ON $2+2$-DIMENSIONAL SPACETIMES, STRINGS, BLACK-HOLES AND MAXIMAL ACCELERATION IN PHASE SPACES

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Abstract

We study black-hole-like solutions (spacetimes with singularities) of Einstein field equations in $3+1$ and $2+2$-dimensions. In the $3+1$-dim case, it is shown how the horizon of the standard black hole solution at $r = 2G_N M$ can be *displaced* to the location $r = 0$ of the point mass M source, when the radial gauge function is chosen to have an ultra-violet cutoff $R(r = 0) = 2G_N M$ if, and only if, one embeds the problem in the Finsler geometry of the spacetime tangent bundle (or in phase space) that is the proper arena where to incorporate the role of the physical point mass M source at $r = 0$. We find three different cases associated with hyperbolic homogeneous spaces. In particular, the *hyperbolic* version of Schwarzschild's solution contains a conical singularity at $r = 0$ resulting from pinching to *zero* size $r = 0$ the throat of the hyperboloid $\mathcal{H}^2$ and which is quite different from the static spherically symmetric $3+1$-dim solution. Static circular symmetric solutions for metrics in $2+2$ are found that are singular at $\rho = 0$ and whose asymptotic $\rho \to \infty$ limit leads to a flat $1+2$-dim boundary of topology $S^1 \times R^2$. Finally we discuss the $1+1$-dim Bars-Witten *stringy* black-hole solution and show how it can be embedded

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into our $3+1$-dimensional solutions with a *displaced* horizon at $r=0$ and discuss the plausible *stringy* nature of a point-mass, along with the maximal acceleration principle in the spacetime tangent bundle (maximal force in phase spaces). Black holes in a $2+2$-dimensional "spacetime" from the perspective of complex gravity in $1+1$ complex dimensions and their quaternionic and octonionic gravity extensions deserve furher investigation. An appendix is included with the most general Schwarzschild-like solutions in $D \geq 4$.

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1.- Introduction

Through the years it has become evident that the 2+2-signature is not only mathematically interesting [1]-[2] (see also Refs. [3]-[5]) but also physically. In fact, the 2+2−signature emerges in several physical context, including self-dual gravity *a la* Plebanski (see Ref. [6] and references therein), consistent $N=2$ superstring theory as discussed by Ooguri and Vafa [7], $N=(2,1)$ heterotic string [8]-[10]. Moreover, it has been emphasized [11]-[12] that Majorana-Weyl spinor exists in spacetime of $2+2-$signature. Even cosmologically there is a wisdom [13] that the $2+2-$signature is interesting.

In [21] it was shown how a $\mathcal{N}=2$ Supersymmetric Wess-Zumino-Novikov-Witten model valued in the area-preserving (super) diffeomorphisms group is Self Dual Supergravity in 2+2 and 3+1 dimensions depending on the signatures of the base manifold and target space. The interplay among $\mathcal{W}_\infty$ gravity, $\mathcal{N}=2$ Strings, self dual membranes, $SU(\infty)$ Toda lattices and $SU^*(\infty)$ Yang-Mills instantons in $2+2$ dimensions can be found also [21] .

More recently, using the requirement of the $SL(2,R)$ and Lorentz symmetries it has been proved [14] that $2+2$-target spacetime of a 0-brane is an exceptional signature. Moreover, following an alternative idea to the notion of worldsheets for worldsheets proposed by Green [15] or the 0−branes condensation suggested by Townsend [16] it was also proved in Ref. [14] that special kind of 0-brane called quatl [17]-[18] leads to the result that the $2+2$-target spacetime can be understood either as $2+2$-worldvolume spacetime or as $1+1$-matrix-brane.

Another recent motivation for a physical interest in the $2+2$-signature has emerged via the Duff's [19] discovery of hidden symmetries of the Nambu-Goto action. In fact, this author was able to rewrite the Nambu-Goto action in a 2+2-target spacetime in terms of a hyperdeterminant, reveling apparently new hidden symmetries of such an action. More recently the Duff's observation has been linked with the matrix-brane idea [20].

Considering seriously the possibility that the 2+2-dimensional "spacetime" is an exceptional signature one may wonder what is the connection between 2+2-dimensional "spacetime" and other exceptional structures in physics such as black-holes. In this respect it becomes convenient to discuss "black-holes" physics from modern perspective (see Section 2). In particular, it become convenient to clarify the many subtleties behind the introduction of a true point-mass source at $r = 0$ [26] and the admissible family of radial functions $R(r)$ in the static spherically symmetric solutions of Einstein field equations [23]-[25] (see also Ref. [29]).

With these modern developments at hand one may proceed to find "black-hole" type solutions of the Einstein field equations for a 2+2-dim "spacetime". In section 3 we present static hyperbolic solutions in a 2+2-dimensional "spacetime" and describe its differences with the corresponding solution in $3 + 1$ dimensions. In section 4, 5, we present the straightforward computations of the static circular symmetric solutions of Einstein field equations in $2 + 2$-dimensions. Finally, in section 6 we show how the $1 + 1$ Bars-Witten *stringy* black-hole solution can be embedded into the $3 + 1$-dim solution of sections **2** and discuss the "stringy" nature behind a point-mass. Black holes in a $2 + 2$-dimensional "spacetime" from the perspective of complex gravity in $1 + 1$ complex dimensions and its quaternionic and octonionic gravity extensions deserve furher investigation. In the appendix we construct Schwarzschild-like solutions in dimensions $D \geq 4$.

2. Static Spherically Symmetric Solutions of Einstein's Equations : Black Holes, Stringy Nature of Point Masses and Maximal Acceleration

We begin by writing down the class of static spherically symmetric (SSS) vacuum solutions of Einstein's equations [30] studied by Abrams [23] (where there are *no* mass sources *anywhere*) given by a $infinite$ family of solutions parametrized by a family of admissible radial functions $R(r)$

$$(ds)^2 = g_{00}\ (dt)^2\ - g_{RR}\ (dR)^2\ - R^2\ (d\Omega)^2 =$$

$$g_{00}\ (dt)^2\ - g_{RR}\ (\frac{dR}{dr})^2 dr^2\ - R^2\ (d\Omega)^2 = g_{00}\ (dt)^2\ - g_{rr}\ (dr)^2\ - (R(r))^2\ (d\Omega)^2 \quad (2.1)$$

where the solid angle infinitesimal element is

$$(d\Omega)^2 = (d\phi)^2 + sin^2(\phi)(d\theta)^2, \qquad (2.2a)$$

and

$$g_{00} = 1 - \frac{\alpha}{R(r)}; \quad g_{RR} = \frac{1}{g_{00}} = \frac{1}{1 - \alpha/R(r)}.$$

$$g_{rr} \ = \ g_{RR} \ (\frac{dR}{dr})^2 \ = \ (1 - \frac{\alpha}{R(r)})^{-1} \ (\frac{dR(r)}{dr})^2. \qquad (2.2b)$$

where α is an arbitrary constant that *happens* to have dimensions of *mass* when $c = 1$ (but there are no masses at all in this vacuum case). When a point mass source is present at the location $r = 0$, then $\alpha = 2G_N M$ and one must replace everywhere $r \rightarrow |r|$ as required when point-mass sources are inserted. We know that the Newtonian gravitational potential due to a point-mass source at $r = 0$ is given by $-G_N M/|r|$.

Notice that the vacuum SSS solutions of Einstein's equations, with and without a cosmological constant, do *not* determine the form of the radial function $R(r)$. In the appendix we present the Schwarzcshild-like solutions in any dimensions $D > 3$ and show that the radial function $R(r)$ is completely *arbitrary*.

There are two cases to study based on the boundary conditions obeyed by $R(r)$: (i) the Hilbert textbook (black hole) solution based on the choice $R(r) = r$ obeying $R(r = 0) = 0$ and $R(r \rightarrow \infty) \rightarrow r$. (ii) the case based on choosing the cutoff $R(r = 0) = 2G_N M$ such that $g_{tt}(r = 0) = 0$ which apparently seems to "eliminate" the horizon and $R(r \rightarrow \infty) \rightarrow r$. This was the original solution of 1916 found by Schwarzschild. This ultra-violet cutoff $2GM$ for the radial function $R(r)$ does *not* violate Birkoff's theorem since the radial coordinate $0 \leq r \leq \infty$ and corroborates the numerical findings in [23], [34], [35], [36], [37] , [29], [42] among others, that Einstein's equations do *not* determine the functional form of $R(r)$ in the same vein that they do not fix the topology of spacetime. For example, $R(r)$ can be chosen to be an $infinite$ family of functions like

$$R(r) = r + \alpha; \quad R(r) = [r^3 + \alpha^3]^{1/3}; \quad R(r) = [r^n + \alpha^n]^{1/n}; \quad R(r) = \frac{\alpha}{1 - e^{-\alpha/r}}. \qquad (2.3a)$$

found by Brouillin [32] , Schwarzschild [31], Crothers [35], and Fiziev-Manev [29] respectively obeying the conditions that

$$R(r = 0) = \alpha = 2G_N M; \quad and\ when \ \ r >> 2G_N M \ \Rightarrow \ R(r) \ \rightarrow r. \qquad (2.3b)$$

there exist an infinite class of solutions to the vacuum SSS Einstein's equations $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$ for an *arbitrary* family of radial functions $R(r)$. In particular for functions of the type displayed above (but the curvature Riemnan tensor $\mathcal{R}^{\mu}_{\nu\rho\sigma} \neq 0$).

However, there is one serious problem (riddle) when a cutoff $R(r = 0) = 2G_N M$ is introduced, riddle that was *never* properly solved by any of these authors above and is : How is it *possible* for a point-mass at $r = 0$ to have a non-zero area $4\pi(2G_N M)^2$ and a *zero* volume *simultaneously* ? It is the purpose of this section to solve this riddle by showing the reason why the non-zero proper area of the point mass at $r = 0$ (while the volume is zero) may be due to the *stringy* nature of a "point" mass. A string world-sheet has an area but a zero volume.

Since the radial function may be arbitrary we may choose functions like $R(r) = r + 2G_N M\ \Theta(r)$ or $R(|r|) = |r| + 2GNM\Theta(|r|)$, where the Heaviside Step function [3] is defined $\Theta(r) = 1$ when $r > 0$ and $\Theta(r) = 0$, when $r \ \leq 0$. Since $r = \pm\sqrt{x^2 + y^2 + z^2}$, a negative r branch is mathematically possible and fits the *double* covering inherent in the Fronsdal-Kruskal-Szekeres analytical continuation in terms of the u, v coordinates. Each point of spacetime inside $r < 2G_N M$ is represented *twice*. Therefore, by using $R(r) = r + 2G_N M\Theta(r)$, (or the one with the modulus $|r|$) now we safely have that $R(r) = r + 2G_N M$, when $r > 0$ and $R \sim r$ for $r >> 2G_N M$; while $R(r) = r$, when $r = 0$ and such that now we can satisfy the required condition $R(r = 0) = r = 0$, consistent with our intuitive notions that the spatial area and spatial volume of a point $r = 0$ is *zero*.

We can see that due to the discontinuity of $R(r)$ at $r = 0$, when one takes first and second derivatives of the metric, one will generate the desired delta functions (and derivatives) as one should to obtain the scalar curvature delta function singularity of the point mass source at $r = 0$. The derivative of the step function is a delta function. In the bulk region of spacetime ($r > 0$) , the metric is smooth and differentiable and one will have $\mathcal{R}_{\mu\nu} = \mathcal{R} = 0$ (in the region empty of matter). For instance, $g_{tt}(r) = 1 - (2G_N M/r + 2G_N M\Theta(r))$ is well defined and smooth for all $r > 0$ and tends to zero when r tends to $0 + \epsilon$. It tends to minus infinity when $r = 0$ (discontinuity at the location of the point mass $r = 0$). It is only in the $r = 0$ boundary (the singularity) where the metric is discontinuous. Colombeau's theory of nonlinear distributions is the proper way to deal with point-mass sources in nonlinear theories and where one may rigorously solve the problem without having to introduce a boundary

[3] We thank Michael Ibison for pointing out the importance of the Heaviside step function and the use of the modulus $|r|$ to account for point mass sources at $r = 0$.

at $r = 0$ [40].

