

Towards a Set Theoretic Axiomatization of Quantum Theory and Related Topics

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Abstract

Motivated by Hilbert's sixth problem on axiomatization of physics, the author is proposing a rather provocative abstract axiomatic framework called S-formalization, where S is an arbitrary physical system. The proposal is an attempt to provide a general axiomatic framework, from which mathematical frameworks of new physical theories can be formulated on firm axiomatic basis and the mathematical frameworks of subjects such as standard (nonrelativistic) quantum mechanics are naturally derived as special cases of this general axiomatic framework.

Introduction

Quantum mechanics is without doubts a very successful subject in its own right, its predictions have been successfully consistent with experimental data over a vast number of observed phenomenon in the last 87 years. Quantum mechanics had also some impact in the development of mathematics in 20th century, new mathematical theories such as operator algebra theory were born as a consequence of the need of new mathematical tools to deal with the subject.

Yet decades after the first axiomatization of quantum mechanics due to von Neumann, the foundations of the subject still somehow puzzling and obscure! In fact, von Neumann himself was not completely convinced about his axiomatization as he famously wrote "I would like to make a confession which may seem immoral: I do not believe absolutely in Hilbert space any more". So he spent some time looking for a more general alternative to the Hilbert space approach!

The reader may now ask, how come the foundations of quantum mechanics is obscure and puzzling despite the powerful experimental support of the subject?

There are quite few possible answers to the question, but I will just consider the following one:

- Prior to quantum mechanics, from Newton to Einstein, a fundamental physical theory has always evolved from simple and intuitive physical principles. Take for example Newton's principle "An object remains at rest or uniform motion unless acted upon by an external force" or Einstein's principle "The speed of light is the same in all inertial frames". With these simple principles, both physicists were able to successfully derive the mathematical frameworks of their respective theories. On the other hand, quantum mechanics as formulated by the likes of Heisenberg, Schrodinger, Dirac and von Neumann, does not provide any physical principle that justifies the use of its abstract mathematical framework. In fact, many attempts have been made to find the physical principles (if there is any!) underlying the abstract mathematical framework of the subject, but so far none of the attempts have gained a universal acceptance in the physics community.

As far as S-formalization is concerned, the reader may ask, why do we need to add more abstraction to the already abstract mathematical framework? I will answer the question as follows;

- First in my view, quantum mechanics is just a very special case of a general axiomatic scheme that hasn't been yet clarified or fully understood by physicists! This axiomatic scheme is abstract by nature, it often lacks the so called "physical principles", but nonetheless it has an astonishing power of prediction. Second, I think abstraction is the only way to clarify and give this scheme a firm foundations. In this regard, let us take the example of classical probability theory. Before Kolmogorov, the foundations of the subject was very obscure, indeed some thought it would remain obscure forever! However, after Kolmogorov came along with his abstract axiomatic treatment of the subject, the nature of the subject become more clear and all the mysteries vanished! I intend to follow the example of Kolmogorov, and although I don't claim to solve or clarify all the mysteries that surrounds quantum mechanics, I hope the axiomatic framework presented in this manuscript will provide a better understanding of the subject and its relationship with other subjects such as classical mechanics. I also hope the framework will provide a new insight on Hilbert's sixth problem, especially in regard to mechanics.

I will finish the introduction with the following quote :
 “ If geometry is to serve as a model for the treatment of physical axioms, we shall try first by a small number of axioms to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories..... The mathematician will have also to take into account not only of those theories coming near reality, but also , as in geometry, of all logically possible theories. He must be always alert to obtain a complete survey off all conclusions derivable from the system of axioms assumed.“ David Hilbert.

Why Set Theory In Foundations Of Physics ?

Set theory (ZFC) has without doubts established itself at the centre of modern mathematics. It is widely used as the starting point for the axiomatic foundations of subjects such as abstract algebra , topology , mathematical analysis ,.....etc. The various mathematical structures encountered in these subjects are formally defined within the language of set theory.

The mathematical frameworks of our best physical theories are formulated around mathematical structures such as manifolds , vector spaces , ...etc. For instance , the mathematical framework of Classical mechanics is formulated around a very special type of manifold (symplectic manifold) and the mathematical framework of standard Quantum mechanics is formulated around a very special type of vector space (Hilbert space). All these mathematical structures have the set structure underlying them. So it seems obvious (at least to me) that , if one wishes to construct a general axiomatic framework , from which the mathematical frameworks of physical theories that rely on particular mathematical structures are to be derived as special cases , then a natural starting point to construct such a general framework must be set theory ! Therefore in my view , set theory can provide a good axiomatic foundations for theoretical physics , just as it did for mathematics in the 20th century !

In recent years , there has been a lot of enthusiasm around category theory in both mathematics and physics , a lot of attempts have been made to apply categorical concepts to physics (quantum mechanics in particular). Although I am not enthusiastic about category theory , the axiomatic framework we develop in this paper is in my opinion flexible enough for a generalization to categories , so that one may talk about S - formalization on an abstract category C !

Convention

- \emptyset will denote the empty set.
- Given two sets X_1 and X_2 , their union will be written $X_1 \cup X_2$ as usual and their intersection written $X_1 \cap X_2$ as usual. In particular $X_1 \subseteq X_2$ means “ X_1 is contained in X_2 “ and $\text{Map}(X_1, X_2)$ denotes the set of all maps $f : X_1 \rightarrow X_2$.
- Given a finite sequence of nonempty sets X_1, \dots, X_n , $\prod_{i=1}^n X_i = X_1 \times \dots \times X_n$ denotes their standard Cartesian product.
- \mathbb{R} denotes the set of real numbers and \mathbb{C} denotes the set of complex numbers.
- M will denote a symplectic manifold and $C^\infty(M)$ will denote the set of all real valued smooth maps $f : M \rightarrow \mathbb{R}$. $C^\infty(M)$ has a natural structure of a Poisson algebra.
- \mathcal{H} will denote a separable complex Hilbert space and $\mathcal{A}(\mathcal{H})$ will denote the set of all self - adjoint operators on \mathcal{H} . Also $B(\mathcal{H})$ denotes the set of all bounded operators on \mathcal{H} .
- A^* will denote a unital \mathbb{C}^* - algebra and $F^+(A^*, \mathbb{C})$ will denote the set of all positive linear functionals $\mu : A^* \rightarrow \mathbb{C}$. Also $\mathcal{A}(A^*)$ will denote the set of all self - adjoint elements in A^* .
- $\hbar = \frac{h}{2\pi}$, where h denotes the Planck's constant . We shall set $\hbar = 1$ whenever convenient to do so.

1 The Mathematical Frameworks of Classical and Quantum mechanics

The Classical and quantum mechanical frameworks for describing a physical system S consist in the following two schemes :

1. Classical mechanics. We first identify a symplectic manifold M (phase space), then we identify the states of our system S with points in M and the observables of S with real valued smooth maps on M , i.e. elements of $C^\infty(M)$. The dynamics of the system is given by the flow generated by the Hamiltonian vector field on M . This scheme works so well in describing macroscopic objects such as nasa rockets and baseballs, however the scheme fails to describe subatomic objects such as electrons !

