DARK PARTICLES

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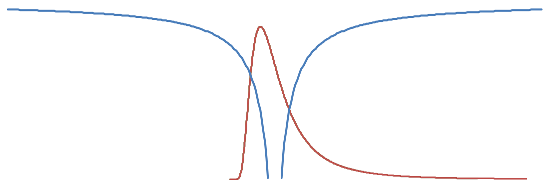

ABSTRACT

"Dark" particles, a candidate theory to explain dark matter and dark energy, are hypothesized. A change in one of the assumptions of Langevin's equation of statistical mechanics (namely making the noisy driving force anti-correlated with the particle's location instead of un-correlated, justified by the gravitational potential of the particle) results in an exact solution shared by Freidmann's equation of general relativity. The anti-correlated force, or spring force, places the "dark" particle in the ground state of the harmonic oscillator, consistent with the interpretation of it being the vacuum energy density. A density of black body radiation is found gravitationally bound to the "dark" particle. When hydrogen atoms are present (e.g. near Galaxies) the "dark" particles can exchange heat with ordinary energy, making the radiation pressure positive, $w = 1/3$, similar to dark matter. But when no coupling mechanism is available, the "dark" particles' temperature is frozen with a negative radiation pressure, $w = -1$. If the temperature of "dark" particles in intergalactic space was frozen after re-ionization ($z = 10.5$ when hydrogen atoms were lost), the bound BBR density is well within the tight confidence levels of the accepted relic density of dark energy.

I. INTRODUCTION

A. Background

Cosmological observations early last century indicate the Universe is expanding. These observations came by measuring the speed at which objects are moving away from Earth and noticing the strong correlation with their distance, known as Hubble's Law. However it was not until the end of the last century when we had observations of Type Ia supernova that indicated the Universe is also accelerating [Liddle 2003].

The most popular explanation for these findings is an elusive energy density with an equation of state, $w < -1/3$ [Peebles & Ratra 2003] coined "Dark Energy" making up ~73% [Larson, Dunkley, et al. 2010] of the energy of the Universe. Despite many attempts to explain dark energy's origin [Rugh & Zinkernagel 2002], those attempts have fallen short [Peebles & Ratra 2003, Greene 2004]. Yet there are still many theories under review [Papantonopoulos 2007]; with the most accepted being the Lambda Cold Dark Matter model, $\Lambda\mathrm{CDM}$ where the Lambda, the Cosmological Constant provides a negative pressure $w = -1$ [Peebles & Ratra 2003].

Another set of observations (that seem to be independent of the expansion and acceleration observations) show a discrepancy between the amounts of luminous matter we can visually account for and the amount of mass we can infer from gravitational effects, such as speed of galaxy rotation [Liddle 2003], even if we triple the number of stars. "Dark Matter," the explanation for this discrepancy, is a positive pressure energy density that clumps near other baryonic matter, is devoid of interactions with photons [Liddle 2003], and makes up ~22% of the energy of the Universe [Larson, Dunkley, et al. 2010].

We hypothesize that under different conditions experienced in the Universe, "dark" particles can explain both dark energy and dark matter. A "dark" particle is hypothesized by introducing a spring force that acts to keep the particle in the ground state of the harmonic oscillator. Justification for the "dark" particle being the vacuum energy density is found by uniting Langevin's equation from statistical mechanics (which governs the microscopic diffusion of particle) and Freidmann's equation from general relativity (which governs the gravitational length scale of a system) through a solution that is common to both equations.

We will see that despite the high gravitational fields of the "dark" particle, which come from it carrying the reduced Planck mass, the effects are cancelled by curvature of space around the particle. Yet a hidden (bound) density of black body radiation remains who's behavior mimics that of both dark matter and dark energy depending on if the temperature of the black body radiation is variable or constant, respectively.

B. Ansatzs and Definitions

Two Ansatzs are used in this paper. The first one was already introduced, i.e. the anti-correlated spring force. Defining the proportionality of the spring force is the classical mass times acceleration (i.e. displacement over the time squared).

$$F_{spring} = \frac{-mx}{\tau^2}$$

The time τ will later be shown equal to one over twice the temperature by applying the equipartition theorem to the energy.