Another possibility is to formulate the problem in *phase space*, in particular within the framework of the Finsler geometry associated with the (co) tangent bundle of spacetime. A point mass may have a zero area from the spacetime perspective but a non-zero area from the phase space point of view due to the incorporation of the momentum degrees of freedom into the picture ; i.e. in the static case $p^\mu = (E = M, 0, 0, 0)$ there is a non-vanishing phase space area element (setting aside the nature of curved phase space for the moment) $\mathcal{A} = E\ t = M\ t$. A compactification of the temporal direction t along a circle S^1 gives an Euclidean time coordinate interval of $2\pi t_E$ which is defined in terms of the Hawking temperature T_H and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi G_N M$. From which we infer that $t_E = 4 G_N M$. Therefore the area element in phase space, $\mathcal{A}$, after equating $G_N = L^2_{Planck}$, (in natural units $\hbar = c = 1$) is

$$\mathcal{A} = E\ t_E = M \times 4 G_N M = 4 G_N M^2 = \frac{4\pi (2 G_N M)^2}{4\ \pi L_P^2} = \frac{Horizon\ Area}{4\ Planck\ Disc}. \tag{2.4a}$$

where the area of the Planck disc is $\pi (L_P)^2$. Therefore, the phase space area element $E\ t_E$, in units of $\hbar$, is the same as the Black Hole Entropy (one quarter of the area of the spherical horizon at $r = 2 G_N M$) in units of the area of a Planck disc. This fact may have some relation to the Holographic principle and warrants investigation.

We will show in this section how the horizon of the standard black hole solution at $r = 2 G_N M$ (when the Hilbert textbook choice is taken $R(r) = r$) can be *displaced* to the location of the point mass $r = 0$, when the radial function is chosen to have a cutoff $R(r = 0) = 2 G_N M$, if, and only if, one embeds the problem in *phase space* (or the spacetime tangent bundle) that is the proper arena to incorporate the role of the physical point mass M at $r = 0$. Relativity in phase space is the arena where one may unify space, time *and* matter due to the equivalence between mass and energy. In the Kruskal-Fronsdal-Szekeres coordinates u, v description, to describe what happens when one crosses the horizon $r = 2 G_N M$ of topology $R \times S^2$ and whose spatial slice is a sphere of radius $r = 2GM$, one has a *null* hyper-surface at $r = 2 G_N M$ due to the *tipping* of the lightcone as one approaches the horizon. When the radial function obeys a different boundary condition $R(r = 0) = 2 G_N M$ than the Hilbert textbook one, one may displace the null horizon from $r = 2 G_N M$ to a null horizon in $r = 0$ but such horizon lives in the spacetime tangent bundle (phase space) to account for the presence of a point-mass M at $r = 0$.

Furthermore, to corroborate our proposal, in section **6** we will show the relationship between our description of the field of a point mass, within the framework of phase spaces (Hamilton-Cartan Geometry) or in the spacetime tangent bundle (Finsler geometry), and the Bars-Witten *stringy* black hole in $1+1$-dim that has a *null* horizon at $r = 0$. The stringy black hole singularity occurs in the complex realm when r is extended to the field of complex numbers $r = 0 + i(2G_N M)(\pi/2)$. Whereas the horizon (the null surface) actually lives at $r = 0$.

The physical motivation of embedding the problem in a larger space (phase space) was already evisioned by Max Born [44] who was the first to propose a Reciprocal (or Dual) Relativity Principle in Phase Spaces, where in addition to a limiting speed given by the speed of light, there is a limiting proper force (acceleration). Since speed is the rate of change of position, and force is the rate of change of momentum, then the reciprocal principle of Relativity in Phase Spaces requires a limiting speed given by the speed of light and a maximal force experienced by a *fundamental* particle that can be conjectured to be $F = m_{Planck}\ c^2/L_{Planck} = M(Universe)\ c^2/R_{Hubble}$ and which leads to the Weyl-Dirac-Eddington large numbers coincidence in Cosmology [46]. A maximal acceleration c^2/L_{Planck} is also consistent with the Finsler geometry of the spacetime tangent bundle [45] and the *stringy* minimal Planck length uncertainty relations [47]

$$\Delta X \ \geq \ \frac{\hbar}{\Delta P} \ + \ L^2_{Planck}\ \Delta P. \qquad (2.4b)$$

The most general p-brane uncertainty relations based on a unified treatment of p-branes, for all values of p, in Clifford spaces was derived in [48].

The physical interpretation of the *phase space* null horizon at $r = 0$, null from the perspective of the full fledged *phase space metric* $g_{\mu\nu}(x,p)$, or Finsler metric $g_{\mu\nu}(x,v)$ is that it is the "attractor" region where a test particle (of mass m) approaches asymptotically as it moves in the gravitational background produced by the point mass M located at $r = 0$ (when $m <<< M$) . As the test particle approaches the horizon at $r = 0$, its speed and acceleration approach asymptotically the speed of light c and the maximal acceleration c^2/L_P. This is very reminiscent of what occurs when one uniformly accelerates a massive test particle in flat Minkowski spacetime, the trajectory is a hyperbola which asymptotically approaches the light cone passing through the Minkowski spacetime origin $r = 0$. The speed tends asymptotically to the speed of light.

To reinforce our calculations, in the final section **6** it will be shown how the Bars-Witten *stringy* $1+1$-dim black-hole solution can be embedded into the $conformally$ re-scaled metrics of the form in eq-(2.1) for a unique choice of the radial function given by the tortoise radial coordinate :

$$R \ + \ 2G_N M \ ln \ (\frac{R - 2G_N M}{2G_N M}) \ = \ 2G_N M \ ln \ [\ sinh \ \frac{r}{2G_N M} \ ]. \qquad (2.5a)$$

such that

$$R(r = 0) \ = \ 2G_N M; \quad R(r \to \infty) \ \to \ R \ \sim \ r. \qquad (2.5b)$$

and which *precisely* has the same behaviour at $r = 0$ and ∞ as the radial functions displayed in eqs-(2.4) when $\alpha = 2G_N M$. The radial function R in (2.5a) has also a lower (ultraviolet cutoff) bound given by $2G_N M$. An interesting analysis of how a string (an extended object) can probe space-time points was presented by Aspinwall [22]. This requires altering our classical conceptions of Topology and Geometry at very small scales.

The geometric proper displacement in the spacetime tangent bundle involving coordinates and velocities typical of Finsler metrics is

$$(d\sigma)^2 \ = \ (ds)^2 \ + \ L_P^2 \ (dv^\mu)^2 \ = \ (ds)^2 \ [\ 1 \ + \ L_P^2 (\frac{dv^\mu}{ds})^2 \ ] \ = \ (1 - \frac{a^2(s)}{a_o^2}) \ (ds)^2.$$

$$- \ a^2(s) \ \equiv \ (\frac{dv^\mu}{ds})^2 \ < \ 0. \qquad (2.6a)$$

the acceleration is spacelike when the velocity is timelike, this accounts for the minus sign in the last term of (2.6a).

We are naturally assuming that the test particle does *not* follow a geodesic in the base spacetime manifold, otherwise $a = 0$. For example, when the test particle remains *static*, the acceleration is the force per unit mass required to maintain the test particle at a fixed position (since the metric is static) and prevent it from falling into the point mass M source at $r = 0$. The closer it gets to $r = 0$, the greater the force is required to hold it in that place. The force (acceleration) has an upper limit in our case due to Born's relativity principle in phase space. If one ignores the back-reaction of the gravitational field on the point mass M at $r = 0$, the world line of the very own point mass M, as it is inmersed in its own gravitational field background, is a timelike geodesic at $r = 0$ such it does *not* experience an acceleration. So the world line interval corresponding to the point mass location $(d\sigma)^2$ coincides with $(ds)^2$ in this case.

The conformal factor in front of the standard spacetime interval $(ds)^2$ vanishes when the acceleration of the test particle moving in such background is $a^2 = a_o^2$, where a_o is the maximal acceleration c^2/L_{Planck} . When $(ds)^2$ is given by the Schwarzschild solutions of eq-(2.1) and when the radial functions $R(r)$ is subjected to the cutoff $R(r=0) = 2G_N M$ we have a divergence of $(ds)^2(r=0) = \infty$, due to $g_{rr}(r=0) = \infty$, while the conformal factor is *zero*, since a test particle attains the limiting upper value of the proper acceleration when it asymptotically reaches $r = 0$. Its speed also asymptotically tends to the speed of light as it approaches $r = 0$. One then ends up with an interval $(d\sigma)^2$ of the form $0 \times \infty$ which nevertheless tends to *zero*.

In section **6** we will encounter a similar behaviour of the conformal factor in the embedding process of the *stringy* black hole in $1+1$-dim into the *conformally* rescaled $3+1$-dim metric of the form in eq-(2.1). Such conformal factor vanishes also at $r = 0$ ensuring that the conformally rescaled interval is *null* at $r = 0$ compatible with the existence of a null *stringy* black hole horizon at $r = 0$.

The most salient feature of (2.6a) is that at $r = 0$ we end up with

$$(1 - \frac{a^2(s)}{a_o^2})\ (ds)^2 = 0 \times \infty \rightarrow 0. \qquad (2.6b)$$

implying that the *conformally rescaled* Schwarzschild metrics (2.1) yield a *null* interval, a *null* surface $(d\sigma)^2 = 0$, in the spacetime tangent bundle at the precise location of the point mass source $r = 0$. In the stringy black hole case the location $r = 0$ is also a null surface and coincides with the Bars-Witten *stringy* black hole horizon. Hence, the horizon that a test particle (of mass $m << M$) experiences as it approaches $r = 0$ asymptotically, is a *null* surface that lives in the spacetime tangent bundle corresponding to the coordinates $x^\mu(s), v^\mu(s)$ associated with the world-line of the test particle. This is because the *conformally* rescaled area $(1 - \frac{a^2(s)}{a_o^2})\ 4\pi R(r)^2$ that the test particle sees, as it approaches $r = 0$, tends to *zero* due to the vanishing of the conformal factor when the maximal acceleration is attained, despite the fact that $4\pi(R(r=0))^2 = 4\pi(2G_N M)^2 \neq 0$. This is a peculiar feature of Finsler geometry when a metric is velocity (momentum) dependent in addition to position dependent.

To see why phase space metrics can behave like stringy black hole metrics, let us look for the analog of a static spherically symmetric metric in phase space (or in the spacetime tangent bundle), which is for example, $g_{tt}(r, p_r); g_{rr}(r, p_r)$ where p_r is the radial conjugate momentum to the radial variable r. This dependence on the conjugate pair of variables (r, p_r) resembles the Bars-Witten stringy-black hole metric $(ds)^2 = g_{uv}(u, v)dudv$ depending on the pair of variables u, v and studied in detail in section **6** . Thus, the maximal proper acceleration $a_o = c^2/L_{Planck}$ acts as regulator in spacetime [45] in the same vein that there is a maximum value of tidal forces (acceleration) in string theory [47] due to the minimal length string uncertainty relations. This maximal acceleration regulator is consistent with the introduction of an ultra-violet cutoff $R(r = 0) = 2G_N M$.

In [26] we studied the many subtleties behind the introduction of a true point-mass source at $r = 0$ (that couples to the vacuum field) and the physical consequences of the delta function singularity (of the scalar curvature) at the location of the point mass source $r = 0$. Those solutions were obtained from the vacuum SSS solutions simply by replacing r for $|r|$ and α for $2G_N M$. For instance, the Laplacian in spherical coordinates in flat space of $1/|r|$ is equal to $-(1/r^2)\delta(r)$, but the Laplacian of $1/r$ is *zero*. Thus, to account for the presence of a true mass-point source at $r = 0$ one must use solutions depending on the modulus $|r|$ instead of r [38]. The scalar curvature was $\mathcal{R} = -[2G_N M/R^2(dR/dr)]\delta(r)$ [26] . It is interesting that for the Hilbert $R = r$ and Schwarzschild choices $R^3 = r^3 + (2G_N M)^3$, the scalar curvature is the *same* : $\mathcal{R} = -(2G_N M/r^2)\delta(r)$ and also the measures $4\pi R^2 dR\ dt = 4\pi r^2 dr\ dt$. This deserves further investigation. The study of the SSS solutions with cosmological constant based on the introduction of an admissible family of radial functions $R(r)$ allowed us [25] to obtain the observed value of the vacuum energy density.