2. Quantum mechanics (Hilbert space approach). We first identify a separable complex Hilbert space \mathcal{H} , then we identify the states of our system S with vectors (but sometimes with rays) in \mathcal{H} and the observables of S with self-adjoint operators on \mathcal{H} . Now in respect to the dynamics, there are two pictures : the Schrodinger picture where the state vectors evolve in “time“ according to the so-called Schrodinger equation, and the Heisenberg picture where the observables evolve according to Heisenberg’s equations. The abstract Hilbert space approach to quantum mechanics was due to von Neumann, when he realized that the two versions of quantum mechanics, namely Heisenberg’s matrix mechanics and Schrodinger’s wave mechanics had something very deep in common. That deep thing was the mathematical structure of Hilbert space that nobody had yet defined ! More precisely, Heisenberg was using the space l^2 of square-summable sequences of complex numbers while Schrodinger was using the space $L^2(\mathbb{R}^3)$ of square integrable complex functions, so interestingly both physicists were already working with Hilbert spaces without being aware of it !

Another approach to quantum mechanics, is the algebraic approach. In this approach, we identify a unital C^* -algebra A^* , then we associate the states of our system S with positive linear functional on A^* and the observables with the self-adjoint elements of A^* .

1.1 The relationship between the Classical and Quantum mathematical frameworks

As we can see from above, the language of the mathematical framework of classical mechanics is symplectic geometry, while the language of quantum framework is basically linear algebra (with some sophisticated analysis). However, according to a certain physicists convention, if $\psi \in \mathcal{H}$ is a vector representing some state of system S and $\beta \in \mathbb{C}$ is non zero, then $\beta\psi$ represents the same state as ψ ! If we accept this convention, then the quantum mechanical state space of S is actually the projective Hilbert space $\mathbb{P}\mathcal{H}$, i.e. $\mathbb{P}\mathcal{H} = \mathcal{H} - \{0\} / \sim$, where \sim is the equivalence relation $\psi_1 \sim \psi_2$ iff there exists a non zero $\beta \in \mathbb{C}$ such that $\psi_1 = \beta\psi_2$. Hence the states of S are actually represented by rays (, i.e. equivalence classes).

Now the projective space $\mathbb{P}\mathcal{H}$ has obviously very interesting geometrical properties, for instance it can be made into a Kahler manifold. Indeed many attempts have been made to geometrize quantum mechanics via the projective Hilbert spaces of the systems under consideration, however all the attempts are undermined by the lack of another physicists convention, namely the superposition principle, which is a central conceptual tool for very important practical applications of quantum mechanics. Because of this obstacle, some argue that the superposition principle should be generalized in a non trivial geometrical fashion, while others argue that the principle is not essential and so not necessary at all ! Anyway for most practical applications of quantum mechanics, the linear algebraic structure of the Hilbert space is more predominant than the projective structure of the Hilbert itself.

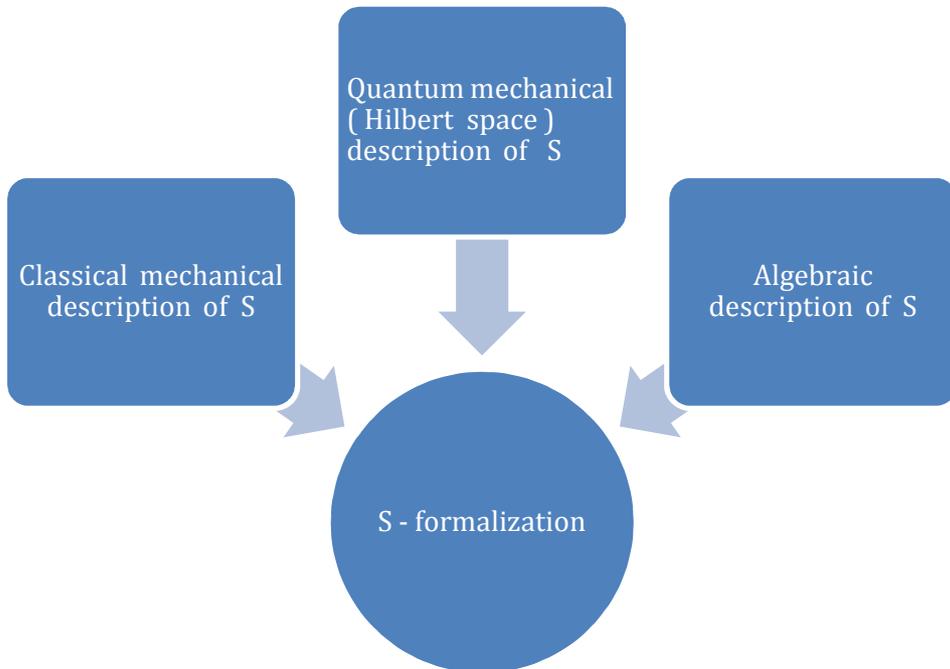
Quantization. Given the differences between the classical and quantum mathematical frameworks, an attempt known as quantization was introduced in the literature. Roughly speaking, the goal of quantization is to find a general procedure that produces a quantum mechanical description of a system S from its classical description. Roughly speaking, given a system S and its classical phase

space M with symplectic form ω , one can view quantization (in a naive Dirac sense) as a procedure Q satisfying conditions such as the following :

- $Q(M) = \mathcal{H}_M$, where \mathcal{H}_M is a certain separable complex Hilbert space.
- $Q(f) \in \mathcal{A}(\mathcal{H}_M)$, where $f \in C^\infty(M)$ represents some observable of system S .
- $Q(\beta \cdot f) = \beta \cdot Q(f) \quad \forall f \in C^\infty(M)$ and $\forall \beta \in \mathbb{R}$.
- $Q(f_1 + f_2) = Q(f_1) + Q(f_2) \quad \forall f_1, f_2 \in C^\infty(M)$.
- $Q(\{f_1, f_2\}) = i\hbar [Q(f_1), Q(f_2)]$, where $\{, \}$ is the standard Poisson bracket defined as $\{f_1, f_2\} = \omega(X_{f_1}, X_{f_2})$ and $[,]$ is the standard commutator of operators given as $[Q(f_1), Q(f_2)] = Q(f_1)Q(f_2) - Q(f_2)Q(f_1)$.
- $Q(\text{id}_{C^\infty(M)}) = I_{\mathcal{H}_M}$, where $\text{id}_{C^\infty(M)}$ is the identity in $C^\infty(M)$ and $I_{\mathcal{H}_M}$ is the identity operator in \mathcal{H}_M .

Unfortunately there isn't a general (and physically plausible) quantization procedure on phase spaces of arbitrary physical systems. In fact there are the so-called "No go theorems" against the possibility of a mathematically and physically plausible general quantization procedure over certain phase spaces including the standard phase space \mathbb{R}^{2n} ! Geometric quantization and deformation quantization, are currently very popular approaches amongst mathematicians and mathematical physicists.

From a foundational point of view, our goal is to make the S -formalization framework a common ground for both classical and quantum mathematical frameworks as illustrated in the following diagram :



The easiest and obvious way of uniting the classical and quantum mathematical frameworks into a single general axiomatic framework, is to first identify the fundamental notions that are essential in both mathematical frameworks, then to formulate the general axiomatic framework around these fundamental notions. This is indeed my general strategy towards the S -formalization framework. In particular, the notions of "state and observable" are in my view the most fundamental notions of both classical and quantum mathematical frameworks! Therefore, the S -formalization framework will be formulated around these two central notions of both classical and quantum mathematical frameworks.

2 Mathematical concepts towards the general axiomatic framework

Definition 2. Let X_1 and X_2 be sets. A bridge from X_1 to X_2 is a pair written $(X_1, \mathring{A}(X_1, X_2))$ such that $\mathring{A}(X_1, X_2) \subseteq \text{Map}(X_1, X_2)$.

