{10^8 \times 10^{30}} = \frac{10^{35}}{10^{38}} = 10^{-3} \tag{577}$$

$$\varphi = \frac{10^{32}}{r_0^4} = \frac{10^{32}}{10^{40}} = 10^{-8} \tag{578}$$

All the second order terms except for φ in the third case are in the range of the detection capabilities of the European Space Agency Satellite GAIA and will affect the Gravitational Bending of Light.In order to get a correct result for the Gravitational Bending of Light by the Sun using Artificial Satellites we need to use the Aliev-Talazan formula for the rotating case and not the stationary one.

7 Conclusion

In this work we tried to develop a concise treatment of the physical nature of the $5D$ Extra Dimension and how it could be detected from an experimental point of view.Since this can be considered a huge task we divided this work into the following Sections we will briefly resume here:

- Section 1

In Section 1 we compared the Klein Compactification Mechanism with the Kaluza Cylindrical Condition and we explained why the $5D$ Extra Dimension remains "invisible".We also presented a resume of what appears in the other Sections.

- Section 2

In Section 2 we compared the Shiromizu-Maeda-Sasaki formalism for the Randall-Sundrum BraneWorld with the Basini-Capozziello-Overduin-Wesson formalism for the Induced Kaluza-Klein BraneWorld in order to demonstrate that both will produce the same Weyl Tensor that generates in the $4D$ Spacetime a Tidal Charge as a consequence of the projection of the $5D$ Spacetime into the $4D$ one.The Weyl Tensor is the same because we used two apparently "different" formalisms to describe the same $5D$ Extra Dimension so in the end and as expected both formalism converge to the same result.

- Section 3

In Section 3 we analyzed the Hamilton-Jacobi equation using the Ponce De Leon formalism.We demonstrated how the $5D$ Spacetime generates as a projection in the $4D$ Spacetime the masses and electrical charges of all elementary particles and antiparticles and we explained why antiparticles have the same rest-mass of particles but electrical charges of opposite signs also using the Hamilton-Jacobi equation.

- Section 4

In Section 4 we analyzed the structure of a BrsneWorld Star under the Metric of Dadhich,Maartens,Papadopolous and Rezania and we computed a numerical value for the Tidal Charge of the Sun.Also we demonstrated that when a BraneWorld Star reaches the Schwarzschild Radius the sign of the Tidal Charge is inverted.

- Section 5

In Section 5 we compared the Gravitational Bending of Light formulas for the Schwarzschild and Reissner-Nordstrom Metrics with the Dadhich,Maartens,Papadopolous and Rezania Metric for the stationary case.While we observed the Schwarzschild and Reissner-Nordstrom Metrics being affected by the terms $\frac{G}{c^2}$ and $\frac{G}{c^4}$ that reduces the capability to detect experimentally the Extra Terms generated by the $5D$ Extra Dimension in the Gravitational Bending of Light formulas we observed that the absence of the terms $\frac{G}{c^2}$ and $\frac{G}{c^4}$ in the Extra Terms generated by the $5D$ Extra Dimension in the Gravitational Bending of Light formulas of the Dadhich,Maartens,Papadopolous and Rezania Metric makes these Extra Terms suitable for experimental detection making the Dadhich,Maartens,Papadopolous and Rezania Metric the best candidate to demonstrate the Higher Dimensional Nature of the Universe.

- Section 6

In Section 6 we studied the Gravitational Bending of Light formula of Aliev-Talazan for the rotating case of the Dadhich,Maartens,Papadopolous and Rezania Metric because in an experimental context of Artificial Satellites measuring the Gravitational Bending of Light of the Sun the Sun possesses Angular Momentum and the Extra Terms generated by the $5D$ Extra Dimension in the Gravitational Bending of Light are also affected by the Angular Momentum so the rotational case is the corrct one to be used in a real scenario.

It is true that the Physics of Extra Dimensions started in the year of 1918 with the works of Theodore Kaluza and Oskar Klein but however since then we still dont have an experimental decisive and convinced proof that the $5D$ Extra Dimension really exists and the Universe have really a Higher Dimensional Nature.The European Space Agency Satellite GAIA have a Gravitational Bending of Light measuring capability of 5×10^{-7} (see pg 4 in [28]) by far more than enough to detect the presence of the Extra Dimensions in our Universe and can detect the Extra Terms generated by the $5D$ Extra Dimension in the Gravitational Bending of Light formula of Aliev-Talazan even considering second order terms.These two things coupled together can perhaps change the whole picture. The experimental discovery of Extra Dimensions would imply in one of the biggest scientific revolutions. If this happens someday then the Tidal Charges From BraneWorld Black Holes As An Experimental Proof of the Higher Dimensional Nature Of The Universe would not only change the way we understand the Laws of Physics but above everything else it would change and change forever the way we look to the Supreme Beauty of All The Creation

8 Epilogue

- "The only way of discovering the limits of the possible is to venture a little way past them into the impossible."-Arthur C.Clarke[138]
- "The supreme task of the physicist is to arrive at those universal elementary laws from which the cosmos can be built up by pure deduction. There is no logical path to these laws; only intuition, resting on sympathetic understanding of experience, can reach them"-Albert Einstein[139][140]