A different and detailed treatment of point masses, point charges, delta function sources and the physical implications of the many different choices of the radial functions $R(r)$ in General Relativity has been given by Fiziev [29]. A thorough rigorous mathematical analysis on the theory of tensor-valued distributions, point-mass sources and delta function singularities in nonlinear theories (like General Relativity) based on Colombeau's theory of *nonlinear* distributions can be found in [40]. Colombeau developed the rigorous mathematical treatment of nonlinear distributions in General Relativity and other nonlinear theories because the old Schwarz theory of distributions was restricted to linear theories. A modern treatment of singularities in Riemann-Finsler geometries can be found in [41]. For the historical implications of the most general SSS solutions of Einstein's equations see [23] and the book by [34]. The solutions for a mass point that we have been all accustomed to were those given by Hilbert-Droste-Weyl [33] and can be recovered from eqs-(2.1-2.4) by setting $R(r) = r$.

There are many *physical* differences among the Hilbert and Schwarzschild 1916 solution, in particular in the global properties. We will explain below why the 1916 Schwarzschild solution is *not* a radial reparametrization of the Hilbert textbook solution. In particular, because the radial function $R = [|r|^3 + (2G_N M)^3]^{1/3}$ can *never zero*. The absolute value $|r|$ properly accounts for the field of a point mass *source* at $r = 0$. Thus, the lower bound of R is given by $2G_N M$. The Frondsdal-Kruskal-Szekeres analytical continuation [39] of the Hilbert solution for $r < 2G_N M$ yields a *spacelike* singularity at $r = 0$ and the roles of t and r are interchanged when one crosses $r = 2G_N M$; so the interior region $r < 2G_N M$ is *no* longer static. The Schwarzschild solution is static for $r < 2G_N M$, and there is a *timelike* singularity at $r = 0$, the true location of the point mass source. Notice that when $r >> 2G_N M$ the Schwarzschild solution reduces to the Hilbert solution and one has the correct Newtonian limit in the asymptotic region.

It is very important to emphasize that despite the fact that one can always *relabel* the variable r for R in such a way that the metric in eq-(2.1) has exactly the $same\ functional\ form$ as the standard Hilbert textbook solution [33] (black-holes solutions with a horizon at $r = 2G_N M$) this does *not* mean that the Hilbert textbook metric is $diffeomorphic$ to the metric in eq-(2.1). The reason is that the values of r range from 0 to ∞ while the values of R range from $2G_N M$ to ∞. The physical explanation why there is an ultraviolet cutoff at $R = 2G_N M$ was provided long ago by Abrams [23], and rather than imposing this cutoff $R = 2G_N M$ by fiat (by decree, by hand) there is a deep physical reason for doing so; namely that the Hilbert textbook solution $R(r) = r$ does not furnish the static gravitational field of a point mass centered at the origin $r = 0$ since the Hilbert textbook solution is not *static* in the region $0 < r < 2G_N M$ after performing the Fronsdal-Kruskal-Szekeres analytical continuation.

One can also explain the physical meaning of this UV cutoff $R(r = 0) = 2G_N M$ resulting from the Noncommutativity of the spacetime coordinates within scales of the Planck length. Since the point $r = 0$ is fuzzy and delocalized, it has an area due to the noncommutativity of coordinates because points cannot be resolved below the Planck scale. To illustrate the crucial role of the momentum degrees of freedom inherent in the point-mass source, let us recur to the standard noncommutative algebra (there are far more fundamental algebras like Yang's algebra in noncommutative phase spaces) of the form

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}. \quad [p^\mu, p^\nu] = 0 \quad [x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{2.7}$$

where $\eta^{\mu\nu}$ is a flat space metric and the structure constants (c-numbers) $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$ are c-numbers that commute with x, p and that have dimensions of $length^2$; the $\Theta^{\mu\nu}$ are proportional to the L^2_{Planck}. A change of coordinates

$$x'^\mu = x^\mu + \frac{1}{2}\Theta^{\mu\rho}\ p_\rho. \quad p'^\mu = p^\mu. \tag{2.8}$$

leads to an algebra with commuting coordinates

$$[x'^{\mu}, x'^{\nu}] = 0. \quad [p'^{\mu}, p'^{\nu}] = 0. \quad [x'^{\mu}, p'^{\nu}] = i\eta^{\mu\nu}. \tag{2.9}$$

Due to the *mixing* of coordinates and momentum in the new commuting variables x' one can envisage coordinate and momentum dependent metrics in phase space, in particular Finsler geometries, and whose average over the momentum coordinates $< \pi_{\mu\nu}(x,p) >_p \ = \ g_{\mu\nu}(x)$ yield the effective spacetime metric. This momentum averaging procedure is very similar to the averaging of the momentum-scale dependent metrics employed in the Renormalization Group flow of the effective average action employed in Nonperturbative Quantum Einstein Gravity by [52]. Morever, the momentum dependence of the new coordinates x' leads to a momentum dependent radial coordinate $r' = \sqrt{x'^i \ x'_i}$ involving commuting x'^{μ} coordinates

$$r' \ = \ \sqrt{(x^i + \frac{1}{2}\Theta^{i\rho} \ p_\rho) \ (x_i + \frac{1}{2}\Theta_{i\tau} \ p^\tau)}. \tag{2.10}$$

Similar attempts to study the Noncommutative effects on black holes by modifying $r \rightarrow r'$ have been made by many other authors , [49], [50] however, to our knowledge its relation to phase spaces and Finsler geometries has not been explored. The impending question is to find another interpretation of the radial function $R(r)$ and the physical meaning of the cutoff $R(r=0) = 2G_N M$ in terms of the momentum dependent radial coordinate r'.

When $x^i = 0 \Rightarrow r = 0$, and (2.10) becomes

$$r' \ = \ \frac{1}{2}\sqrt{\Theta^{i\rho} \ p_\rho \ \Theta_{i\tau} \ p^\tau}. \tag{2.11}$$

The expression inside the square root can be written in terms of $p_\mu p^\mu = M^2$, in the *static* case when $|\vec{p}| = p^i = 0, \ i = 1,2,3$, after the following steps. Firstly, due to the static condition $p^i = 0, \ p_0 = E = M$ one has

$$\Theta^{i\rho} \ \Theta_{i\tau} \ p_\rho \ p^\tau \ = \ \Theta^{\mu\rho} \ \Theta_{\mu\tau} \ p_\rho \ p^\tau \ - \ \Theta^{0i} \ \Theta_{0j} \ p_i \ p^j \ = \ \Theta^{\mu\rho} \ \Theta_{\mu\tau} \ p_\rho \ p^\tau, \tag{2.12}$$

this last expression may be recast as

$$\Theta^{\mu\rho} \ \Theta_{\mu\tau} \ p_\rho \ p^\tau \ = \ \lambda \ p_\tau \ p^\tau \ = \ \lambda \ M_o^2. \tag{2.13}$$

if, and only if, the 4×4 antisymmetric matrix $\Theta^{\mu\nu}$ obeys the eigenvalue condition :

$$\Theta^{\mu\rho} \ \Theta_{\mu\tau} \ p_\rho \ = \ \lambda \ p_\tau. \tag{2.14}$$

In the static case $p_\rho = (M_o, 0, 0, 0)$, the eigenvalue condition yields the following 4 conditions

$$\Theta^{\mu 0} \ \Theta_{\mu 0} \ p_0 \ = \ \lambda \ p_0, \quad \Theta^{\mu 0} \ \Theta_{\mu i} \ p_0 \ = \ p_i \ = \ 0, \ i = 1,2,3. \tag{2.15}$$

that will restrict the values of the 6 components of the 4×4 antisymmetric matrix $\Theta^{\mu\nu}$; i.e. the 6 components are *not* all independent from eachother.

Therefore, in the *static* case $p^i = \vec{p} = 0$, upon imposing the eigenvalue condition and after adjusting the value of the constant $\lambda = 16\ L^4_{Planck} = 16\ G^2_N$, gives then the ultra-violet cutoff

$$r'(r=0) \ = \ \frac{1}{2}\sqrt{\Theta^{i\rho}\ p_\rho\ \Theta_{i\tau}\ p^\tau} \ = \ 2\ L^2_P\ M \ = \ 2\ G_N\ M. \qquad (2.16)$$

consistent with $R(r=0) = 2G_N M$ with the only subtlety that that $r = \sqrt{x^i x_i}$ involves now noncommuting coordinates x^μ.

When $r \neq 0$, the terms

$$\Theta^{\mu\rho}\ p_\rho\ x_\mu \ + \ \Theta_{\mu\tau}\ x^\mu\ p^\tau \ = \ \Theta^{\mu\rho}\ p_\rho\ x_\mu \ + \ \Theta^{\mu\tau}\ x_\mu\ p_\tau \ =$$

$$\Theta^{\mu\rho}\ p_\rho\ x_\mu \ + \ \Theta^{\mu\rho}\ x_\mu\ p_\rho \ = \ \Theta^{\mu\rho}\ (\ x_\mu\ p_\rho - i\ \eta_{\mu\rho}\ ) \ + \ \Theta^{\mu\rho}\ x_\mu\ p_\rho \ = \ 2\ \Theta^{\mu\rho}\ x_\mu\ p_\rho. \qquad (2.17)$$

due to the antisymmetric property of $\Theta^{\mu\rho}$, one has $\Theta^{\mu\rho}\ \eta_{\mu\rho} = 0$.

The quantity $\Theta^{i\rho}\ x_i\ p_\rho$ involving the angular momentum operator, $x_i\ p_\rho - x_\rho\ p_i$ does not preserve the spherically symmetry unless one imposes a condition (constraint) in phase space like

$$\Theta^{i\rho}\ x_i\ p_\rho \ \sim \ L^2_{Planck}\ M\ \omega(r)\ r^2 \ = \ G_N\ M\ \omega(r)\ r^2 \qquad (2.18)$$

where $\omega(r)$ is a scale-dependent frequency. Concluding, in the most general case one has :

$$r' = r'(r) \ = \ \sqrt{r^2 \ + \ 2\Theta^{i\rho}\ x_i\ p_\rho \ + \ (2G_N M)^2}. \qquad (2.19)$$

Since eq-(2.17) involves the phase space variabes x, p, the question is to see whether or not phase space metrics solutions of the form $g_{\mu\nu}(x,p) = g_{\mu\nu}(x^\mu + \Theta^{\mu\rho} p_\rho)$ solve the field equations corresponding to Moyal-Fedosov star product deformations of Noncommutative Finsler Gravity associated with the contangent bundle [41]. For a recent status of Noncommutative Riemannian gravity see [43] and references therein. However, we believe that it is Finslerian geometry the appropriate one to study and the proper arena to quantize gravity. When $r = 0$ one recovers the cutoff $r'(r=0) = 2G_N M$. Therefore this procedure to relate the effects of the Noncommutativity of coordinates with the ultra-violet cutoff $R(r=0) = 2G_N M$ is quite promising . We shall leave it for future work.

Another controversy that we must address is the dispute by many authors as to whether or not the radial function $R(r)$ is just a radial change of coordinates; i.e. a naive radial reparametrization that has *no* effect at all on the physics. We will see now why strictly speaking $R(r)$ is *not* a naive radial reparametrization. One can have an infinite number of metrics parametrized by a family of arbitrary radial functions $R(r)$ with the desired behaviour at $r = 0$ and $r = \infty$, and $different\ functional$ forms of the scalar curvature (which is a coordinate invariant) given by $\mathcal{R} = -(2G_N M\ \delta(r)/R^2\ (dR/dr))$ [26]. Since the scalar curvature $\mathcal{R} \neq 0$ (when $r = 0$) is a coordinate invariant quantity, this result for $\mathcal{R} \sim \delta(r)$ whose proportionality factor depends explicitly on the $functional$ form of the family of radial functions $R(r)$, corroborates once more that one *cannot* view the role of the radial function $R(r)$ as a naive change of radial coordinates from r to R, because if it were, then it should also leave invariant the *proportionality* factors in front of the delta function. Therefore, one must not confuse having an infinite family of metrics parametrized by the functions $R(r)$ with an infinite number of radial reparametrizations $r \rightarrow r'(r)$ of a given fiduciary metric.