The subset $\mathring{A}(X_1, X_2)$ is called a bridge support. When $X_2 \subseteq X_1$ then $(X_1, \mathring{A}(X_1, X_2))$ is called an auto-bridge on X_1 . For our purpose in this paper, X_1 and X_2 are either mathematical structures or contained in some mathematical structure. We can make a lot of interesting constructions with bridges, but in this manuscript we shall just focus on setting the general framework rather than mathematical constructions, however a patient reader is encouraged to do so! Hopefully there will be a follow up of this manuscript in future that will give more emphasis on mathematical constructions. A category theorist can obviously define the notion of bridge as follows: let X_1 and X_2 be objects in a category \mathbf{C} , a bridge from X_1 to X_2 is a pair written $(X_1, \mathring{A}(X_1, X_2))$ such that $\mathring{A}(X_1, X_2) \subseteq \text{Hom}_{\mathbf{C}}(X_1, X_2)$ (the set of all morphisms from X_1 to X_2). Thus one may recover the notion of bridge on sets by putting $\mathbf{C} = \mathbf{Sets}$ (category of sets).

The set of all bridges from X_1 to X_2 is $\mathcal{B}(X_1, X_2) = \{ (X_1, \mathring{A}(X_1, X_2)) : \mathring{A}(X_1, X_2) \subseteq \text{Hom}_{\mathbf{C}}(X_1, X_2) \}$. A category theorist is invited to have some fun with $\mathcal{B}(X_1, X_2)$!

Some interesting examples

- For $X_1 = M, X_2 = \mathbb{R}$ and $\mathring{A}(M, \mathbb{R}) = C^\infty(M)$, one obtains the bridge $(M, C^\infty(M))$.
- For $X_1 = X_2 = \mathcal{H}$ and $\mathring{A}(\mathcal{H}, \mathcal{H}) = \mathcal{A}(\mathcal{H})$, one obtains the bridge $(\mathcal{H}, \mathcal{A}(\mathcal{H}))$. This is of course an auto-bridge on \mathcal{H} .
- For $X_1 = A^*, X_2 = \mathbb{C}$ and $\mathring{A}(A^*, \mathbb{C}) = F^+(A^*, \mathbb{C})$, one obtains $(A^*, F^+(A^*, \mathbb{C}))$.
- Let \mathcal{M}_1 and \mathcal{M}_2 be smooth manifolds. Now let $\text{Diff}(\mathcal{M}_1, \mathcal{M}_2)$ denote the set of all the diffeomorphisms from \mathcal{M}_1 to \mathcal{M}_2 . Now for $X_1 = \mathcal{M}_1, X_2 = \mathcal{M}_2$ and $\mathring{A}(\mathcal{M}_1, \mathcal{M}_2) = \text{Diff}(\mathcal{M}_1, \mathcal{M}_2)$, one obtains the bridge $(\mathcal{M}, \text{Diff}(\mathcal{M}_1, \mathcal{M}_2))$.
- Let T be a topological space and let $\text{Path}(T)$ be the set of all paths in T . If we set $X_1 = [0, 1], X_2 = T$ and $\mathring{A}([0, 1], T) = \text{Path}(T)$, then one obtains the bridge $([0, 1], \text{Path}(T))$.

Definition 2.1. Let $(X_1, \mathring{A}(X_1, X_2))$ be a bridge and let X_3 be a nonempty set. An extension of $(X_1, \mathring{A}(X_1, X_2))$ on X_3 is defined as $(X_1, \mathring{A}(X_1, X_2) \cup X_3)$.

Some examples

- For $(X_1, \mathring{A}(X_1, X_2)) = (M, C^\infty(M))$ and $X_3 = \mathbb{R}$, one obtains $(M, C^\infty(M) \cup \mathbb{R})$.
- For $(X_1, \mathring{A}(X_1, X_2)) = (\mathcal{H}, \mathcal{A}(\mathcal{H}))$ and $X_3 = \mathbb{R}$, one obtains $(\mathcal{H}, \mathcal{A}(\mathcal{H}) \cup \mathbb{R})$.
- For $(X_1, \mathring{A}(X_1, X_2)) = (A^*, F^+(A^*, \mathbb{C}))$ and $X_3 = \mathcal{A}(A^*)$, one obtains $(A^*, F^+(A^*, \mathbb{C}) \cup \mathcal{A}(A^*))$.

Definition 2.2. $(X_1, \mathring{A}(X_1, X_2) \cup X_3)$ is a proper extension if $\mathring{A}(X_1, X_2) \cap X_3 = \emptyset$, otherwise it's called an improper extension.

Obviously if $X_3 \subseteq \mathring{A}(X_1, X_2)$, then $(X_1, \mathring{A}(X_1, X_2) \cup X_3)$ is an improper extension. Hence a bridge $(X_1, \mathring{A}(X_1, X_2))$ can always be seen as an improper extension by assuming $X_3 \subseteq \mathring{A}(X_1, X_2)$! For instance if $(X_1, \mathring{A}(X_1, X_2)) = (\mathcal{H}, \mathcal{A}(\mathcal{H}))$ and $X_3 \subseteq \mathcal{A}(\mathcal{H})$, then $(\mathcal{H}, \mathcal{A}(\mathcal{H}) \cup X_3) = (\mathcal{H}, \mathcal{A}(\mathcal{H}))$.

Observe that for a Hilbert space \mathcal{H} , $(\mathcal{H}, \mathcal{A}(\mathcal{H}) \cup \mathbb{R})$ is a proper extension iff $\dim \mathcal{H} \geq 2$, this is because self - adjoint operators of one dimensional Hilbert spaces are real numbers and so we obviously can't have $\mathcal{A}(\mathcal{H}) \cap \mathbb{R} = \emptyset$ if $\dim \mathcal{H} = 1$. In fact for our purpose in this manuscript, the Hilbert spaces will be assumed to be of dimension two or greater. The concept of bridge extension is indeed, the mathematical tool from which the S - formalization framework will be elaborated.

3 Towards the axiomatic formalization of physical systems

In this section, we shall use bridge extensions to introduce the notion of formalization of a physical system. The generalization to joint systems is straightforward and it will be done through **sections 6** and **7**.

Before we start with our axiomatic framework, let us review four of the fundamental postulates of quantum mechanics (Schrodinger picture) as normally presented in standard textbooks :

Postulate 1. At a given “time” t , the state of a physical system S is represented by a normalized vector $\Psi_t \in \mathcal{H}$ called state vector, where \mathcal{H} is a separable complex Hilbert space.

Postulate 2. An observable \mathbf{a} is represented by a self - adjoint operator written $\hat{\mathbf{a}}$ and acting on the Hilbert space \mathcal{H} .

Postulate 3. A state vector Ψ_t satisfies the Schrodinger equation $i\hbar \frac{d}{dt} \Psi_t = \hat{h} \Psi_t$, where \hat{h} is the self - adjoint operator representing the Hamiltonian (i.e. total energy) of S .

Postulate 4. If an observable \mathbf{a} is represented by a self - adjoint operator $\hat{\mathbf{a}}$ and a state is represented by $\Psi_t \in \mathcal{H}$, then the expectation value of measurement \mathbf{a} is $\langle \Psi_t, \hat{\mathbf{a}} \Psi_t \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product in \mathcal{H} .

The above are the fundamental postulates of quantum mechanics, physicists have learned how to use them to make some of the most fantastic predictions in mankind history. However, as far as the postulates are concerned, it's not an exaggeration to say that the physics community is currently divided in two groups :

- **Shut up and calculate group.** In this group, the general attitude is to not be worried or ask about the meaning of the postulates above, but to just use them to calculate until there is a strong experimental evidence that the predictions generated by the calculations are wrong !