The second Ansatz is "dark" particles come in pairs. This assumption is used to define two length scales, a general relativistic length scale, L, and a quantum mechanical length scale, ℓ. L is defined by the distance the pair of particles can cover if they travel in opposite directions at the speed of light for time τ.

$$L = 2c\tau$$

ℓ is defined by the square root of the variance of the distance between pair. As long as the pair has independent wavefunctions we can write,

$$\ell = \sqrt{2}\Delta x_{Quantum}$$

We will find that when the mass of the "dark" particle is the reduced Planck mass, these two length scales will be equal if L is the solution to Freidmann's equation and ℓ is a solution to Langevin's equation.

II. DERIVATION OF SOLUTION

A. Energy Density

Before we solve the Freidmann equation, we need to first derive the energy density. While a thorough derivation is available in Appendix A, a straightforward derivation is here.

With the introduction of a spring force, F_{spring}

$$F_{spring} = \frac{-mx}{\tau^2}$$

And it's associated potential energy,

$$\hbar\omega_x = \frac{m}{2}\left(\frac{x}{\tau}\right)^2$$

We find the harmonic osscilator with 1-D groud state energy, E_{x0} [Feynman 1965].

$$E_{x0} = \frac{\hbar\omega_0}{2} = \frac{\hbar}{2}\sqrt{\frac{m/\tau^2}{m}} = \frac{\hbar}{2\tau} = \frac{\hbar c}{L}$$

In Appendix A we find $L = \hbar c/k_B T$ by using the equipartition theorem to show $\tau = \hbar/2k_B T$; leading to

$$E_{x0} = k_B T$$

Looking at all three dimensions (x, y, z) we have,

$$E_0 = 3E_{x0} = 3k_B T$$

This shows us that with the introduction of the proposed spring force, the particle has just the right amount of thermal energy (not more, nor less) to exist in the in the ground state of the harmonic oscillator. Being in the ground state is consistent with *an interpretation of this density as the vacuum energy*.

Diving by the volume in question, $V = L^3$, the energy density is,

$$\rho = \frac{E_0}{V} = \frac{3(k_B T)^4}{\hbar^3 c^5} = \frac{3\hbar}{c(L)^4}$$

In Appendix B, we show the details how this energy density has three terms; one where the temperature T is variable with an equation of state $w = 1/3$, one where the temperature T_0 is constant with an equation of state $w = -1$, and a third term with equation of state $w = -1/3$ that can be traced back to the curvature of space. This third term is from the gravitational potential energy and is shown to be the source of the resistive spring force.

We confirm the correct form for each term of the density by showing the corresponding solution to Friedmann's equation is shared by a solution to Heisenberg's uncertainty equation when the mass of the "dark" particle is the reduced Planck mass.

With details found in Appendix B, the resulting holistic density for "dark" particles is,

$$\rho_{DP}(L) = 3\left(\frac{\hbar^2}{(L(t))^4} - \frac{2mk_BT_0}{(L(t))^2} + \frac{m^2(k_BT_0)^2}{\hbar^2}\right)$$

B. Friedmann Equation (General Relativistic Solution)

Directly substituting $\rho_{DP}(L)$ into Friedmann's equation,

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{\hbar^2}{m^2(L(t))^4} - \frac{2k_BT_0}{m(L(t))^2} + \frac{(k_BT_0)^2}{\hbar^2}$$

The three terms have an equation of state, $w = \frac{1}{3}, -\frac{1}{3}, -1$ respectivly. This equation is easily solved.

$$\frac{(\dot{L})}{L} = \left(\frac{\hbar}{mL^2} - \frac{k_BT_0}{\hbar}\right)$$

With a little calculus the solution is

$$L^2 = \frac{\hbar^2}{mk_BT_0}\left(1 - e^{-2k_BT_0t/\hbar}\right)$$

With $k_BT_0 = \hbar/(2\tau_0)$ and the diffusion constant from appendix B, $D = \hbar/(2m)$, this is re-written,

$$L^2 = 4D\tau_0\left(1 - e^{-t/\tau_0}\right)$$

C. Langevin Equation (Statistical/Quantum Mechanical Solution)

The Langevin equation begins with Newton's 2^{nd} Law (the force equation) with a resistive force proportional to the velocity and a noise driving force [Kubo 1966].

$$m(\ddot{x}) = -\frac{m}{\tau}(\dot{x}) + F_{Noise}$$

In the classical case F_{Noise} is uncorrelated with x. However here the force is anti-correlated with x and is proportional to the mass times the acceleration where acceleration is the change in position over the change in time squared.

$$F_{Noise} = F_{spring} = \frac{-mx}{\tau^2}$$

$$m(\ddot{x}) = -\frac{m}{\tau}(\dot{x}) - \frac{m}{\tau^2}(x)$$

Detailed in Appendix B we have,

$$\overline{x(\dot{x})} = Ce^{-t/\tau}$$

Where C is a constant of integration.