The source of the controversy was due to the fact that because for all values of $r > 0$ the scalar curvature is always $\mathcal{R} = 0$, and since 0 is an invariant, one might conclude that $R(r)$ has to be just a radial reparametrization because it leaves $\mathcal{R}$ invariant (equal to zero for $r > 0$). However, one must not forget the crucial delta function singularity of the scalar curvature and whose proportionality factor depends explicitly on the functional forms of the radial functions $R(r)$. The impending question is : If this is the case, *what* is the quantity which remains *invariant* for all the infinite choices of $R(r)$? We will show now that the relevant *invariant* physical quantity, independent of the any arbitrary choice of $R(r)$, is precisely the Einstein-Hilbert action. In particular, the Euclideanized action after a compactification of the temporal interval yields an invariant quantity which is precisely equal to the black hole entropy in Planck area units.

We shall see that the source of entropy is due entirely to the scalar curvature delta function singularity at the location of the point mass source given by $\mathcal{R} = -[2G_N M/R^2(dR/dr)]\delta(r)$ [26] after using the 4-dim measure

$$4\pi R^2\ |g_{RR}|^{1/2} dR\ |g_{tt}|^{1/2} dt \ = \ 4\pi R^2\ dR\ dt. \tag{2.19}$$

in the Euclidean Einstein-Hilbert action. The Euclideanized Einstein-Hilbert action associated with the scalar curvature delta function is obtained after a compactification of the temporal direction along a circle S^1 giving an Euclidean time coordinate interval of $2\pi t_E$ and which is defined in terms of the Hawking temperature T_H and Boltzman constant k_B as $2\pi t_E = (1/k_B T_H) = 8\pi G_N M$. The measure of integration is $4\pi R^2\ dR\ dt_E$, leading to :

$$S_E = -\frac{1}{16\pi G_N}\int \left(-\frac{2G_N M}{R^2(dR/dr)}\delta(r) \right) (4\pi R^2\ dR\ dt) =$$

$$-\frac{1}{16\pi G_N}\int \left(-\frac{2G_N M}{r^2}\delta(r) \right) (4\pi r^2\ dr\ dt) = \frac{4\pi(G_N M)^2}{L^2_{Planck}} =$$

$$\frac{4\pi\ (2G_N M)^2}{4\ L^2_{Planck}} = \frac{Area}{4\ L^2_{Planck}}. \tag{2.20}$$

when equating $G_N = L_P^2$. It is interesting that the Euclidean action (2.20) is the same as the black hole entropy in Planck area units.

The $action - entropy$ connection has been obtained from a different argument by Padmanabhan [51] by showing how it is the $surface$ term added to the action which is related to the entropy, interpreting the horizon as a boundary of spacetime. The surface term is given in terms of the trace of the *extrinsic* curvature of the boundary. The surface term in the action is directly related to the observer-dependent-horizon entropy, such that its variation, when the horizon is moved infinitesimally, is equivalent to the change of entropy $d\mathcal{S}$ due to the virtual work. The variational principle is equivalent to the thermodynamic identity $Td\mathcal{S} = dE + pdV$ due to the variation of the matter terms in the right hand side.

Notice that this result (2.20) remains *invariant* for any *arbitrary* choice of the radial function $R(r)$, whether or not it is the Hilbert textbook choice $R(r) = r$. Since the action is invariant for any choices of $R(r)$, it is in this sense that we can argue that the use of a particular radial function $R(r)$ is just equivalent to choosing a *radial gauge* that does not change the value of the action. Furthermore, this result that the Euclidean action is equal to the entropy in Planck units can be generalized to higher dimensions upon using the results in the Appendix for metrics in higher dimensions. Instead of areas we have $D-2$-dim regions. In this higher dimensional case we can see in the Appendix how one has two classes of solutions. There are the usual black hole solutions (with horizons enclosing the point $r = 0$) when $R(r = 0) = 0$, and the second class of solutions occurs when there is an ultra-violet cutoff due to the presence of matter $R(r = 0) = [16\pi G_D M/(D-2)\Omega_{D-2}]^{1/D-3}$. The point $r = 0$ is $p - brany$ in nature such that the $p+1$ world-volume of a p-brane spans the $D-2$-dim surface associated with the hyper-area $\sim R(r=0)^{D-2}$ of the point. Black p-branes are not the same as $p - brany$ black-holes (the generalization of *stringy* black holes metrics) which resemble the behaviour of these higher-dim metrics when $R(r = 0) \neq 0$. Once again , a proper treatment requires Finsler geometries in higher dimensions.

To end this section we briefly explain why the ultra-violet cutoff $R(r = 0) = 2G_N M$ is compatible with the exact Nonperturbative Renormalization Group flow of the Newtonian coupling $G = G(r)$ and the mass parameter $M = M(r)$ in Quantum Einstein Gravity [52] . For further details we refer to [53]. The presence of an ultra-violet cutoff $R = 2G_N M_o$ originates from the mere presence of matter and permits to relate $g_{tt} = 1 - 2G_N M_o/R(r)$ to $g_{tt} = 1 - 2G(r)M(r)/r$ such that $g_{tt}(r = 0) = 0$ and which is compatible with the ultra-violet cutoff of the radial function $R(r = 0) = 2G_N M_o$. G_N is the value of the Newtonian coupling in the deep infrared and $M = M_o$ is the Kepler mass as seen by an observer at asymptotic infinity. The non-perturbative exact Renormalization Group program for Quantum Einstein Gravity helps to determine the choice $R(r)$ uniquely from the infinity family of plausible radial functions $R(r)$. The momentum dependence of $G(k^2)$ was found by Reuter et al [52] to be

$$G(k^2) = \frac{G_N}{1 + \gamma\ G_N\ k^2}. \tag{2.21a}$$

Setting

$$k^2 \ = \ (\frac{\beta}{D(R)})^2. \tag{2.21b}$$

in terms of the proper radial distance $D(R)$ defined by

$$D(R) = \int_{2G_N M_o}^{R} \sqrt{g_{RR}}\ dR \ = \ \int_{2G_N M_o}^{R} \frac{dR}{\sqrt{1 - (2\ G_N\ M_o/R)}} \ =$$

$$\sqrt{R\ (\ R - 2\ G_N\ M_o)} \ + \ 2\ G_N\ M_o\ ln\ [\ \sqrt{\frac{R}{2G_N M_o}} \ + \sqrt{\frac{R - 2G_N M_o}{2G_N M_o}}\ ]. \tag{2.22}$$

where the lower (ultra-violett cutoff) is $R(r = 0) = 2G_N M_o$. Notice that $D(R = 2G_N M_o) = 0$ as it should since the proper distance from $r = 0$ is zero when one is located at $r = 0$. Hence,

$$G \ = \ G(R) \ = \ \frac{G_N}{1 + \gamma\ G_N\ k^2} \ = \ \frac{G_N\ D(R)^2}{D(R)^2 + \gamma\beta^2\ G_N}. \tag{2.23}$$

since $R = R(r)$, by imposing the conditions for *all* values of r

$$(1 - \frac{2\ G_N\ M_o}{R(r)}) \ = \ (1 - \frac{2\ G(r)\ M(r)}{r}). \tag{2.24}$$

$$\frac{(\frac{dR}{dr})^2}{(1 - \frac{2\ G_N\ M_o}{R(r)})} \ = \ \frac{1}{(1 - \frac{2\ G(r)\ M(r)}{r})}. \tag{2.25}$$

from eqs-(2.23, 2.24, 2.25) one infers that

$$\frac{dR}{dr} \ = 1 \ \Rightarrow R(r) \ = \ r + 2G_N M_o. \tag{2.26}$$

which is the Brillouin choice for the radial function, as well as the relation

$$G(r) \ = G_N \ (\frac{r}{R}) \ (\frac{M_o}{M(r)}) \ = \ (\frac{G_N \ D(R)^2}{D(R)^2 + \gamma\beta^2 \ G_N}) \ \Rightarrow$$

$$M(r) \ = \ M_o \ (\frac{r}{R}) \ (\frac{D(R)^2 + \alpha\beta^2 \ G_N}{D(R)^2}). \tag{2.27}$$

that allows us to determine the form of the $M(r)$ once the radial function $R(r) = r + 2G_N M_o$ is plugged into $D(R)$ given by eq-(2.22). The constant found by Reuter et al is $\gamma\beta^2 = 118/15\pi$ and the proper distance $D(R)$ is given by eq-(2.22).

When $r = 0$ a careful analysis yields

$$M(r \to 0) \ \sim \ \frac{1}{2 \ G_N \ M_o}. \tag{2.28}$$

therefore, the running mass parameter at $r = 0$, $M(r = 0) \sim 1/R(r = 0) = 1/(2G_N M_o)$ is $finite$ instead of being infinite. The running mass at $r = 0$ has a cutoff given by the inverse of the ultra-violet cutoff $R(r = 0) = 2G_N M_o$ (up to a numerical constant). When $r \to \infty$ one has $M(r \to \infty) \to M_o$ as expected, where M_o is the Kepler mass observed by an observer at asymptotic infinity (deep infrared).

Concluding this section, $R = r + 2G_N M_o$ is the sought after relation between r and R, out of an infinite number of possible functions $R(r)$ obeying the SSS vacuum solutions of Einstein's equations that is consistent with the Renormalization Group flow of the Newtonian coupling in Quantum Einstein Gravity [52].

To summarize : We have explained how the horizon of the standard black hole solution at $r = 2G_N M$ (when the Hilbert textbook choice is taken $R(r) = r$) can be *displaced* to the location of the point mass M source $r = 0$, when the radial function is chosen to have a cutoff $R(r = 0) = 2G_N M$, if, and only if, one embeds the problem in *phase space* (or the spacetime tangent bundle) that is the proper arena to incorporate the role of the physical point mass M at $r = 0$. Thus, the horizon that a test particle (of mass $m << M$) experiences at $r = 0$ is a *null* surface that lives in the *phase space* (spacetime tangent bundle) corresponding to the coordinates $x^\mu(s), p^\mu(s)$ ($v^\mu(s)$) associated with the worldline of the test particle in the base spacetime manifold. The purpose in introducing velocities/momenta into a Finsler metric in the spacetime tangent bundle (phase space) was aimed in resolving the riddle of how is it possible that a point-mass can have a non-zero area $4\pi(2G_N M)^2$ but a zero volume simultaneously when the cutoff $R(r = 0) = 2G_N M$ is imposed on the radial functions $R(r)$ in eq-(2.1). The non-zero area of the point-mass M at $r = 0$ should be thought of as a $phase - space$ area such $\mathcal{A}/\hbar = Mt/\hbar \ = \ 4\pi(2G_N M)^2/4\pi L_p^2$ which matches the black hole entropy in units of πL_P^2, as shown in eq-(2.4a), when the temporal direction is compactified along a circle S^1.

In section **6** we analyze the details of the stringy black hole horizon in $1+1$-dim and its connection to the results of this section. In the next section we study what happens in $2+2$ dimensions since $2+2$ is the natural extension of $1+1$ dimensions.