- **I am not convinced group.** Here, the general attitude is to not take the postulates seriously until their physical meaning is found ! In particular, one must find clear answers for questions such as : “ why Hilbert spaces “, “ what is state vector ? does it tell us something about reality ? “, “ why self - adjoint operators ? “, “ why Schrodinger equation ? “.

I sympathize with the latter group, however as already stated in the introduction, my view is that quantum mechanics is just a special case of a general abstract mathematical scheme that hasn't been yet clarified or fully understood by physicists ! In particular, my view is that abstraction is the only viable option to clarify and give this abstract scheme a clear foundations. This is in contrast to most research projects on quantum foundations, where there is more focus on finding meaningful physical principles (if there is any!) underlying the mathematical frameworks of the subject, just like Einstein found the physical principles underlying the Lorentz transformations that already existed before special relativity !

Now the reader may genuinely ask, if there are “meaningful” physical principles from which the mathematical framework of quantum mechanics can be derived, why is it taking so long to find such principles? One possible answer is that, perhaps there isn't any “meaningful” physical principle underlying the quantum mathematical framework. Another possible answer is that, there are “meaningful” physical principles right under our nose, but we are not sufficiently clever to realize that! In recent years, there have been a huge interest in applying information theoretic methods to quantum foundations, this leads to the so-called quantum information theory. However the big issues on quantum foundations, still remain big issues!

Convention

- For any physical system S , $E(S)$ denotes the set of all states of S and $\mathcal{K}(S)$ denotes the set of all observables of S . $E(S)$ is called the state set of S and $\mathcal{K}(S)$ is the observable's set.

The starting point towards our axiomatic approach is to not assume any particular mathematical structure on $E(S)$ and $\mathcal{K}(S)$, but to just view the two as abstract sets!

Definition 3. Let $(X_1, \mathring{A}(X_1, X_2) \cup X_3)$ be a bridge extension and let S be a physical system. A S -formalization on $(X_1, \mathring{A}(X_1, X_2) \cup X_3)$ is a quadruple $(\mathcal{W}(S), \hat{\mathcal{K}}(S), \Gamma_1, \Gamma_2)$ satisfying the following axioms:

- (i) $\mathcal{W}(S) \subseteq E(S)$.
- (ii) $\hat{\mathcal{K}}(S) \subseteq \mathcal{K}(S)$ such that $h \in \hat{\mathcal{K}}(S)$, where h is the Hamiltonian of S .
- (iii) $\Gamma_1 \in \text{Map}(\mathcal{W}(S), X_1)$ and $\Gamma_2 \in \text{Map}(\hat{\mathcal{K}}(S), \mathring{A}(X_1, X_2) \cup X_3)$.

Remarks :

- the subsets $\mathcal{W}(S)$ and $\hat{\mathcal{K}}(S)$ are called formalization supports for system S .
- the maps Γ_1 and Γ_2 are called formal preparation maps for system S .
- $\forall \alpha \in \mathcal{W}(S)$, $\Gamma_1(\alpha)$ is called a formal state of S and $\forall \mathbf{a} \in \hat{\mathcal{K}}(S)$, $\Gamma_2(\mathbf{a})$ is called formal observable of S . Consequently as we shall see, “state vectors” are special cases of “formal states”.
- For notational convenience, we shall write $\Sigma(S)$ instead of $(\mathcal{W}(S), \hat{\mathcal{K}}(S), \Gamma_1, \Gamma_2)$.

The maps Γ_1 and Γ_2 are formalization of the daily operational procedure that working physicists take to associate the physical concepts of “state and observable” with mathematical entities such as for example “vector, ray or linear operators in a Hilbert space”! This means that, physicists are already using S -formalization in daily basis without being aware of it! Just like Schrodinger and Heisenberg were already using Hilbert spaces before its formal definition. Right now, our setting is too general for us to construct an interesting example of a S -formalization, but we shall construct an interesting example shortly in the next section.

As far as $\Gamma_1(\alpha)$ and $\Gamma_2(\mathbf{a})$ are concerned, my personal view is that, $\Gamma_1(\alpha)$ and $\Gamma_2(\mathbf{a})$ are just abstract tools of obtaining some knowledge (information) about our system S under consideration! Consequently in my personal view, the state vectors in standard quantum mechanics are just abstract tools to obtain some knowledge about physical systems!

Definition 3.1. An observable $\mathbf{a} \in \hat{\mathcal{K}}(S)$ is called a functional observable if $\Gamma_2(\mathbf{a}) \in \mathring{A}(X_1, X_2)$, otherwise it's called a non-functional observable.

The main motivation for the definition above is the following : there are observables in nonrelativistic quantum mechanics that are not represented by operators (functions) ! For instance , the **mass** of a nonrelativistic particle moving in line is not represented by an operator .On the other hand , the **position** and the **Hamiltonian** of the particle are represented by operators on the relevant Hilbert space of the particle.

Also in classical mechanics , the **mass** of the particle is not represented by a real valued smooth map on the phase space , it appears instead as a structural quantity. However the **position** and the **Hamiltonian** of the particle are represented by real valued smooth maps on the phase space.

Hence in our abstract language , the **mass** of a nonrelativistic particle is a non - functional observable , while the **position** and the **Hamiltonian** of the particle are functional observables !

Remark. We shall now adopt the following change of notation :

- we shall write $\Gamma_1(\alpha)$ as Ψ^α (or sometimes as Ψ when α is evident from the context).
- we shall write $\Gamma_2(\mathbf{a})$ as $\hat{\mathbf{a}}$ if $\Gamma_2(\mathbf{a}) \in \mathring{A}(X_1, X_2)$.

Definition 3.2. Let $\Sigma(S)$ be a S - formalization. $\Sigma(S)$ is called an ideal S - formalization if the Hamiltonian h is a functional observable , i.e. $\Gamma_2(h) \in \mathring{A}(X_1, X_2)$.

Ideal S - formalizations are very interesting , because as we shall see next , both the classical and quantum mathematical frameworks are special cases of ideal S - formalization.

4 Classical formalization and von Neumann formalization

In this section we shall formally derive the kinematical structure of classical and quantum mechanics.

Definition 4. Let $\Sigma(S)$ be an ideal S - formalization. $\Sigma(S)$ is called a classical formalization if the following inclusions hold :

- $X_1 \subseteq M$ and $X_2 \subseteq \mathbb{R}$, where M is a certain symplectic manifold called phase space.
- $\mathring{A}(X_1, X_2) \subseteq C^\infty(M)$.

Thus , the definition above implies the following :

- $\Psi^\alpha \in M \quad \forall \alpha \in \mathcal{W}(S)$, i.e. the formal states are points in the symplectic manifold M .
- $\hat{\mathbf{a}} \in C^\infty(M) \quad \forall \hat{\mathbf{a}}$, i.e. the formal observables of the functional observables are real valued smooth maps on M . Therefore $\hat{h} \in C^\infty(M)$.

Hence we have derived the kinematical structure of classical mechanics ! Since our main focus is quantum mechanics , we shall leave the classical formalization for now. However the reader is encouraged to reconstruct other features of classical mechanics such as the classical dynamics !

Definition 4.1. Let $\Sigma(S)$ be an ideal S - formalization. $\Sigma(S)$ is called a von Neumann formalization if the following inclusions hold :

- $X_2 \subseteq X_1 \subseteq \mathcal{H}$, where \mathcal{H} is a certain separable complex Hilbert space called von Neumann state space.
- $\mathring{A}(X_1, X_2) \subseteq \mathcal{A}(\mathcal{H})$.