Again we break from the classical derivation in the initial condition, $\overline{x(\dot{x})}(t = 0)$. Classically this initial condition is zero, however this would lead to a trivial result. Plus we want the Langevin equation to be quantum mechanically correct and obey the Heisenberg uncertainty relation.

$$\overline{x(\dot{x})}(t = 0) = \frac{\Delta x \Delta p_x}{m} = \frac{\hbar}{2m}$$

$$\overline{x(\dot{x})} = \frac{\hbar}{2m}e^{-t/\tau}$$

Calculus and substitution of the diffusion constant, $D = \hbar/(2m)$ completes the derivation,

$$\overline{x^2} = \Delta x^2_{Langevin} = 2D\tau\left(1 - e^{-t/\tau}\right)$$

$$\ell^2 = 2\Delta x^2_{Langevin} = 4D\tau\left(1 - e^{-t/\tau}\right)$$

We can clearly see that ℓ (the solution to Langevin's equation) is equal to L (the solution to Freidmann's equation, giving justification that "dark" particles obey both statistical & quantum mechanics and general relativity in the same way.

III. BLACKBODY RADIATION

A. Balanced (Zero) Density

One amazing feature of the density inclusive of all three terms is that it approaches zero as time passes. When time first starts out, the density is huge given the inverse dependence on the fourth power of the length scale.

$$\rho_{DP}(L) = 3\left(\frac{\hbar^2}{(L(t))^4} - \frac{2mk_BT_0}{(L(t))^2} + \frac{m^2(k_BT_0)^2}{\hbar^2}\right)$$

However at $L's$ asymptotic value the three separate densities perfectly cancel each other.

$$\lim_{L\to\hbar/\sqrt{mk_BT_0}} \rho_{DP}(L) = 0$$

One might ask however if this energy density is a candidate for the missing energy of the Universe: i.e. dark matter under the the temperature is variable where $w = 1/3$ [Liddle 2003]; and dark energy when the temperature is constant where $w = -1$ [Peebles & Ratra 2003]. The answer is not directly. As we just argued the density approaches zero as time passes. However indirectly we find that the strong local gravitation fields produce a radiation density of bosons that separately adds to the density.

B. RADIATION FIELD

Due to the high mass of the "dark" particle, it will be dominated by gravitational effects if its charge is less than, $\sqrt{\hbar c\varepsilon_0/2} \cong 2.34q_{electron}$. In this environment, one would expect the particle to be coupled to a radiation field [Hartle & Hawking 1976]; and this is what is found (as detailed in appendix C).

We implicitly assumed in the beginning of the paper that the dark particles have a non-zero mass. However a density of photons will have zero mass. In this case only the first term in the "dark" particle energy density remains.

$$\rho_{zero-mass} \propto \frac{3\hbar}{c(L(t))^4}$$

Furthermore when the probability distribution on the energy of the harmonic oscillator hypothesized in appendix A is associated with bosons instead of fermions (where multiple particles can occupy each mode), the density needs to be summed over these multiple states. Doing so and further accounting for time dilation and red-shift effects of radiating in the high gravitational fields of the particle, the resulting radiation density bound by the gravitational fields is exactly the black body distribution (as detailed in appendix C). When T_{DP} is the temperature of the radiation trapped by the dark particles we have:

$$\rho_{radiation}(\omega)d\omega = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{\left(e^{\hbar\omega/kT_{DP}} - 1\right)}d\omega$$

$$\rho_{radiation} = \int_0^\infty \rho_{radiation}(\omega)d\omega = \frac{\pi^2(k_BT_{DP})^4}{15\hbar^3c^3}$$

In normal black body radiation, a macroscopic cavity provides the confinement of the radiation [Reif 1965]. Here it is the gravitational effects from the mass of the particle that confines the radiation as illustrated in figure 1 below.