3.- Static Hyperbolic Symmetric Solution in $2+2$-dimensions

Consider the vacuum static spherically symmetric solutions of Einstein field equations in a spacetime of $3+1$-signature

$$\mathcal{R}_{\mu\nu} = 0. \tag{3.1}$$

of the form

$$ds^2 = -e^{\mu(r)}(dt_1)^2 + e^{\alpha(r)}dr^2 + R^2(r)d\Omega^2, \tag{3.2}$$

where

$$d\Omega^2 = d\phi^2 + \sin^2\phi \ d\theta^2. \tag{3.3}$$

The solutions are

$$ds^2 = -(1 - \frac{\alpha}{R})(dt_1)^2 \ + \ \frac{(dR/dr)^2}{(1-\alpha/R)} \ dr^2 \ + \ R^2(r)d\Omega^2. \tag{3.4}$$

where α is a parameter that has mass dimensions. When a point mass source is present at the location $r = 0$, then $\alpha = 2M$, $r \to |r|$ as discussed in the previous section and the radial function $R(r) = [|r|^3 + (2M)^3]^{1/3}$ for the genuine horizonless Schwarzschild solution and $R(r) = |r|$ for the Hilbert text book solution.

Several remarks are now in order pertaining whether or not a Wick rotation of the metric (3.4) furnishes solutions to the vacuum field equations for the signature $2+2$. A naive Wick rotation of the angle coordinate $\phi \rightarrow i\phi = \chi$ in the above solutions (3.4) yields

$$sin^2(\phi) \rightarrow sin^2(i\phi) = -sinh^2(\chi). \quad d\phi^2 \rightarrow -d\chi^2. \tag{3.5}$$

and due to the *two* sign changes in (3.5) one would have a $1+3$ signature instead of a split $2+2$ signature.

A Wick rotation of $\theta \rightarrow i\theta = \chi$, $(d\theta)^2 \rightarrow -(d\chi)^2$ yields a $2+2$ signature but since the range of the only remaining angle ϕ is $[0,\pi]$, instead of $[0,2\pi]$, and one will no longer cover the space completely. Furthermore, since there is a signature change (a sign change in one of the metric components $g_{\theta\theta}$) the connection and curvature expressions will be *modified* accordingly and there is no reason now why the vacuum field equations should be satisfied. In the next section we will find explicit solutions in the static circular symmetric case :

$$ds^2 = -e^{\ \tilde\mu(R(\rho))}(dt_1)^2 - e^{\ \tilde\nu(R(\rho))}(dt_2)^2 + e^{\ \tilde\alpha(R(\rho))}(dR(\rho))^2 + (R(\rho))^2 d\theta^2.$$

where the rho function $R(\rho)$ is now a function of ρ, the radius of a circle $\rho^2 = x^2 + y^2$.

In order to construct solutions with topology $\mathcal{H}^3 \times \mathcal{R}$ where $\mathcal{H}^3$ is a 3-dim pseudo-sphere (a hyperboloid) of radius R parametrized by the coordinates ψ, θ, χ as

$$x = R\ cosh\ \chi\ cos\ \theta. \quad y = R\ cosh\ \chi\ sin\ \theta.$$

$$t_1 = R\ sinh\ \chi\ cos\ \psi. \quad t_2 = R\ sinh\ \chi\ sin\ \psi. \tag{3.6}$$

where $-\infty \leq \chi \leq \infty$ and $0 \leq \theta \leq 2\pi$; $0 \leq \psi \leq 2\pi$ such that the flat spacetime metric in $2+2$ dimensions is

$$ds^2 = -(dt_1)^2 - (dt_2)^2 + (dx)^2 + (dy)^2 =$$

$$(dR)^2 \ + \ R^2\ [\ cosh^2\chi\ (d\theta)^2 \ - sinh^2\chi\ (d\psi)^2 \ - (d\chi)^2\ ]. \tag{3.7a}$$

From eq-(3.6) we infer that the 3-dim pseudo-sphere $\mathcal{H}^3$ is represented analytically by :

$$-(t_1)^2 - (t_2)^2 + x^2 + y^2 = R^2. \tag{3.7b}$$

The curved spacetime metric we are interested involve the two functions $\Sigma = \Sigma(R)$ and $\tilde{f} = \tilde{f}(\Sigma(R)) = f(R)$ such that

$$ds^2 = e^{\tilde{f}(\Sigma)}(d\Sigma)^2 + \Sigma^2 [cosh^2\chi (d\theta)^2 - sinh^2\chi (d\psi)^2 - (d\chi)^2] =$$

$$e^{f(R)} (\frac{d\Sigma}{dR})^2 (dR)^2 + \Sigma^2(R) [cosh^2\chi (d\theta)^2 - sinh^2\chi (d\psi)^2 - (d\chi)^2] =$$

$$e^{\mu(R)} (dR)^2 + \Sigma^2(R) [cosh^2\chi (d\theta)^2 - sinh^2\chi (d\psi)^2 - (d\chi)^2]. \quad (3.8)$$

where we have defined $e^{\mu(R)} \equiv e^{f(R)} (d\Sigma/dR)^2$. The flat spacetime metric (3.7) is recovered from (3.8) in the limit $R \rightarrow \infty$ such that $\mu(R) \rightarrow 0$ and $\Sigma(R) \sim R$.

Another interesting parametrization $r \geq 0$, and $-\infty \leq \xi \leq \infty$; $0 \leq \theta \leq 2\pi$ is

$$t_2 = r\ sinh\ \xi; \quad x = r\ cosh\ \xi\ cos\ \theta; \quad y = r\ cosh\ \xi\ sin\ \theta. \quad (3.9)$$

where r is the throat size of the 2-dim hyperboloid $\mathcal{H}^2$ defined in terms of t_2, x, y as

$$-(t_2)^2 + x^2 + y^2 = r^2. \quad (3.10)$$

and the flat spacetime metric $-(dt_1)^2 - (dt_2)^2 + (dx)^2 + (dy)^2$ can be recast as

$$ds^2 = -(dt_1)^2 + (dr)^2 + r^2[cosh^2\ \xi\ (d\theta)^2 - (d\xi)^2]. \quad (3.11)$$

Notice that we have a $2 + 2$ signature in eq-(3.11), as one should, and that there is a *difference* between the forms of the metric in eq-(3.7) and eq-(3.11). The topology corresponding to eq-(3.7) is $\mathcal{H}^3 \times \mathcal{R}^*$ where $\mathcal{H}^3$ is a 3-dim hyperboloid (a 3-dim pseudo-sphere); whereas, instead, the topology corresponding to eq-(3.11) is $\mathcal{R} \times \mathcal{R}^* \times \mathcal{H}^2$.

$\mathcal{R}^*$ is the half-interval $[0, \infty]$ representing the values of the radial coordinates. In eq-(3.7) the 3-dim hyperboloid (pseudosphere) of fixed radius R is spanned by the 3 coordinates θ, ψ, χ as indicated by eq-(3.6). Whereas in eq.(3.11), one temporal variable t_1 is characterized by the real line $\mathcal{R}$ and whose values range from $-\infty, +\infty$, and the other temporal variable t_2 is one of the 3 coordinates (t_2, x, y) which parametrized the two-dim hyperboloid $\mathcal{H}^2$ described by eq-(3.10).

A curved spacetime version of eq-(3.11) is :

$$ds^2 = -e^{\mu(r)}(dt_1)^2 + e^{\nu(r)}(dr)^2 + (R(r))^2[cosh^2\ \xi\ (d\theta)^2 - (d\xi)^2]. \quad (3.12a)$$

The metric in eq-(3.12a) whose signature is $2+2$ is the hyperbolic version of the Schwarzschild metric. One can replace $r \rightarrow R(r)$ since Einstein's equations do not determine the form of the radial function $R(r)$ as explained in section **2**. The global topology of the solutions depends on the choices of $R(r)$. We still must determine what are the functional forms of $\mu(r)$ and $\nu(r)$. In order to go from the solid angle $(d\Omega)^2 = sin^2(\phi)\ (d\theta)^2 + (d\phi)^2$ to $cosh^2\ \xi\ (d\theta)^2\ -\ (d\xi)^2$ one must first perform the change of coordinates $\phi \rightarrow \pi/2 + \phi$ such that $sin^2\phi \rightarrow cos^2(\phi)$ and then Wick rotate $\phi \rightarrow \phi = i\xi$ so that $cos^2(\phi) \rightarrow cosh^2\xi$ and $(d\phi)^2 = -(d\xi)^2$

In the appendix we find the solutions to Einstein's vacuum field equations in D-dimensions for metrics whose signature is $(D-2)+2$ (two times) associated with a $D-2$-dim homogeneous space of constant positive (negative) scalar curvature. In particular when $D=4$ and the two-dim homogeneous space $\mathcal{H}^2$ has a constant positive scalar curvature, like two-dim de Sitter space, the metric components, in natural units $G=\hbar=c=1$, are given by

$$g_{t_1t_1} \ = \ -(1-\frac{\beta M}{R(r)}); \quad g_{rr} \ = \ \frac{(dR/dr)^2}{(1-\beta M/R(r))}. \quad \beta = constant. \qquad (3.12b)$$

which are almost identical to the components appearing in the Schwarzchild solutions for signature $3+1$. The 2-dim hyperboloid defined by eq-(3.10) coincides with a 2-dim de Sitter space of constant positive scalar curvature. Anti de Sitter space has a constant negative scalar curvature.

There is a physical singularity at $r=0$, the location of the point mass source, when the hyperboloid $\mathcal{H}^2$ degenerates to a *cone* since the throat size r has been pinched to zero. When the radial function is chosen to be $R^3 = r^3 + (\beta M)^3\ \Rightarrow R(r=0) = \beta M$ then $g_{rr}(r=0) = \infty$ and $g_{t_1t_1}(r=0) = 0$. The proper circumference for this choice $R^3 = r^3 + (\beta M)^3$ is

$$C(r,\xi) = 2\pi\ R(r)\ cosh\ \xi` \Rightarrow\ C(r=0,\xi) = 2\pi\ \beta M\ cosh\ \xi. \qquad (3.13)$$

The proper area for a given value of r is

$$\mathcal{A}(r) = 2\pi\ R^2(r)\ \int_{-\infty}^{+\infty}\ cosh\ \xi\ d\xi\ = 2\pi\ R^2(r)\ 2\ sinh\ \xi \rightarrow \infty \qquad (3.14)$$

and diverges as $\xi \rightarrow \infty$ because the 2-dim hyperboloid is *not* compact. If one chooses $R(r) = r$, then $R(r=0) = 0$, so the proper circumference is zero (for finite ξ) and the proper area corresponding to $r=0$ is $0 \times \infty = \infty$ since $sinh\ \xi$ approaches infinity faster than r^2 approaches zero.

Another parametrization is :

$$t_2 = r\ cosh\ \xi; \quad x = r\ sinh\ \xi\ cos\ \theta; \quad y = r\ sinh\ \xi\ sin\ \theta. \tag{3.15}$$

where the thoat size r is defined in terms of t_2, x, y as

$$-(t_2)^2\ + x^2\ + y^2\ =\ - r^2. \tag{3.16}$$

which can be obtained from eq-(3.10) by $r^2 \rightarrow -r^2$. Eq-(3.16) represents analytically the two disconnected branches of a two-dim hyperboloid.

$$ds^2 = -(dt_1)^2 - (dt_2)^2 + (dx)^2 + (dy)^2 =$$

$$-(dt_1)^2 - (dr)^2 + r^2[\ sinh^2\ \xi\ (d\theta)^2\ +\ (d\xi)^2\ ]. \tag{3.17}$$

Notice the sign change $-dr^2$ in eq-(3.15) as one must have if one persists in having a $2+2$ signature. In this case the coordinate r must be interpreted as a "radial time".

The curved spacetime version of (3.17) would be :

$$ds^2 = -e^{\alpha(r)}(dt_1)^2 - e^{\beta(r)}(dr)^2 + (R(r))^2\ [\ sinh^2\ \xi\ (d\theta)^2\ +\ (d\xi)^2\ ]. \tag{3.18}$$

where $\alpha(r)$ and $\beta(r)$ are two functions to be determined by solving Einstein's equations. The functional form of $\alpha(r), \beta(r)$ $differs$ from the functions $\mu(r), \nu(r)$ in eqs-(3.12a, 3.12b) due to a crucial sign change in the g_{rr} component of the metric in eq-(3.18).

Concluding, we have 3 interesting cases described by the metrics of $2+2$ signature given by eqs-(3.8, 3.12, 3.18). The 2+2 hyperbolic-symmetric version of Schwarzschild's $3+1$ solution is given by eqs-(3.12a, 3.12b).