Thus , the definition above implies the following :

- $\Psi^\alpha \in \mathcal{H} \quad \forall \alpha \in \mathcal{W}(S)$, i.e. the formal states are members of the separable Hilbert space \mathcal{H} .
- $\hat{\mathbf{a}} \in \mathcal{A}(\mathcal{H}) \quad \forall \hat{\mathbf{a}}$, i.e. the formal observables of the functional observables are self - adjoint operators on \mathcal{H} . Therefore $\hat{h} \in \mathcal{A}(\mathcal{H})$.

Hence we have derived the kinematical structure of standard quantum mechanics ! Now we just need to derive the dynamics and the other essential features !

Simple example of von Neumann formalization

Let the system S be an electron in a magnetic field. We may construct a von Neumann formalization $\Sigma(S)$ as follows :

- Let $\mathcal{W}(S) = \{\alpha^\uparrow, \alpha^\downarrow\}$, where α^\uparrow is the state “spin up” of the electron and α^\downarrow is the state “spin down” ! Now let $\hat{\mathbf{K}}(S) = \{S_x, S_y, S_z, h\}$, where S_x is the spin along x ; S_y the spin along y ; S_z the spin along z and h is the Hamiltonian.

If we now consider the improper extension $(X_1, \hat{\mathbf{A}}(X_1, X_2) \cup X_3) = (\mathbb{C}^{2 \times 1}, \mathcal{A}(\mathbb{C}^{2 \times 1}))$, where $\mathbb{C}^{2 \times 1}$ is the two dimensional Hilbert space of column matrices over \mathbb{C} , then we can construct the maps Γ_1 and Γ_2 as follows :

- $\Gamma_1(\alpha^\uparrow) = \Psi^{\alpha^\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Gamma_1(\alpha^\downarrow) = \Psi^{\alpha^\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- $\Gamma_2(S_x) = \hat{S}_x = \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$; $\Gamma_2(S_y) = \hat{S}_y = \begin{pmatrix} 0 & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & 0 \end{pmatrix}$;

- $\Gamma_2(S_z) = \hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$; $\Gamma_2(h) = \hat{h} = \begin{pmatrix} \frac{\hbar\omega_x}{2} & \frac{\hbar\omega_x - i\hbar\omega_y}{2} \\ \frac{\hbar\omega_x + i\hbar\omega_y}{2} & \frac{-\hbar\omega_z}{2} \end{pmatrix}$ where $\omega_x, \omega_y, \omega_z$ are certain real numbers associated with the magnetic field along the axis x, y and z respectively.

Definition 4.2. Let $\Sigma(S)$ be a von Neumann formalization. For all $\hat{\mathbf{a}}$ and $\Psi^\alpha \in D(\hat{\mathbf{a}})$, we define the fraction $\hat{\mathbf{a}}\langle \Psi^\alpha \rangle = \langle \Psi^\alpha, \hat{\mathbf{a}} \Psi^\alpha \rangle / \|\Psi^\alpha\|^2$, where $\langle \cdot, \cdot \rangle$ is the inner product in \mathcal{H} ; $\|\cdot\|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$ and $D(\hat{\mathbf{a}})$ is the domain of $\hat{\mathbf{a}}$.

$\hat{\mathbf{a}}\langle \Psi^\alpha \rangle$ is called the expectation value of $\hat{\mathbf{a}}$ over Ψ^α . Now since $\hat{\mathbf{a}}$ is a self-adjoint operator, then obviously $\hat{\mathbf{a}}\langle \Psi^\alpha \rangle \in \mathbb{R} \quad \forall \Psi^\alpha \in D(\hat{\mathbf{a}})$.

Example

Taking the example of the electron above, one can compute the following :

- If $\hat{\mathbf{a}} = \hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$ and $\Psi^\alpha = \Psi^{\alpha^\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $\hat{S}_z\langle \Psi^{\alpha^\uparrow} \rangle = \frac{\hbar}{2}$.

- If $\hat{\mathbf{a}} = \hat{S}_z = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$ and $\Psi^\alpha = \Psi^{\alpha^\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $\hat{S}_z\langle \Psi^{\alpha^\downarrow} \rangle = -\frac{\hbar}{2}$.

Ψ^{α^\uparrow} and Ψ^{α^\downarrow} are obviously eigenvectors of \hat{S}_z with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively. In fact, the formal states Ψ^{α^\uparrow} and Ψ^{α^\downarrow} form an orthonormal basis in $\mathbb{C}^{2 \times 1}$.

Definition 4.3. We shall define $\Delta(\hat{\mathbf{a}}, \Psi^\alpha) = \sqrt{\hat{\mathbf{a}}^2\langle \Psi^\alpha \rangle - \hat{\mathbf{a}}\langle \Psi^\alpha \rangle^2}$ for all $\hat{\mathbf{a}}$ and $\Psi^\alpha \in D(\hat{\mathbf{a}})$.

The number $\Delta(\hat{\mathbf{a}}, \Psi^\alpha)$ is called von Neumann's dispersion of $\hat{\mathbf{a}}$ on Ψ^α . Obviously we have that $\Delta(\hat{\mathbf{a}}, \Psi^\alpha) = 0$ iff Ψ^α is an eigenvector of $\hat{\mathbf{a}}$. Thus if we consider the previous example,

then we have $\Delta(\hat{S}_z, \psi^{\alpha^\dagger}) = \Delta(\hat{S}_z, \psi^{\alpha^\dagger}) = 0$! Now given two formal observables $\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2$ such that $\psi^\alpha \in D(\hat{\mathbf{a}}_1)$ and $\psi^\alpha \in D(\hat{\mathbf{a}}_2)$, we have that $\Delta(\hat{\mathbf{a}}_1, \psi^\alpha) \Delta(\hat{\mathbf{a}}_2, \psi^\alpha) \geq \frac{1}{2} |[\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2] \langle \psi^\alpha \rangle|$, this is what physicists call generalized uncertainty relation. When $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are position and momentum operators respectively, one has the so called Heisenberg uncertainty relation !

5 Unitary States and Schrodinger formalization

According to a postulate in quantum mechanics (Schrodinger picture), at given “time” t , a state vector Ψ_t satisfies the Schrodinger equation $i\hbar \frac{d}{dt} \Psi_t = \hat{H} \Psi_t$. The formal solution of this equation is of course $\Psi_t = \exp\left(\frac{-it\hat{H}}{\hbar}\right) \Psi_0$, where Ψ_0 is the initial state vector. Our goal in this section is to find an abstract and natural derivation of Schrodinger picture within our axiomatic scheme. This also means that, we shall find an abstract and natural derivation of “time” within our axiomatic scheme !

Definition 5. Let $\Sigma(S)$ be a S -formalization and let Ω be a nonempty set. $\Sigma(S)$ is called a state index formalization on base Ω if the subset $\mathcal{W}(S)$ is indexed by a subset $T \subseteq \Omega$, i.e.

$$\mathcal{W}(S) = \{ \alpha_t \}_{t \in T \subseteq \Omega} .$$

The index t is called Ω -state index (or just state index). As obvious, we shall also write Ψ^{α_t} to denote $\Gamma_1(\alpha_t)$. Now Ω doesn't have to be just a set, we may require Ω to be a mathematical structure or contained in some mathematical structure. For instance Ω can be an ordinary space-time manifold or contained on it !

Definition 5.1. $\Sigma(S)$ is called a classical state index formalization if $\Omega \subseteq \mathbb{R}$.

When $\Sigma(S)$ is a classical state index formalization, the state index t is then renamed as “classical time” index ! Recall that in quantum mechanics (Schrodinger picture), “time” appears as an external label attached to the formal states (state vectors) of the system under consideration.