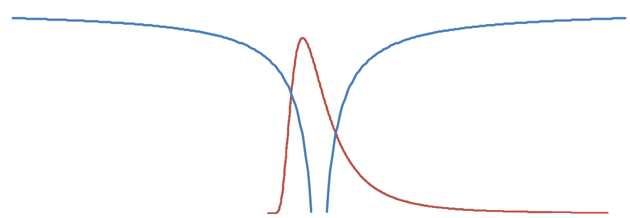

Figure 1 – Black Body Radiation trapped within the gravitational potential defined by the Schwarzschild metric.

IV. DARK MATTER AND DARK ENERGY

The density and length scales discussed above are associated with individual dark particle pairs. However with their theoretical derivation behind us we can get to a discussion of their application to the open questions of dark energy and dark matter by extrapolating to the density and length scale of the Universe.

We hypothesize that under different conditions experienced in the Universe, dark particles can explain both dark energy and dark matter.

A. Two Regimes

Now hypothesize that a local group of dark particle's are able to exchange heat with the local surroundings when hydrogen atoms or other sinks are nearby to capture the radiation from its gravitational binding. In this case the temperature of the dark particles is variable and thus inversely proportional to the length scale of the local Universe leading to an aggregate energy density local to the hydrogen atoms with an equation of state, $w = 1/3$.

Dark particles randomly walk as they exchange heat with available local hydrogen atoms, thus allowing the temperature of its trapped radiation to equilibrate with the external radiation field. Figure 3 below is an artist rendition of what this process might look like.

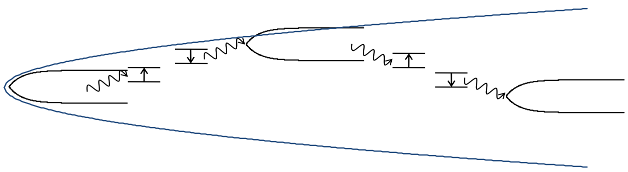

Figure 2 – Artist radiation of "dark" particles with variable temperature

However when the dark particles are isolated away from any sinks, no radiation can escape and the dark particles radiation has no way to release heat or change their temperature. In this case the particles' radiation density is constant and the equation of state is $w = -1$. The total energy scales linearly with the volume (which is exponentially increasing) as work is done on the system as it expands. The radiated photons in turn exponentially generate new dark particles to fill the space as expressed in Figure 3.

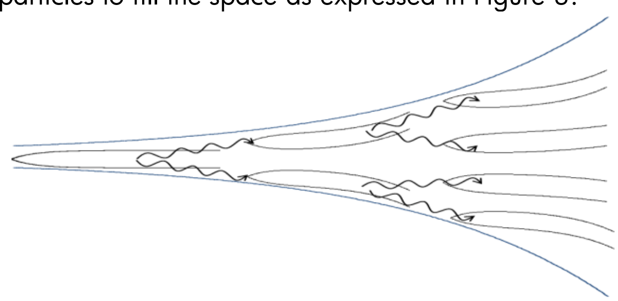

Figure 3 – Artist radiation of "dark" particles with constant temperature

B. Dark Energy in Our Universe (Constant Temperature Regime)

During the dark ages, the time between decoupling and re-ionization [Barkana & Loeb 2001], the Universe was filled with hydrogen atoms that provided the coupling mechanism between the dark particles and regular matter. In these conditions the dark particles were coupled to the Cosmic Microwave Background (CMB). However after re-ionization the hydrogen was ionized and the dark particles and its associated radiation energy density became frozen. The red-shift of re-ionization and the current temperature of the CMB provide an estimate of the temperature of the dark particle before they moved into the dark energy regime after re-ionization.

$$T_{\mathrm{DP}} = (1 + z_{Re-ionization})T_{today} = constant$$

If we know the temperature of the dark particles at re-ionization, then we should have an idea for the total energy that contributes to the Cosmological constant.

