To the memory of Daihachiro SATO

A NEW FORMULA FOR THE SUM OF THE SIXTH POWERS OF FIBONACCI NUMBERS

HIDEYUKI OHTSUKA AND SHIGERU NAKAMURA

Abstract. Sloane's On-Line Encyclopedia of Integer Sequences incorrectly states a lengthy formula for the sum of the sixth powers of the first n Fibonacci numbers. In this paper we prove a more succinct formulation. We also provide an analogue for the Lucas numbers. Finally, we prove a divisibility result for the sum of certain even powers of the first n Fibonacci numbers.

Sloane's On-Line Encyclopedia of Integer Sequences records, as A098532;

$$\sum_{k=1}^{n} F_k^6 = (1/500)\times$$
$$(F_{6n+1} + 3F_{6n+2} - (-1)^n(16F_{4n+1} + 8F_{4n+2})) - 60F_{2n+1} + 120F_{2n+2} - (-1)^n \times 40.$$

But this is incorrect. The correct formula should be;

$$\sum_{k=1}^{n} F_k^6 = (1/500)\times$$
$$(F_{6n+1} + 3F_{6n+2} - (-1)^n(16F_{4n+1} + 8F_{4n+2}) - 60F_{2n+1} + 120F_{2n+2} - (-1)^n \times 40).$$

Because this is rather lengthy we were motivated to find a simpler elegant formulation. Our formulation is given in Theorem 1. Theorem 2 gives an analogous result for the Lucas numbers. Since its proof is analogous to the proof of Theorem 1, we state Theorem 2 without proof. Finally we prove Theorem 3, in which we give divisibility results for $\sum_{k=1}^{n} F_k^{4p-2}$, where p is a positive integer.

Theorem 1.

$$\sum_{k=1}^{n} F_k^6 = \frac{F_n^5 F_{n+3} + F_{2n}}{4}.$$

Proof. We have

$$\begin{aligned}
0 &= \sum_{k=0}^{n} F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2}F_{k+3} - \sum_{k=1}^{n+1} F_{k-3}F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2} \\
&= \sum_{k=1}^{n} F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2}(F_{k+3} - F_{k-3}) - F_{n-2}F_{n-1}F_nF_{n+1}F_{n+2}F_{n+3} \\
&= 4\sum_{k=1}^{n} F_k^2(F_k^2 + (-1)^k)(F_k^2 + (-1)^{k-1}) - F_nF_{n+3}\big(F_n^2 + (-1)^n\big)(F_n^2 + (-1)^{n-1}\big) \\
&= 4\sum_{k=1}^{n}(F_k^6 - F_k^2) - F_nF_{n+3}(F_n^4 - 1).
\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} F_k^6 &= \frac{F_nF_{n+3}(F_n^4-1)}{4} + \sum_{k=1}^{n} F_k^2 \\ &= \frac{F_n^5F_{n+3} - F_nF_{n+3}}{4} + F_nF_{n+1} \\ &= \frac{F_n^5F_{n+3} - F_n(F_{n+2}+F_{n+1}) + 4F_nF_{n+1}}{4} \\ &= \frac{F_n^5F_{n+3} + F_n(3F_{n+1} - F_{n+2})}{4}.\end{aligned}$$

Finally, Theorem 1 follows from the identity

$$3F_{n+1} - F_{n+2} = 2F_{n+1} - F_n = F_{n+1} + F_{n-1} = L_n.$$

□

Similarly, we have the following 6th power sum formula for the Lucas numbers.

Theorem 2.
$$\sum_{k=1}^{n} L_k^6 = \frac{L_n^5L_{n+3} + 125F_{2n}}{4} - 32.$$

Theorem 3. *Let* $S = \sum_{k=1}^{n} F_k^{4p-2}$ *for a positive integer* p*. Then,*

(1) $F_{n+1} \mid S$ *if* n *is even;*
(2) $F_n \mid S$ *if* n *is odd, respectively.*

Proof. We will use the following identity,

$$F_k^2 + F_{2m-k+1}^2 = F_k^2 + F_{2m-2k+1}F_{2m+1} + (-1)^{2m-2k+1}F_k^2 = F_{2m-2k+1}F_{2m+1}.$$

This identity can be obtained from identities from [1] and [2].

(1) **The case n even.**

Put $n = 2m$, and we have

$$\begin{aligned}S = \sum_{k=1}^{2m} F_k^{4p-2} &= \sum_{k=1}^{m}(F_k^{4p-2} + F_{2m-k+1}^{4p-2}) \\ &= \sum_{k=1}^{m}\Big\{(F_k^2 + F_{2m-k+1}^2)\sum_{i=0}^{2p-2}(-1)^iF_k^{4p-2i-4}F_{2m-k+1}^{2i}\Big\} \\ &= F_{2m+1}\sum_{k=1}^{m}\Big(F_{2m-2k+1}\sum_{i=0}^{2p-2}(-1)^iF_k^{4p-2i-4}F_{2m-k+1}^{2i}\Big).\end{aligned}$$

Thus $F_{2m+1} \mid S$.

(2) **The case n odd.**

Put $n = 2m + 1$, and we have

$$S = \sum_{k=1}^{2m+1} F_k^{4p-2} = \sum_{k=1}^{2m} F_k^{4p-2} + F_{2m+1}^{4p-2}.$$

By (1), $F_{2m+1} \mid \sum_{k=1}^{2m} F_k^{4p-2}$. □

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References

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Bunkyo University High School, 1191-7, Kami, Ageo-city, Saitama Pref., 362-0001, JAPAN,

E-mail address: otsukahideyuki@gmail.com

Prof. emeritus, Tokyo University of marine science and engineering, 3-29-5, Sakuradai, Nerimaku, Tokyo, 176-0002, JAPAN

E-mail address: shigeru-nn@jcom.home.ne.jp