Definition 5.2. Let $\Sigma(S)$ be a classical state index formalization. $\Sigma(S)$ is a von Neumann state index formalization if it is also a von Neumann formalization.

Definition 5.3. Let $\Sigma(S)$ be a von Neumann state index formalization. A formal state Ψ^{α_t} is called unitary state if the following two conditions hold :

(i) $\Psi^{\alpha_t} \in D(\hat{H})$, where $D(\hat{H})$ is the domain of \hat{H} .

(ii) There exists a formal state $\Psi^{\alpha_{\ddot{t}}} \in D(\hat{H})$ such that $\Psi^{\alpha_t} = \exp\left(\frac{-it\hat{H}}{\hbar}\right) \Psi^{\alpha_{\ddot{t}}}$.

The formal state $\Psi^{\alpha_{\ddot{t}}}$ is called an initial state to Ψ^{α_t} . Obviously Ψ^{α_t} satisfies the Schrodinger equation $i\hbar \frac{d}{dt} \Psi^{\alpha_t} = \hat{H} \Psi^{\alpha_t}$. However please observe that, there may be formal states that are not unitary and thus don't satisfy the Schrodinger equation ! These kind of formal states may naturally arise when for instance \hat{H} is an unbounded operator (i.e. not defined for every vector on von Neumann state space \mathcal{H}) ! For practical and esthetical reasons, a physicist would rather have all the formal states satisfying the Schrodinger equation, indeed this is the motivation for definition 5.4 below.

Definition 5.4. Let $\Sigma(S)$ be a von Neumann state index formalization. $\Sigma(S)$ is a Schrodinger formalization if every formal state is unitary, i.e. every formal state satisfies the Schrodinger equation.

Hence Schrodinger formalization \Leftrightarrow Standard quantum mechanics (Schrodinger picture), and so we've derived the whole standard quantum mechanics from our general scheme!

Although a little bit tricky, the Heisenberg picture can also be derived by indexing the subset $\hat{\mathcal{K}}(S)$ with the real numbers! In fact, indexing the subset $\hat{\mathcal{K}}(S)$ with points of a manifold (say space-time manifold) may be a natural starting point to bring field theory into our scheme! Please see **Appendix A** for the derivation of the Algebraic approach.

6 Towards an Axiomatic Formalization of Joint Physical Systems

The description of Joint Physical Systems in Classical and Quantum mechanics

As we already know, in classical mechanics the formal states of a physical systems S are points in a symplectic manifold M and the formal (functional) observables are real valued smooth maps on M . In standard quantum mechanics on the other hand, the formal states are vectors (but sometimes rays) in a separable Hilbert space \mathcal{H} and the formal (functional) observables are self-adjoint operators in \mathcal{H} . Now given a finite sequence of physical systems S_1, \dots, S_n , the standard frameworks of classical and quantum mechanics tell us the following respectively:

- If M_1, \dots, M_n are state spaces of the respective systems, then the state space of the joint system is the standard Cartesian product $\prod_{i=1}^n M_i$ of the respective state spaces.
- If $\mathcal{H}_1, \dots, \mathcal{H}_n$ are Hilbert spaces of the respective systems, then the Hilbert space of the joint system is the standard tensor product $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ of the respective Hilbert spaces.

Convention

- For a finite sequence of physical systems S_1, \dots, S_n , let $E(S_i)$ denote the set of all states of S_i and let $\mathcal{K}(S_i)$ denote the set of all observables of S_i . Also here, we shall not assume any particular mathematical structure on the sets $E(S_i)$ and $\mathcal{K}(S_i)$, we shall just view the two as abstract sets!

- $E(S_1, \dots, S_n) = \prod_{i=1}^n E(S_i)$ and $\mathcal{K}(S_1, \dots, S_n) = \prod_{i=1}^n \mathcal{K}(S_i)$. The sets $E(S_1, \dots, S_n)$ and $\mathcal{K}(S_1, \dots, S_n)$ are called Cartesian composition of the systems S_1, \dots, S_n .

- We define the n -tuple $\alpha(1, n) = (\alpha(1), \dots, \alpha(n)) \in E(S_1, \dots, S_n)$, where $\alpha(i) \in E(S_i)$ for all $1 \leq i \leq n$. $\alpha(1, n)$ is called a Cartesian state of the systems S_1, \dots, S_n .

- We now define the n -tuple $\mathbf{a}(1, n) = (\mathbf{a}(1), \dots, \mathbf{a}(n)) \in \mathcal{K}(S_1, \dots, S_n)$, where obviously $\mathbf{a}(i) \in \mathcal{K}(S_i)$ for all $1 \leq i \leq n$. The n -tuple $\mathbf{a}(1, n)$ is called Cartesian observable of the systems S_1, \dots, S_n .

There is a special Cartesian observable of the systems called Cartesian Hamiltonian and it's obviously written as $\mathbf{h}(1, n) = (\mathbf{h}(1), \dots, \mathbf{h}(n))$, where $\mathbf{h}(i)$ denotes the Hamiltonian of system S_i .

Definition 6. A joint Cartesian formalization of systems S_1, \dots, S_n on a bridge extension $(X_1, \mathring{A}(X_1, X_2) \cup X_3)$ is a quadruple $(\mathcal{W}(S_1, \dots, S_n), \hat{\mathcal{K}}(S_1, \dots, S_n), \Gamma_1, \Gamma_2)$ satisfying the following axioms :

- (i) $\mathcal{W}(S_1, \dots, S_n) \subseteq \mathbb{E}(S_1, \dots, S_n)$.
- (ii) $\hat{\mathcal{K}}(S_1, \dots, S_n) \subseteq \mathcal{K}(S_1, \dots, S_n)$ such that $h(1, n) \in \hat{\mathcal{K}}(S_1, \dots, S_n)$.
- (iii) $\Gamma_1 \in \text{Map}(\mathcal{W}(S_1, \dots, S_n), X_1)$ and $\Gamma_2 \in \text{Map}(\hat{\mathcal{K}}(S_1, \dots, S_n), \mathring{A}(X_1, X_2) \cup X_3)$.

Remarks :

- the subsets $\mathcal{W}(S_1, \dots, S_n)$ and $\hat{\mathcal{K}}(S_1, \dots, S_n)$ are called formalization supports of the systems S_1, \dots, S_n .
- the maps Γ_1 and Γ_2 are called formal preparation maps for the joint system.
- $\forall \alpha(1, n) \in \mathcal{W}(S_1, \dots, S_n)$, $\Gamma_1(\alpha(1, n))$ is called a formal state of the joint system and $\forall a(1, n) \in \hat{\mathcal{K}}(S_1, \dots, S_n)$, $\Gamma_2(a(1, n))$ is called formal observable of the joint system.
- we shall write $\Sigma(S_1, \dots, S_n)$ instead of $(\mathcal{W}(S_1, \dots, S_n), \hat{\mathcal{K}}(S_1, \dots, S_n), \Gamma_1, \Gamma_2)$.

Definition 6.1. A Cartesian observable $a(1, n) \in \hat{\mathcal{K}}(S_1, \dots, S_n)$ is called functional observable if $\Gamma_2(a(1, n)) \in \mathring{A}(X_1, X_2)$, otherwise it's a non-functional observable.

Remark. We shall now adopt the following change of notation :

- we shall write $\Gamma_1(\alpha(1, n))$ as $\psi^{(1, n)}$.
- we shall write $\Gamma_2(a(1, n))$ as $\hat{a}(1, n)$ when $\Gamma_2(a(1, n)) \in \mathring{A}(X_1, X_2)$.