[138] special thanks to Maria Matreno from Residencia de Estudantes Universitas Lisboa Portugal for providing the Second Law Of Arthur C.Clarke

[139] "Ideas And Opinions" Einstein compilation, ISBN $0-517-88440-2$, on page 226."Principles of Research" ([Ideas and Opinions],pp.224-227), described as "Address delivered in celebration of Max Planck's sixtieth birthday (1918) before the Physical Society in Berlin"

[140] appears also in the Eric Baird book Relativity in Curved Spacetime ISBN $978-0-9557068-0-6$

9 Legacy

This work is dedicated to the Memory of the physicists Theodore Kaluza and Oskar Klein.Both in 1918 studied for the first time the possibility of a Universe of Higher Dimensional Nature

References

[1] *Basini G. and Capozziello S. (*2005*).Gen Rel Grav* 37 115 .

[2] *Basini G. and Capozziello S. (*2003*).Gen Rel Grav* 35 2217.

[3] *Ponce De Leon J. (*2004*).Gen Rel Grav* 36 1335*,arXiv:gr-qc/0310078.*

[4] *Ponce De Leon J. (*2003*).Gen Rel Grav* 35 1365*,arXiv:gr-qc/0207108*

[5] *Kar S. and Sinha M. (*2003*).Gen Rel Grav 35* 1775.

[6] *Loup F (*2006*).Gen Rel Grav 38 1423,arXiv:gr-qc/0603106*

[7] *Loup F (*2008*).,ISBN 978-1-60692-264-4 (*2009*) Chapter IX/arXiv:0710.0924[physics.gen-ph]*

[8] *Overduin J.M. and Wesson P. (*1997*). Phys.Rept. 283 303-380,arXiv:gr-qc/9805018*

[9] *Carroll S (*1997*)./arXiv:gr-qc/9712019*

[10] *Boehmer C.,Harko T and Lobo F. (*2008*).Class.Quant.Grav.25:045015,arXiv:0801.1375[gr-qc]*

[11] *Briet J. and Hobill D. (*2008*),arXiv:0801.3859[gr-qc]*

[12] *Dadlich N.,Maartens R.,Papadopoulos P and Rezania V. (*2000*).Phys.Lett.B487-1,arXiv:hep-th/0003061*

[13] *Maartens R.(*2004*).LivingRev.Rel.7:7,arXiv:gr-qc/0312059*

[14] *Shiromizu T.,Maeda K and Sasaki M. (*2000*).Phys.Rev D62.024012,arXiv:gr-qc/9910076*

[15] *Shiromizu T.,Maeda K and Sasaki M. (*2000*).Phys.Rev D62.024008,arXiv:hep-th/9912233*

[16] *Ponce De Leon J. (*2003*).Int.J.Mod.Phys.D12 1053-1066,arXiv:gr-qc/0212036*

[17] *Ponce De Leon J. (*2004*).Gen Rel Grav* 36 923*,arXiv:gr-qc/0212058*

[18] *Ponce De Leon J. (*2003*).Int.J.Mod.Phys. D12 757-780,arXiv:gr-qc/0209013*

[19] *Berman S.M. (*2007*).,arXiv:physics/0702014 [physics.gen-ph]*

[20] *Pal S. and Kar S.(*2008*).Class.Quant.Grav.25:045003 ,arXiv:0707.0223[gr-qc]*

[21] *Gergely L.,Keresztes Z., Dwornik M.(*2009*).Class. Quant. Grav.26:145002,arXiv:0903.1558([gr-qc]*

[22] *Gergely L.,Darzs B.,(*2006*).Publ.Astron.Dep.Eotvos Univ. 17/213-219 ,arXiv:astro-ph/0602427*

[23] *Eiroa E.,Romero F.,Torres D.(*2002*).Phys.Rev.D66:024010, arXiv:gr-qc/0203049*

[24] *Aliev.A.N,Talazan P.(*2009*).,Phys.Rev.D80:044023,arXiv:0906.1465([gr-qc]*

[25] *Kotrlova A.,Stuchlik Z. and Torok G.(*2008*).Class.Quant.Grav.25:225016,arXiv:0812.0720[gr-qc]*

[26] *Kotrlova A.,Stuchlik Z.,(*2009*).Gen Rel Grav* 41 1305*,arXiv:0812.5066v1 [astro-ph]*

[27] *Germani C.,Maartens R.,(*2001*).Phys.Rev.D64:124010,arXiv:hep-th/0107011*

[28] *Perryman M.A.C,Pace.O,(*2000*), ESA Bulletin 103*

[29] *Aliev.A.N,Gümrükçüoğlu A.E ,(*2005*).Phys.Rev.D71:104027,arXiv:hep-th/0502223*

[30] *Boehmer C.,De Risi G.,Harko T and Lobo F. (*2009*),arXiv:0910.3800[gr-qc]*

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