4.- Static Circular Symmetric Solution in $2+2$-dimensions

Let us look for a solution of the field equations of the form

$$ds^2 = -e^{\ \tilde{\mu}(R)}(dt_1)^2 - e^{\ \tilde{\nu}(R)}(dt_2)^2 + e^{\ \tilde{\alpha}(R)}dR^2 + R^2 d\theta^2 =$$

$$-e^{\mu(\rho)}(dt_1)^2 - e^{\nu(\rho)}(dt_2)^2 + e^{\alpha(\rho)}d\rho^2 + R^2(\rho)d\theta^2. \tag{4.1a}$$

where

$$\tilde{\mu}(R(\rho)) = \mu(\rho). \quad \tilde{\nu}(R(\rho)) = \nu(\rho). \quad e^{\ \tilde{\alpha}(R(\rho))}(\frac{dR}{d\rho})^2 = e^{\alpha(\rho)}. \tag{4.1b}$$

The only nonvanishing Christoffel symbols are

$$\begin{aligned}
&\Gamma^1_{31} = \tfrac{1}{2}\mu', &&\Gamma^2_{32} = \tfrac{1}{2}\nu', &&\Gamma^4_{34} = \tfrac{R'}{R}, \\
&\Gamma^3_{11} = \tfrac{1}{2}\mu' e^{\mu-\alpha}, &&\Gamma^3_{22} = \tfrac{1}{2}\nu' e^{\nu-\alpha}, &&\Gamma^3_{44} = -e^{-\alpha}RR', \\
&\Gamma^3_{33} = \tfrac{1}{2}\alpha',
\end{aligned} \tag{4.2}$$

and the only nonvanishing Riemann tensor are

$$\begin{aligned}
&\mathcal{R}^1_{212} = \tfrac{1}{4}\mu'\nu' e^{\nu-\alpha}, && \mathcal{R}^1_{414} = -\tfrac{1}{2}\mu' e^{-\alpha}RR', \\
&\mathcal{R}^2_{121} = \tfrac{1}{4}\mu'\nu' e^{\mu-\alpha}, && \mathcal{R}^2_{424} = -\tfrac{1}{2}\nu' e^{-\alpha}RR', \\
&\mathcal{R}^4_{141} = \tfrac{1}{2}\mu' e^{\mu-\alpha}\tfrac{R'}{R}, && \mathcal{R}^4_{242} = \tfrac{1}{2}\nu' e^{\nu-\alpha}\tfrac{R'}{R}, \\
&\mathcal{R}^1_{313} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\alpha'\mu', && \mathcal{R}^2_{323} = -\tfrac{1}{2}\nu'' - \tfrac{1}{4}\nu'^2 + \tfrac{1}{4}\alpha'\nu', \\
&\mathcal{R}^4_{343} = -\tfrac{R''}{R} + \tfrac{1}{2}\alpha'\tfrac{R'}{R}, && \mathcal{R}^3_{131} = e^{\mu-\alpha}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\alpha'\mu'), \\
&\mathcal{R}^3_{232} = e^{\nu-\alpha}(\tfrac{1}{2}\nu'' + \tfrac{1}{4}\nu'^2 - \tfrac{1}{4}\alpha'\nu'), && \mathcal{R}^3_{434} = e^{-\alpha}R(\tfrac{1}{2}\alpha' R' - R'').
\end{aligned} \tag{4.3}$$

The field equations are

$$\mathcal{R}_{11} = e^{\mu-\alpha}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' - \frac{1}{4}\alpha'\mu' + \frac{1}{2}\mu'\frac{R'}{R}) = 0, \tag{4.4}$$

$$\mathcal{R}_{22} = e^{\nu-\alpha}(\frac{1}{2}\nu'' + \frac{1}{4}\nu'^2 + \frac{1}{4}\mu'\nu' - \frac{1}{4}\alpha'\nu' + \frac{1}{2}\nu'\frac{R'}{R}) = 0, \tag{4.5}$$

$$\mathcal{R}_{33} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\alpha' - \frac{1}{2}\nu'' - \frac{1}{4}\nu'^2 + \frac{1}{4}\alpha'\nu' + \frac{1}{2}\alpha'\frac{R'}{R} - \frac{R''}{R} = 0, \tag{4.6}$$

and

$$\mathcal{R}_{44} = e^{-\alpha}R(-\frac{1}{2}\mu' R' - \frac{1}{2}\nu' R' + \frac{1}{2}\alpha' R' - R'') = 0. \tag{4.7}$$

From (4.7) we get

$$\alpha' = \mu' + \nu' + \frac{2R''}{R'}. \qquad .(4.8)$$

Substituting (4.8) into (4.4) and (4.5) we obtain

$$\frac{\mu''}{\mu'} = (\frac{R''}{R'} - \frac{R'}{R}) \qquad (4.9)$$

and

$$\frac{\nu''}{\nu'} = (\frac{R''}{R'} - \frac{R'}{R}), \qquad (4.10)$$

respectively. The Eqs. (4.9) and (4.10) can be integrated to give

$$\mu' = a\frac{R'}{R} \qquad (4.11)$$

and

$$\nu' = b\frac{R'}{R}, \qquad (4.12)$$

respectively, where a and b are constants. Substituting (4.11) and (4.12) into (4.8) leads to

$$\alpha' = a\frac{R'}{R} + b\frac{R'}{R} + \frac{2R''}{R'}. \qquad (4.13)$$

The expressions (4.11), (4.12) and (4.13) can be solved. We get

$$\mu = a \ln R/c \qquad (4.14)$$

$$\nu = b \ln R/d, \qquad (4.15)$$

and

$$\alpha = a \ln R/c + b \ln R/d + 2 \ln R' + f, \qquad (4.16)$$

where c, d and f are arbitrary constants. If we substitute (4.14), (4.15) and (4.16) into (4.6) we find

$$-\tfrac{1}{2}a(\tfrac{R''}{R} - \tfrac{R'^2}{R^2}) - \tfrac{1}{4}a^2\tfrac{R'^2}{R^2} + \tfrac{1}{4}(a\tfrac{R'}{R})(a\tfrac{R'}{R} + b\tfrac{R'}{R} + \tfrac{2R''}{R'}) - \tfrac{1}{2}b(\tfrac{R''}{R} - \tfrac{R'^2}{R^2})$$

$$-\tfrac{1}{4}b^2\tfrac{R'^2}{R^2} + \tfrac{1}{4}(b\tfrac{R'}{R})(a\tfrac{R'}{R} + b\tfrac{R'}{R} + \tfrac{2R''}{R'}) + \tfrac{1}{2}(a\tfrac{R'}{R} + b\tfrac{R'}{R} + \tfrac{2R''}{R'})\tfrac{R'}{R} - \tfrac{R''}{R} = 0. \qquad (4.17)$$

This can be reduced to

$$\left(a+\frac{1}{2}ab+b\right)\frac{R'^2}{R^2}=0. \tag{4.18}$$

Excluding the solutions

$$R = Const. \tag{4.19}$$

eq-(4.18) gives

$$a+\frac{1}{2}ab+b=0. \tag{4.20}$$

Therefore we have shown why the form of $R = R(\rho)$ can be completely *arbitrary* while one must have the following constraint among the constants

$$b=-\frac{2a}{(a+2)}, \tag{4.21}$$

where we assumed that $a+2 \neq 0$.

A trivial solution of eq-(4.20) is $a = b = 0$ which leads to $\mu = \nu = 0$ and $\alpha = 2ln\ (dR/d\rho)$, when $f = 0$, yielding the metric :

$$ds^2 = -(dt_1)^2 - (dt_2)^2 + dR(\rho)^2 + R^2(\rho)d\theta^2. \tag{4.22}$$

the flat spacetime metric is attained when $R(\rho) = \rho$, and also for any function $R(\rho)$ with the asymptotic property such that for very large values of ρ it behaves $R \sim \rho$.

5.- An Explicit Nontrivial Solution

We have seen that the trivial flat spacetime solutions (4.22) are obtained when $a = b = f = 0$ and when $R(\rho) = \rho$. In order to find interesting nontrivial solutions we should have a nontrivial rho function $R(\rho) \neq \rho$. Let us consider two particular cases of (4.21). In the first case taking $a = 2$ from eq-(4.21) we get $b = -1$. Similarly, in the second case by setting $a = -1$in eq- (4.21) implies $b = 2$. Thus in the first case (4.14), (4.15) and (4.16) become

$$\mu = 2\ln R/c, \tag{5.1}$$

$$\nu = -\ln R/d \tag{5.2}$$

and

$$\alpha = 2 \ln R/c - \ln R/d + 2 \ln R' + f. \tag{5.3}$$

While in the second case we find

$$\mu = - \ln R/c, \tag{5.4}$$

$$\nu = 2 \ln R/d \tag{5.5}$$

and

$$\alpha = - \ln R/c + 2 \ln R/d + 2 \ln R' + f. \tag{5.6}$$

An interesting possibility arises by setting $c = d = M$ and $f = 0$. In the first case we get that the metric in $2+2$ dimensions ends up being expressed in the R-variable as :

$$ds^2 = -(R/M)^2 (dt_1)^2 - (M/R)(dt_2)^2 + (R/M)(dR)^2 + R^2 (d\theta)^2, \tag{5.7}$$

while in the second case we obtain

$$ds^2 = -(M/R)(dt_1)^2 - (R/M)^2 (dt_2)^2 + (R/M)(dR)^2 + R^2 (d\theta)^2. \tag{5.8}$$

Notice that in both solutions (5.7) and (5.8) there is a kind of duality in the two times t_1 and t_2 factors.

Eqs-(5.7, 5.8) can be written as :

$$ds^2 = -(M/R)(dt_2)^2 + (R/M)(dR)^2 + R^2 [(d\theta)^2 - (dt_1)^2 / M^2]. \tag{5.9a}$$

$$ds^2 = -(M/R)(dt_1)^2 + (R/M)(dR)^2 + R^2 [(d\theta)^2 - (dt_2)^2 / M^2]. \tag{5.9b}$$

As announced earlier, the form of the rho function $R(\rho)$ is undetermined. Any arbitrary choice of $R(\rho)$ solves Einstein's equations.

A study reveals that a rho function $R(\rho)$ given by

$$\frac{1}{R} = \frac{1}{\rho} + \frac{1}{M}, \tag{5.10}$$

in units of $G = \hbar = c = 1$ is an appropriate choice. When $\rho = 0$, $R = 0$ and when $\rho = \infty$ we have $R(\rho = \infty) = M$, so we do recover an asymptotically *flat* spacetime metric at spatial $\rho = \infty$ given by

$$ds^2 = -(dt_1)^2 - (dt_2)^2 + (dR)^2 + R^2(d\theta)^2 = -(dt_1)^2 - (dt_2)^2 + M^2(d\theta)^2. \quad (5.10)$$

Asymptotic infinity is defined by the condition $R(\rho = \infty) = M$. It is the 3-dimensional asymptotic boundary of the $2+2$-spacetime. It is a 3-dim manifold of topology $S^1 \times R^2$. The radius of S^1 is $R = M$. When $\rho = 0$ we have in eq-(5.7) that $R(\rho = 0) = 0$, so the metric component $g_{22}(\rho = 0) = \infty$ and there is a metric singularity at $\rho = 0$ as expected. Conversely, in eq-(5.8) the singularity occurs in the component $g_{11}(\rho = 0) = \infty$, instead.

6.- Stringy $1+1$ black holes embedded in $3+1$ and $2+2$ dimensions

One of the main topics of the present work has been to link the 2+2 signature with the black-hole concept; i.e. spacetimes with singularities. We have shown that there are many different interesting ways to do this. In section 3 we presented three very diferent cases associated with hyperboloids. In particular, in the static hyperbolic-symmetric version of the Schwarschild case given by eqs-(3.12a, 3.12b), there is singularity at $r = 0$ which is associated with the *conical* geometry resulting from having pinched to *zero* size $r = 0$ the throat of the hyperboloid $\mathcal{H}^2$ and which is quite different from the spherically symmetric case in 3+1 dimensions, discussed in section 2. In the static circular symmetric case developed in sections 4 and 5 we obtained solutions with singularties at $\rho = 0$ and whose asymptotic $\rho \to \infty$ limit leads to a flat $1+2$-dim boundary of topology $S^1 \times R^2$ where the radius of S^1 is $R(\rho = \infty) = M$.