$$\begin{aligned}\rho_{\mathrm{Dark\ Particles-BBR}} &= \frac{\pi^2}{15}\frac{(k_B T_{DP})^4}{\hbar^3 c^5} \\ &= \frac{\pi^2}{15}\frac{\left(k_B(1 + z_{Re-ionization})T_{today}\right)^4}{\hbar^3 c^5} \\ &= constant\end{aligned}$$

Because we have estimates of today's z value of re-ionization and today's temperature of the CMB we can estimate the density, ρ_{DP-BBR}. Without going into the details, the Lambda Cold Dark Matter model, ΛCDM, provides a completely independent estimate of the density of dark energy, $\rho_{\Lambda\mathrm{CDM}}$ [Liddle 2003], which we can estimate using the parameter, Ω_Λ, and today's Hubble constant.

$$\rho_{\Lambda CDM} = \Omega_\Lambda \cdot \rho_{\mathrm{critical}} = \Omega_\Lambda \cdot \frac{3H^2}{8\pi G}$$

The source of our estimates will be from the 7 year Wilkinson Microwave Anisotropy Probe [Larson, Dunkley, et al. 2010]. However we must keep in mind that these estimates are best fit parameters and come with confidence ranges. The 68% confidence ranges and estimates are compiled below in Table 1 and shown in Figure 1 for the Dark Particles BBR model and the Lambda Cold Dark Matter model.

	Low	Average	High		Low	Average	High
$\rho_{\mathrm{DP-BBR}}$ (kg/m^3)	5.22E-27	8.12E-27	1.21E-26	$\rho_{\Lambda CDM}$ (kg/m^3)	6.21E-27	6.95E-27	7.74E-27
T_{today} $(degrees)$	2.725	2.725	2.725	H $\left(\frac{km}{sec \cdot Mpc}\right)$	68.5	71.0	73.5
$z_{re-ionization}$	9.3	10.5	11.7	Ω_Λ	0.705	0.734	0.763

Table 1 Estimate and confidence rages of Dark Energy from the Dark Particles BBR model and the Lambda Cold Dark Matter model using 7 year WMAP data

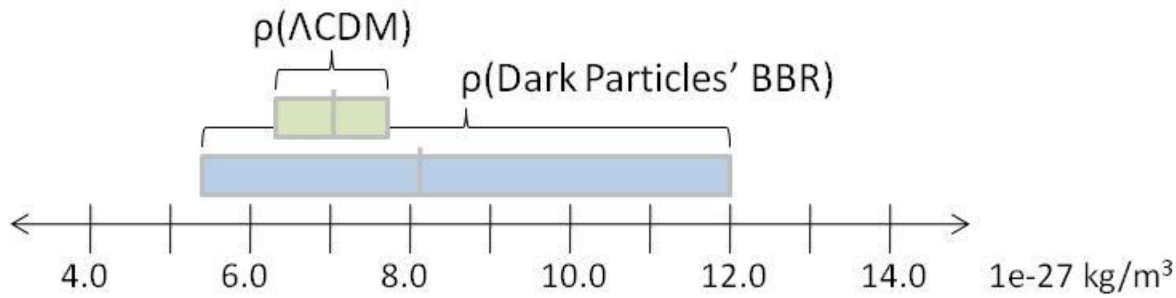

Figure 2 Visualization of estimate and confidence rages of Dark Energy from the Dark Particles BBR model and the Lambda Cold Dark Matter model using 7 year WMAP data

With this development we can explain the previous 10^{120} discrepancy between theory and observation. In previous theory the energy cutoff was set at the Planck temperature, however we see that if "dark" particles are the cause of vacuum density there is a temperature associated with that density. In the case of our Universe, that temperature was frozen in at re-ionization which is 10^{30} times smaller than the Planck temperature. As the density is to the fourth power of temperature, the discrepancy is explained.

It is also supportive to examine how this theory holds up to the inflationary period just after the big bang. The inflationary period that follows the grand unified period lasts for $\sim 10^{-34}$ seconds and during this time, the scale factor of the Universe grows exponentially by a factor of $\sim 10^{26}$[Liddle & Lyth 2000]. Assuming that dark particles where able to release heat thereby maintaining equilibrium with the rest of the Universe's energy during the time of grand unification (immediately preceding inflation), but that once the inflation period began the dark particles became isolated, then the dark particles will have a constant energy density during inflation leading to exponential expansion.