Definition 6.2. Let $\Sigma(S_1, \dots, S_n)$ be a joint Cartesian formalization. $\Sigma(S_1, \dots, S_n)$ is an ideal formalization if $h(1, n)$ is a functional observable, i.e. $\Gamma_2(h(1, n)) \in \mathring{A}(X_1, X_2)$.

7 Classical formalization and von Neumann formalization

In this section, we shall naturally derive the kinematical structures of classical and quantum mechanical framework for joint systems S_1, \dots, S_n within our axiomatic scheme.

Definition 7. Let $\Sigma(S_1, \dots, S_n)$ be an ideal formalization. $\Sigma(S_1, \dots, S_n)$ is a classical formalization if the following inclusions hold :

- (i) $X_1 \subseteq M$ and $X_2 \subseteq \mathbb{R}$, where $M = \prod_{i=1}^n M_i$ is a manifold product of certain symplectic manifolds M_1, \dots, M_n called phase spaces.
- (ii) $\mathring{A}(X_1, X_2) \subseteq C^\infty(M)$.

Thus, the definition above implies the following :

- $\psi^{(1, n)} \in M = \prod_{i=1}^n M_i \quad \forall \alpha(1, n) \in \mathcal{W}(S_1, \dots, S_n)$, i.e. the formal states are points in the manifold product M .
- $\hat{\alpha}(1, n) \in C^\infty(M) \quad \forall \hat{\alpha}(1, n)$, i.e. the formal observables of the functional observables are real valued smooth maps on the manifold product M . Therefore $\hat{h}(1, n) \in C^\infty(M)$.
Hence we've derived the classical kinematical structure for joint systems S_1, \dots, S_n !

Definition 7.1. Let $\Sigma(S_1, \dots, S_n)$ be an ideal formalization. $\Sigma(S_1, \dots, S_n)$ is a von Neumann formalization if the following inclusions hold :

- (i) $X_2 \subseteq X_1 \subseteq \mathcal{H}$, where \mathcal{H} is a certain separable Hilbert space called von Neumann state space.
- (ii) $\hat{A}(X_1, X_2) \subseteq \mathcal{A}(\mathcal{H})$.

As obvious, the definition above implies the following :

- $\psi^{(1, n)} \in \mathcal{H} \quad \forall \alpha(1, n) \in \mathcal{W}(S_1, \dots, S_n)$, i.e. the formal states are members of the separable Hilbert space \mathcal{H} .
- $\hat{\alpha}(1, n) \in \mathcal{A}(\mathcal{H}) \quad \forall \hat{\alpha}(1, n)$, i.e. the formal observables of the functional observables are self-adjoint operators on \mathcal{H} . Therefore $\hat{h}(1, n) \in \mathcal{A}(\mathcal{H})$.

Please observe that, because there is no mentioning of tensor products in definition 7.1 above, the definition by itself is not sufficient to derive the quantum mechanical kinematical structure of the joint systems S_1, \dots, S_n , thus we need to add tensor products and this motivates definition 7.2 below !

Definition 7.2. Let $\Sigma(S_1, \dots, S_n)$ be a von Neumann formalization. $\Sigma(S_1, \dots, S_n)$ is a tensor formalization if $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$, where for all $1 \leq i \leq n$, \mathcal{H}_i is a separable complex Hilbert space.

Remarks :

- A formal state $\psi^{(1, n)}$ is called a tensor state if $\psi^{(1, n)} = v_1 \otimes \dots \otimes v_n$, where $v_i \in \mathcal{H}_i$. Otherwise $\psi^{(1, n)}$ is called an entangled state.

Please observe that, the entanglement of $\psi^{(1, n)}$ depends on how the map Γ_1 is constructed, the same is true if $\psi^{(1, n)}$ is a tensor state !

- The inner product of two tensor states $\psi^{(1, n)} = v_1 \otimes \dots \otimes v_n$ and $\dot{\psi}^{(1, n)} = w_1 \otimes \dots \otimes w_n$ is obviously defined as $\langle v_1, w_1 \rangle_{\mathcal{H}_1} \times \dots \times \langle v_n, w_n \rangle_{\mathcal{H}_n}$, where $\langle \cdot, \cdot \rangle_{\mathcal{H}_i}$ denotes the inner product in \mathcal{H}_i and \times is the ordinary multiplication of complex numbers.

- If $\hat{\alpha}(1, n) = F_1 \otimes \dots \otimes F_n$, where $F_i \in \mathcal{A}(\mathcal{H}_i)$ and $\psi^{(1, n)} = v_1 \otimes \dots \otimes v_n$ such that $v_i \in D(F_i)$, then $\hat{\alpha}(1, n)\psi^{(1, n)} = (F_1 v_1) \otimes \dots \otimes (F_n v_n)$.

Hence we have derived the kinematical structure of quantum mechanical framework for joint physical systems ! Please see Appendix B for the dynamics of joint systems. When the spaces \mathcal{H}_i are finite

dimensional then $\mathcal{H} = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ is isomorphic to the space of complex column matrices $\mathbb{C}^{k \times 1}$, where $k = \prod_{i=1}^n \dim \mathcal{H}_i$. So it may be better sometimes to just use $\mathbb{C}^{k \times 1}$ directly for the sake of simplicity !

Simple example of tensor formalization

Let S_1 and S_2 be spin $\frac{1}{2}$ - particles (say two electrons in a magnetic field). If we now chose the improper extension $(X_1, \mathring{A}(X_1, X_2) \cup X_3) = (\mathbb{C}^{2 \times 1} \otimes \mathbb{C}^{2 \times 1}, \mathcal{A}(\mathbb{C}^{2 \times 1} \otimes \mathbb{C}^{2 \times 1}))$, then we may construct a tensor formalization $\Sigma(S_1, S_2)$ of the two joint systems as follows :

- Let $\mathcal{W}(S_1, S_2) = \{\alpha(1, 2) = (\alpha(1)^\uparrow, \alpha(2)^\downarrow)\}$, where $\alpha(1)^\uparrow$ is the state “spin up “ of particle 1 and $\alpha(2)^\downarrow$ is the state “spin down “ of particle 2.

- Let $\hat{\mathcal{K}}(S_1, S_n) = \{a(1, 2) = (S_z(1), S_x(2)), h(1, 2) = (h(1), h(2))\}$, where $S_z(1)$ is spin component of particle 1 along the z axis ; $S_x(2)$ is the spin component of particle 2 along the x axis ; $h(1)$ is the Hamiltonian associated to particle 1 and $h(2)$ is the Hamiltonian associated with particle 2. We may now construct the formalization support maps Γ_1 and Γ_2 as follows ;

- $\Gamma_1(\alpha(1, 2)) = \psi^{(1,2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- $\Gamma_2(a(1, 2)) = \hat{\mathcal{S}}(1, 2) = \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix} \otimes \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$; $\Gamma_2(h(1, 2)) = \hat{h}(1, 2) = \begin{pmatrix} \frac{\hbar\omega_z}{2} & 0 \\ 0 & -\frac{\hbar\omega_z}{2} \end{pmatrix} \otimes \begin{pmatrix} \frac{\hbar\omega_z}{2} & 0 \\ 0 & -\frac{\hbar\omega_z}{2} \end{pmatrix}$, where $\omega_z \in \mathbb{R}$ is a certain parameter associated with the magnetic field along the z axis.