One further interesting possibility may arise if we split the 2+2 metric as the diagonal sum of two $1+1$ metrics in the form

$$ds^2 = g_{ab}(x)dx^a dx^b + g_{mn}(y)dy^m dy^n; \quad a, b = 1, 2; \quad m, n = 3, 4. \quad (6.1)$$

In this case one may look for solutions like

$$ds^2 = \frac{dudv}{1 - uv} + \frac{dwdz}{1 - wz}. \quad (6.2)$$

where we have set the value of the mass parameter $2M = 1$. Such mass parameter is required on physical grounds and also because the denominators in eq-(6.2) must be dimensionless.

The metric of eq-(6.2) can be understood as the diagonal sum of two $1+1$ black-holes solutions [54], [55] and whose singularities are located at $uv = 1$ and $wz = 1$ respectively. There are two horizons. The region outside the first horizon is indicated by $u \geq 0 \geq v$ and $v \geq 0 \geq u$; and the region inside the first horizon is indicated by $1 \geq uv \geq 0$ and $u, v \geq 0$. Similar considerations apply to the second horizon by exchanging $u \leftrightarrow w$ and $v \leftrightarrow z$. The lightcone coordinates are defined by

$$u = \frac{1}{2}\, exp\, [\, x + t_1 + log(1 - e^{-2x})\,] = X + T_1$$

$$v = -\frac{1}{2}\, exp\, [\, x - t_1 + log(1 - e^{-2x})\,] = X - T_1. \qquad (6.3a)$$

$$w = \frac{1}{2}\, exp\, [\, y + t_2 + log(1 - e^{-2y})\,] = Y + T_2$$

$$z = -\frac{1}{2}\, exp\, [\, y - t_2 + log(1 - e^{-2y})\,] = Y - T_2 \qquad (6.3b)$$

Conformally flat Solutions of the form

$$ds^2 = e^{\Upsilon(x,y,t_1,t_2)}\, [\, (dx)^2 - (dt_1)^2 + (dy)^2 - (dt_2)^2\,]. \qquad (6.4)$$

where $\Upsilon(x, y, t_1, t_2)$ has a similar singularity structure as the metric in eq-(6.2) are worth exploring also.

The Bars-Witten black-hole $1 + 1$-dim metric (setting $2M = 1$) is :

$$ds^2 = (dr)^2 - tanh^2(r)\, (dt)^2 = -\frac{dudv}{1 - uv}. \qquad (6.5)$$

with

$$u = \frac{1}{2}\, exp\, [\, r + t + log(1 - e^{-2r})\,]; \quad v = -\frac{1}{2}\, exp\, [\, r - t + log(1 - e^{-2r})\,]. \quad (6.6)$$

the *Euclidean* analytical continuation of the metric in eq-(6.5) is obtained by setting $\theta = it$, such that the metric is $ds^2 = dr^2 + tanh^2\, r\, d\theta^2$ and its *Euclidean* geometry has the shape of a semi-infinite cigar that asymptotically approaches $R^1 \times S^1$ for $r \to \infty$. We should notice that the Lorentzian metric of eq-(6.5) has a singularity at a *complex* value $r = 0 + i\pi/2$ (setting $2M = 1$) since $tanh^2(i\pi/2) = -tan^2(\pi/2) = -\infty$ which is consistent with the singularities at the location where $uv = -\frac{1}{4} e^{2r}(1 - e^{-2r})^2 = 1$, when $r = 0 + i\pi/2$, and a horizon at $r = 0$, since $uv = 0$ when $r = 0$.

However this is not the end of the story. The Bars-Witten black hole in $1+1$-dim is obtained from a gauged $Sl(2,R)/U(1)$ WZNW model with central charge $c = 2 + 6/k$ and *is* a consistent bosonic string background solution in a $1+1$ target background given by the two-dim coset $Sl(2,R)/U(1)$. Namely, the CFT corresponding to the gauged $Sl(2,R)/U(1)$ WZNW model with central charge $c = 2 + 6/k$ *is* a solution of equations derived from the vanishing beta functions required by conformal invariance of the non-linear sigma model. For example, the relevant massless bosonic closed-string fields in a $D = 26$ dim target background (a different CFT) are the antisymmetric tensor $B_{\mu\nu}(X^\rho(\sigma^a))$; the dilaton $\Phi(X^\rho(\sigma^a))$ and the gravitational field $g_{\mu\nu}(X^\rho(\sigma^a)))$; where $\sigma^a = \sigma^1, \sigma^2$ are the world-sheet variables. The conditions for the vanishing of the one loop beta functions, required by Weyl invariance of the non-linear sigma model, to leading order in the string tension α' turn out to be [57]

$$\mathcal{R}_{\mu\nu} + \frac{1}{4} H_\mu^{\lambda\rho} H_{\nu\lambda\rho} - 2 D_\mu D_\nu \Phi \ = \ 0. \tag{6.7a}$$

$$D_\lambda H^\lambda_{\mu\nu} - 2(D_\lambda \Phi) H^\lambda_{\mu\nu} \ = \ 0. \tag{6.7b}$$

$$4(D_\mu\Phi)^2 - 4 D_\mu D^\mu \Phi + \mathcal{R} + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \ = \ 0. \tag{6.7c}$$

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}. \tag{6.7d}$$

is the third rank antisymmetric tensor field strength that is invariant under the transformations $\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$. For details of Quantum Nonlinear Sigma Models, Conformal Field Theory, Supersymmetry, Black Holes and Strings we refer to the monograph by Ketov [56].

The only consistent $2+2$-dim gravitational backgrounds on which $\mathcal{N} = 2$ strings [7] (strings with world-sheet supersymmetry) can propagate are those that are self-dual and which solve the Plebanski heavenly equations in $2+2$ dimensions. Self dual gravitational backgrounds in four dimensions are Ricci flat whose metric is given in terms of a Kahler potential. However, the metric in eq-(6.2) is *not* Ricci flat since the $1+1$-dim black hole metric is *not* Ricci flat. Such metric in eq-(6.5) is *not* a solution of the vacuum Einstein field equations, it is a solution of eqs-(6.7) (without Kalb-Ramond fields $B_{\mu\nu}$) where the role of the dilaton $\Phi = ln(1-uv)$ is essential ! .

Nevertheless, we will show how the Bars-Witten $1+1$-dim black hole metric can be embedded into the $3+1$-dim solutions of section **2** , up to a *conformal* factor e^Υ , since the latter metrics were Ricci flat by construction. The embedding of the $1+1$-dim metric (6.5) into the conformally re-scaled $3+1$-dim solutions of section **2** are obtained by introducing the mass parameter $2M$ (in units of $G = c = 1$) in the appropriate places in order to have consistent units, and by writing :

$$e^{\Upsilon(r)} \ (1 - \frac{2M}{R(r)}) \ = \ tanh^2 \ (\frac{r}{2M}); \quad e^{\Upsilon(r)} \ \frac{(dR/dr)^2}{1 - 2M/R(r)} \ = \ 1. \qquad (6.8)$$

leading to the solutions for $\Upsilon(r)$ and $R(r)$ respectively

$$e^{\Upsilon} \ = \ \frac{1}{1 - 2M/R(r)} \ tanh^2 \ (\frac{r}{2M}). \qquad (6.9a)$$

where

$$\int \frac{dR}{1 - 2M/R} \ = \ R \ + \ 2M \ ln \ (\frac{R - 2M}{2M}) \ = \int \frac{dr}{tanh \ r/2M} \ =$$

$$2M \ ln \ [\ sinh \ \frac{r}{2M} \ ]. \qquad (6.9b)$$

this last equation (6.9b) yields the functional form $R(r)$ (tortoise radial variable) in implicit form for the radial function $R(r)$. Despite that the radial function (6.9b) has a *different* functional form than $R^3 = r^3 + (2M)^3$ one can still make contact with the analysis of section **2** pertaining why a point-mass can have an area of $4\pi(2M)^2$ (and zero volume) as found by Schwarzchild in his original 1916 horizonless solution. From eq-(6.9b) one can infer that

$$R(r = 0) \ = \ 2M; \quad R(r \rightarrow \infty) \ \rightarrow \ R \ \sim \ r. \qquad (6.10)$$

which *precisely* has the same behaviour at $r = 0, \infty$ as the Schwarzschild radial function $R^3 = r^3 + (2M)^3$! The radial function R has a lower (ultraviolet cutoff) bound given by $2M$. The fact that a point-mass can have a non-zero *proper area* but zero volume seems to indicate a "*stringy*" nature underlying the very notion of a point-mass itself. The string worldsheet has area but no volume. Aspinwall [22] has studied how a string (an extended object) can probe space-time points. The fact that the stringy black-hole $1 + 1$-dim solution can be embedded into the conformally rescaled solutions of section **2** , for a very specific functional form of the radial function $R(r)$ in eq-(6.9b), with the same "boundary" conditions (6.10) at $r = 0$ and $r = \infty$ as the radial functions displayed in section **2** , is very appealing.

Notice that if we allow for complex values of r, like $r = 0 + i \ 2M(\pi/2)$, that furnish singularities in the metric (6.5), one must include a constant of integration $R_0 = 2M(1 + i\pi/2)$ to the solution in eq-(6.9b)

$$R \ - \ 2M(1+i\pi/2) \ + \ 2M \ ln \ (\frac{R-2M}{2M}) \ = \ 2M \ ln \ [\ sinh \ \frac{r}{2M} \ ]. \quad (6.11)$$

such that when one plugs in the value $r = 0 + i\ 2M(\pi/2)$ in the right hand side of eq-(6.11), it coincides with the left hand side of (6.11) when the value of the radial function $R(\ r = 0 + i\ 2M\pi/2\ ) = 2M\ (1 + i\ \pi/2)$, after an analytical continuation into the complex plane is performed. This is just a consequence of the relation $ln\ [sinh\ (i\pi/2)] = ln\ [i\ sin\ (\pi/2)] = ln\ i = i\pi/2$.

This complex analytical continuation into regions where r, R are complex-valued roughly speaking amounts to looking into the "interior" of the point-mass. Having complex coordinates to probe into the "interior" of a point-mass is not so farfetched. This suggests that Quantum spacetime might be intrinsically $fractal$, meaning that the Hausdorff topological dimension of an object (let us say of a point) does *not* coincide with the $fractal$ dimension. For a throrough and profound treatment of *complex* dimensions, fractal strings and the zeros of Riemman zeta function see [58]. The interplay among non-extensive statistics, chaos, complex dimensions, logarithmic periodicity in the renormalization group and fractal strings see [59].

The conformal factor was

$$e^{\Upsilon} \ = \ \frac{1}{1 - 2M/R(r)} \ tanh^2 \ (\frac{r}{2M}). \qquad (6.12)$$

where $R(r)$ is given implicitly by (6.10). Notice that from the conditions in (6.10) the conformal factor e^{Υ} becomes *unity* at $r = \infty$ as it should if one wishes to have asymptotic flatness. When $r = 0$ the conformal factor (6.12) is $\frac{0}{0}$ undefined. A careful study reveals that the conformal factor e^{Υ} at $r = 0$ is $zero$ so that $e^{\Upsilon(r=0)} R^2(r = 0) = 0$ and the conformally re-scaled proper area at $r = 0$ is $zero$ as mentioned previously in section **2** . Therefore, at $r = 0$ the conformally rescaled interval ds^2 is $zero$ consistent with the fact that the $1+1$-dim metric exhibits a null horizon at $r = 0$. Concluding, in this fashion, we have shown how one can embed the $1 + 1$-dim Bars-Witten $stringy$ black hole solution into the conformally re-scaled $3 + 1$-dim solutions of section **2** given by :

$$ds^2 \ = \ - \ tanh^2 \ (\frac{r}{2M}) \ (dt)^2 \ + \ (dr)^2 \ + \ e^{\Upsilon(r)} \ R^2(r) \ d\Omega^2. \qquad (6.13)$$

Notice that the conformally re-scaled metric (6.13) is *not* Ricci flat; it has singularities at complex values $r = 0 + i\ 2M\pi/2 \ \Rightarrow e^{\Upsilon} = \infty;\ R = 2M(1+i\pi/2)$

upon using eq-(6.11). There is a difference between the metric (6.13) with the Ricci flat metric (outside the singularity at the point mass source) given in the Fronsdal-Kruskal-Szekeres coordinates by

$$ds^2 \;=\; -\; e^{W(u,v)}\;\frac{du\;dv}{1-uv} \;+\; (R_*(u,v))^2\;[\;sin^2\;\phi\;(d\theta)^2 + (d\phi)^2\;] =$$

$$-\; e^{W(u,v)}\;\frac{du\;dv}{1-uv} \;+\; (R_*(u,v))^2\;d\Omega^2 \qquad (6.14)$$

where $W(u,v)$ and $R_*(u,v)$ are now two complicated functions of the two variables u,v (since when one crosses the horizon the metric is *no* longer static) . Whereas in eq-(6.13) one truly has a *static* metric *everywhere* and two functions of one variable $\Upsilon(r)$, $R(r)$ instead.

Before ending this work we will just add some remarks pertaining complex gravity in 1+1 complex dimensions and its relation to ordinary gravity in 2+2 real dimensions. The properties of geometrical objects in the tangent space (at each point of a curved spacetime) associated to the complex, quaternionic and octonionic algebra permits the construction of Einstein's complexified, quaternionic and octonionic gravity. In particular, Gravity in $2+2$-real dim can be studied from the point of view of Complex Gravity in $1+1$ complex dimensions. Gravity in $4+4$ real dim can be studied from the point of view of Quaternionic Gravity in $1+1$ quaternionic dimensions, and Gravity in $8+8$ real dim can be seen as Octonionic Gravity in $1+1$ octonionic dimensions [60]

To illustrate this, let us write the following complex line element in 4 complex-dimensions :

$$ds^2 \;=\; \frac{dz_1 dz_1 + d\;\tilde{z}_1 d\tilde{z}_1}{1 - z_1 z_1 - \tilde{z}_1\tilde{z}_1} \;+\; \frac{dz_2 dz_2 + d\;\tilde{z}_2 d\tilde{z}_2}{1 - z_2 z_2 - \tilde{z}_2\tilde{z}_2} \qquad (6.15)$$

Complex gravity requires that $g_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]}$ so that now one has $g_{\nu\mu} = (g_{\mu\nu})^*$, [60], [61], which implies that the diagonal components of the metric $g_{z_1 z_1} = g_{z_2 z_2} = g_{\tilde{z}_1\tilde{z}_1} = g_{\tilde{z}_2\tilde{z}_2}$ must be real, and which in turn implies that a *real* slice of the 4-complex-dim space spanned by the 4 complex variables $z_1, z_2, \tilde{z}_1, \tilde{z}_2$ may be taken by imposing the following two constraints :

$$\tilde{z}_1 = z_1^*; \qquad \tilde{z}_2 = z_2^* \qquad (6.16)$$

and upon doing so one ends up with a 4 *real*-dimensional space of signature $2+2$ whose *real* line element is

$$ds^2 \;=\; \frac{dz_1 dz_1 + dz_1^* dz_1^*}{1 - z_1 z_1 - z_1^* z_1^*} \;+\; \frac{dz_2 dz_2 + dz_2^* dz_2^*}{1 - z_2 z_2 - z_2^* z_2^*} \qquad (6.17)$$

where z_1, z_2 are the complex coordinates of the $1+1$ complex dimensional spacetime ($2+2$ real dimensional) while z_1^*, z_2^* are their complex conjugates, respectively. After defining

$$z_1 = \frac{1}{\sqrt{2}}(X+iT_1); \quad z_1^* = \frac{1}{\sqrt{2}}(X-iT_1); \quad z_2 = \frac{1}{\sqrt{2}}(Y+iT_2); \quad z_2^* = \frac{1}{\sqrt{2}}(Y-iT_2). \tag{6.18}$$

the metric in eq-(6.14) coincides *precisely* with the metric in eq-(6.2) comprised of the diagonal sum of two black hole solutions in $1+1$ real dimensions. The quaternionic and octonionic versions of eq-(6.16), in conjunction with the generalized Einstein's field equations, will be the subject of future investigations. The Quaternionic analog of 2-dim Conformal Field theory in four dimensions has been studied by [62]. It is interesting to see (if possible) how one can construct 4-dim Quantum Non-linear sigma models within the context of quantum 3-branes (conformal field theories in the four-dim world volume of the 3-brane) and find the analog of the coupled equations (6.7) associated with the vanishing of the beta functions in 2-dim CFT; namely from the perspective of a 4-dim Quaternionic Conformally invariant Field Theory formulated on Kulkarni four-folds (the four-dim analog of Riemann surfaces) corresponding to 3-branes moving in curved target spacetime backgrounds. The cancellation of the 4-dim conformal anomaly should constrain the type of backgrounds on which 3-branes can propagate.

It is worth mentioning that "black hole" solutions in a two times context have been considered by some authors. In particular Kocinski and Wierzbicki [64] considered Schwarzschild type solution in a Kaluza-Klein theory with two times. In fact, using noncompactified Kaluza-Klein theory with internal signature of the form $2+3$ these authors determine a spherical symmetric solution. Vongehr [65] also considered examples of black holes within the context of the two-times physics formulation of Bars (see [63] and Refs. therein). Their basic examples coreponds essentially to a solutions associated with the signatures $1+1$ and $2+3$.

Finally, the 4-dim Kaluza-Klein approach to General Relativity in $2+2$ as a local product of a $1+1$-dim base manifold and a $1+1$-dim fiber space [66] warrants further investigation in so far that $2+2$ Gravity can be described by a $1+1$-dim Yang-Mills gauge theory of diffeormorphims of the two-dim fiber space coupled to a $1+1$-dim non-linear sigma model and a scalar field; i.e. this formulation of $2+2$ Gravity by [66] is more closely related to the stringy picture of the Bars-Witten black-hole in $1+1$-dimensions. Thus, it seems interesting to pursue further research to see the possible connection between the present work and these other approaches.

Appendix A: Schwarzschild-like solutions in any dimension $D > 3$

Let us start with the line element

$$ds^2 = -e^{\mu(r)}(dt_1)^2 + e^{\nu(r)}(dr)^2 + R^2(r)\tilde{g}_{ij}d\xi^i d\xi. \qquad (A.1)$$

Here, the metric $\tilde{g}_{ij}$ corresponds to a homogeneous space and $i, j = 3, 4, ..., D-2$. The only nonvanishing Christoffel symbols are

$$\begin{aligned} &\Gamma^1_{21} = \tfrac{1}{2}\mu', && \Gamma^2_{22} = \tfrac{1}{2}\nu', && \Gamma^2_{11} = \tfrac{1}{2}\mu' e^{\mu-\nu}, \\ &\Gamma^2_{ij} = -e^{-\nu}RR'\tilde{g}_{ij}, && \Gamma^i_{2j} = \tfrac{R'}{R}\delta^i_j, && \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}, \end{aligned} \qquad (A.2)$$

and the only nonvanishing Riemann tensor are

$$\begin{aligned} &\mathcal{R}^1_{212} = -\tfrac{1}{2}\mu'' - \tfrac{1}{4}\mu'^2 + \tfrac{1}{4}\nu'\mu', && \mathcal{R}^1_{i1j} = -\tfrac{1}{2}\mu' e^{-\nu}RR'\tilde{g}_{ij}, \\ &\mathcal{R}^2_{121} = e^{\mu-\nu}(\tfrac{1}{2}\mu'' + \tfrac{1}{4}\mu'^2 - \tfrac{1}{4}\nu'\mu'), && \mathcal{R}^2_{i2j} = e^{-\nu}(\tfrac{1}{2}\nu' RR' - RR'')\tilde{g}_{ij}, \\ &\mathcal{R}^i_{jkl} = \tilde{R}^i_{jkl} - R'^2 e^{-\nu}(\delta^i_k \tilde{g}_{jl} - \delta^i_l \tilde{g}_{jk}). \end{aligned} \qquad (A.3)$$

The field equations are

$$\mathcal{R}_{11} = e^{\mu-\nu}(\frac{1}{2}\mu'' + \frac{1}{4}\mu'^2 - \frac{1}{4}\mu'\nu' + \frac{(D-2)}{2}\mu'\frac{R'}{R}) = 0, \qquad (A.4)$$

$$\mathcal{R}_{22} = -\frac{1}{2}\mu'' - \frac{1}{4}\mu'^2 + \frac{1}{4}\mu'\nu' + (D-2)(\frac{1}{2}\nu'\frac{R'}{R} - \frac{R''}{R}) = 0, \qquad (A.5)$$

and

$$\mathcal{R}_{ij} = \frac{e^{-\nu}}{R^2}(\frac{1}{2}(\nu' - \mu')RR' - RR'' - (D-3)R'^2)\tilde{g}_{ij} + \frac{k}{R^2}(D-3)\tilde{g}_{ij} = 0, \quad (A.6)$$

where $k = \pm 1$, depending if $\tilde{g}_{ij}$ refers to positive or negative curvature. From the combination $e^{-\mu+\nu}R_{11} + R_{22} = 0$ we get

$$\mu' + \nu' = \frac{2R''}{R'}. \qquad (A.7)$$

The solution of this equation is

$$\mu + \nu = \ln R'^2 + a, \tag{A.8}$$

where a is a constant.

Substituting (A.7) into the equation (A.6) we find

$$e^{-\nu}(\nu' R R' - 2RR'' - (D-3)R'^2 = -k(D-3) \tag{A.9}$$

or

$$\gamma' R R' + 2\gamma R R'' + (D-3)\gamma R'^2 = k(D-3), \tag{A.10}$$

where

$$\gamma = e^{-\nu}. \tag{A.11}$$

The solution of (A.10) for an ordinary D-dim spacetime (one temporal dimension) corresponding to a $D-2$-dim sphere for the homogeneous space can be written as

$$\gamma = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}}) \ (\frac{dR}{dr})^{-2} \Rightarrow$$

$$g_{rr} = e^{\nu} = (1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2} R^{D-3}})^{-1} \ (\frac{dR}{dr})^2. \tag{A.12}$$

where Ω_{D-2} is the appropriate solid angle in $D-2$-dim and G_D is the D-dim gravitational constant whose units are $(length)^{D-2}$. Thus $G_D M$ has units of $(length)^{D-3}$ as it should. When $D = 4$ as a result that the 2-dim solid angle is $\Omega_2 = 4\pi$ one recovers from eq-(A.12) the 4-dim Schwarzchild solution. The solution in eq-(A.12) is consistent with Gauss law and Poisson's equation in $D-1$ spatial dimensions obtained in the Newtonian limit.

For the most general case of the $D-2$-dim homogeneous space we should write

$$-\nu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) - 2\ln R'. \tag{A.13}$$

where β_D is a constant. Thus, according to (A.8) we get

$$\mu = \ln(k - \frac{\beta_D G_D M}{R^{D-3}}) \ + \ constant. \tag{A.14}$$

we can set the constant to zero, and this means the line element (A.1) can be written as

$$ds^2 = -(k - \frac{\beta_D G_D M}{R^{D-3}})(dt_1)^2 \ + \ \frac{(dR/dr)^2}{(k - \frac{\beta_D G_D M}{R^{D-3}})}(dr)^2 \ + \ R^2(r)\tilde{g}_{ij}d\xi^i d\xi. \quad (A.15)$$

One can verify, taking for instance (A.5), that the equations (A.4)-(A.6) do *not* determine the form $R(r)$. It is also interesting to observe that the only effect of the homogeneous metric $\tilde{g}_{ij}$ is reflected in the $k = \pm 1$ parameter, associated with a positive (negative) constant scalar curvature of the homogeneous $D-2$-dim space.

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