Closing Remarks

With elementary set theory, one was able to construct a general axiomatic framework, then one was able to naturally derive the standard mathematical frameworks of classical and quantum mechanics as special cases of this general framework. Because the general framework was elaborated around sets, one is then free to model the state space of a physical system S with any desired mathematical structure, and so one is not just limited to manifolds or vector spaces (Hilbert spaces in particular).

Apart from providing a new insight on foundations of quantum mechanics, I hope that the S - formalization framework will provide the starting point to construct the mathematical frameworks of new physical theories and also provide a starting point for a better mathematical foundations of other well known physical theories such as relativistic quantum field theory !

Finally, I would like to challenge quantum gravity theorists to construct the mathematical framework of their subject within our axiomatic framework ! I am quite convinced that within our abstract axiomatic framework, the “correct “ mathematical framework for quantum gravity falls into the category of von Neumann formalization, i.e. the kinematical structure is built around a separable Hilbert space and self - adjoint operators on this space. However because of the issue of “time “, I doubt the “correct “ mathematical framework for quantum gravity will ever fall into the category of Schrodinger formalization, i.e. the “correct “ dynamics given by the Schrodinger equation !

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Appendix A. Algebraic formalization

Definition 1 A. Let $\Sigma(S)$ be a S -formalization. $\Sigma(S)$ is called a proper formalization if $(X_1, \hat{A}(X_1, X_2) \cup X_3)$ is a proper extension, i.e. $\hat{A}(X_1, X_2) \cap X_3 = \emptyset$.

Definition 2 A. Let $\Sigma(S)$ be a proper formalization. $\Sigma(S)$ is called an anti-ideal formalization if $\Gamma_2(\mathbf{a}) \in X_3 \forall \mathbf{a} \in \hat{\mathcal{K}}(S)$.

This implies that, all the observables are non-functional, i.e. $\Gamma_2(\mathbf{a}) \notin \hat{A}(X_1, X_2) \forall \mathbf{a} \in \hat{\mathcal{K}}(S)$!

Definition 3 A. Let $\Sigma(S)$ be an anti-ideal formalization. $\Sigma(S)$ is called an algebraic formalization if the following inclusions hold :

(i) $X_2 \subseteq X_1 \subseteq F^+(A^*, \mathbb{C})$, where A^* is a certain unital \mathbb{C}^* -algebra called the formal algebraic space of S .

(ii) $X_3 \subseteq \mathcal{A}(A^*)$.

Thus, the definition above implies the following :

- $\Psi^\alpha \in F^+(A^*, \mathbb{C}) \forall \alpha \in \mathcal{W}(S)$, i.e. the formal states are positive linear functional on A^* .

- $\hat{\mathbf{a}} \in \mathcal{A}(A^*) \forall \hat{\mathbf{a}}$, i.e. the formal observables are self-adjoint elements of A^* . Therefore we have $\hat{h} \in \mathcal{A}(A^*)$.

Hence we've derived the kinematics of the algebraic approach to quantum mechanics. Obviously, one can use the GNS construction to establish a relationship between the algebraic formalization and the von Neumann formalization.

Appendix B Unitary States and Schrodinger formalization

Definition 1 B. Let $\Sigma(S_1, \dots, S_n)$ be a Cartesian formalization and let Ω be a nonempty set. $\Sigma(S_1, \dots, S_n)$ is a state index formalization on base Ω if the subset $\mathcal{W}(S_1, \dots, S_n)$ is indexed by a subset $T \subseteq \Omega$, i.e. $\mathcal{W}(S_1, \dots, S_n) = \{ \alpha(1, n)_t \}_{t \in T \subseteq \Omega}$.

The index t is called Ω -state index (or just state index). As obvious, one shall write $\Psi^{(1, n)_t}$ to denote $\Gamma_1(\alpha(1, n)_t)$. Obviously Ω doesn't have to be just a set, we may require Ω to be a mathematical structure or contained in some mathematical structure. For instance Ω can be an ordinary space-time manifold or contained on it!

Definition 2 B. $\Sigma(S_1, \dots, S_n)$ is a classical state index formalization if $\Omega \subseteq \mathbb{R}$.

Definition 3 B. Let $\Sigma(S_1, \dots, S_n)$ be a classical state index formalization. $\Sigma(S_1, \dots, S_n)$ is a von Neumann state index formalization if it is also a von Neumann formalization.

Definition 4 B. Let $\Sigma(S_1, \dots, S_n)$ be a von Neumann state index formalization. A formal state $\Psi^{(1, n)_t}$ is a unitary state if the following conditions hold :

(i) $\Psi^{(1, n)_t} \in D(\hat{h}(1, n))$, where $D(\hat{h}(1, n))$ is the domain of $\hat{h}(1, n)$.

(ii) There exists a formal state $\psi^{(1, n)t} \in D(\hat{h}(1, n))$ such that $\psi^{(1, n)t} = \exp(-i\hat{h}(1, n)t/\hbar)\tilde{\psi}^{(1, n)t}$.

$\tilde{\psi}^{(1, n)t}$ is called an initial state to $\psi^{(1, n)t}$. The formal state $\psi^{(1, n)t}$ clearly satisfies the Schrodinger equation $i\hbar \frac{d}{dt}\psi^{(1, n)t} = \hat{h}(1, n)\psi^{(1, n)t}$. However, there may be formal states that are not unitary and hence don't satisfy the Schrodinger equation! This may naturally happen if for example $\hat{h}(1, n)$ is an unbounded operator.

For practical and esthetical reasons, a physicist would rather have all the formal states satisfying the Schrodinger equation, this is indeed the motivation for the definition below.

Definition 5 B. Let $\Sigma(S_1, \dots, S_n)$ be a von Neumann state index formalization. $\Sigma(S_1, \dots, S_n)$ is a Schrodinger formalization if every formal state is unitary, i.e. every formal state satisfies the Schrodinger equation.

Hence, we have formulated a Schrodinger picture for joint physical systems in a very natural way! Interesting situation arises when $\Sigma(S_1, \dots, S_n)$ is also a tensor formalization and $\psi^{(1, n)t}$ is a tensor state!

Appendix C. On collapse postulate

From all the standard postulates of quantum mechanics, the only one left out of our axiomatic scheme is the collapse postulate! One reason for this is that, I didn't find a way of formally deriving the postulate within our axiomatic scheme. Another reason is that, as far as I know from physicists, the collapse has never been observed in nature, i.e. there is no experimental support that the collapse occurs in nature!

Reading suggestions (the author feels lazy to write the full details of the reading suggestions below, please google the respective suggestions or search in Arxiv.org for more details!).

1. David Hilbert and the axiomatization of physics, Leo Corry.
2. Towards a general operational and realistic framework for quantum theory and relativity theory, Diederik Aerts and Sven Aerts.
3. Representation of State Property Systems, Diederik Aerts and S. Pulmonova.
4. The Linearity of Quantum Mechanics at Stake, Diederik Aerts, F. Valckenborgh.
5. The Description of Joint Quantum Entities and the Formulation of a Paradox, D. Aerts.
6. On the Algebraic Structure of Quantum Mechanics, J. Gunson.
7. Categorical Quantum Mechanics, S. Abramsky and Bob Coecke.
8. Topos Theory in the Formulation of Theories of Physics, Chris Isham and A. Doering.
9. Relational Quantum Mechanics, Carlo Rovelli.
10. On the notion of reconstruction of quantum theory, Alexei Griunbaum.
11. Mathematical Foundations of Quantum Mechanics, John von Neumann.

12. Mathematical Foundations of Quantum Mechanics , George W. Mackey.
13. Principles of Quantum Mechanics , P. M. Dirac.
14. Quantization , Poisson brackets and beyond , Theodore Voronov.