The theory presented here allows us to be precise in the relationship between the temperature of Grand Unification and the inflation ratio during the inflation period. Plugging in accepted estimates of the duration, temperature and inflation ration we find it is consistent [Liddle & Lyth 2000].

$$\frac{R_{post\ inflation}}{R_{pre\ inflation}} = e^{Ht} = e^{\sqrt{\frac{8G\pi^3 (k_B T_{GU})^4}{45\hbar^3 c^5}}.t}$$

The theory also provides insights into reheating, the period after inflation. If you imagine the dark particles were at the temperature of grand unification at the beginning of the inflationary epoch only to become isolated, the dark particles would remain constant during the inflation while the rest of the energy would cool by a factor of $\sim 10^{26^4}$. If quarks, anti-quarks or gluons (which became available at the end of the inflationary period) are able to couple dark particles to the rest of the Universe's energy, heat could flow from the hot dark particles back into the rest of the Universe, reheating it.

C. Dark Matter in Our Universe (Variable Temperature Regime)

After re-ionization, the majority of the dark particles became isolated and the temperature was frozen in as discussed above; however the argument continues for dark matter in local regions near other baryonic matter. For example, the dark particles near galaxies still to this day has have access to neutral hydrogen or other sinks that allow the dark particle to have an equation of state, $w > -1$, or perhaps even $w = 1/3$ if a preponderance of hydrogen atoms (or other sinks) are available.

If $w > -1/3$, the Freidmann acceleration equation shows the gravitation effects of these dark particles to be positive resulting in attractive gravity [Liddle 2003]. If this environment were experienced in our Universe, it would be an explanation for dark matter and why dark matter clumps near other baryonic matter.

We have an observation of a region of space with a preponderance of neutral hydrogen, VIRGOI21, where we can test this hypothesis. In this case a preponderance of neutral hydrogen would imply a preponderance of dark matter, which we see.

We can go even further by using this theory to estimate the temperature of the dark particles (which would be in equilibrium with the neutral hydrogen atoms). First the gravitational energy density for the cloud is found by taking its theorized gravitational mass of the cloud and dividing by its volume. Minchin et al. [2007] estimate the mass of the cloud at $\sim 10^{11}$ solar masses by looking at the clouds effects on a nearby system and other considerations. The actual volume of the cloud is not available, but it is argued that the cloud is a spinning disk which is seen side on with a radius of $\sim 7\ kpc$. If the width of the disk is 1/3 the radius, the mass density is

$$\rho_{V21} \cong \frac{M}{\pi R^2\left(\frac{1}{3}R\right)} \cong 10^{-20}\frac{kg}{m^3}$$

Using the expression for the energy density of dark particles' radiation we have

$$\rho_{\text{Dark Matter}} = \frac{\pi^2 (k_B T_{DP})^4}{15\hbar^3 c^5}$$

The temperature of the hydrogen cloud needs to be, T=~1000 Kelvin and in equilibrium with the dark matter particles for the density of the radiation associated with the dark particles to be equal to observed and estimated density of VIRGOI21. While the Hydrogen 21 line does not provide a precise estimate of the temperature of the hydrogen atoms, the temperature must be greater than a few tens of Kelvin to radiate and less than a few thousand Kelvin to remain neutral [Minisch 2010]. Fortunately, we find an order of magnitude consistency.

The opposite extreme also provides insight. For example Globular Clusters provide a region of the Universe where starlight is very prevalent. In this case we would expect any atoms, or other loose baryonic matter to be fully ionized and thus not able to provide the coupling mechanism to allow the local "dark" particles to cool. In this example the energy density of dark particles would remain constant, the equation of state would be w = -1 and we would not see the effects of any additional positive pressure. Again if this reasoning is correct, it could explain why we don't measure any dark matter within Globular Clusters (Mashchenko & Sills 2005).

Given that current attempts to discover dark matter particles as WIMPS or MACHOS are looking at either way too small an energy scale or way too big, respectively, we have not had much luck in finding observational evidence of dark matter.

V. CONCLUSION

1. Recap

A framework and correct values for the physical parameters necessary to join Freidmann's equation to Langevin's equation and Heisenberg's Uncertainty equation is presented. These equations leverage existing accepted models of the Universe. Mathematics then shows us how these models of general relativity and quantum and statistical mechanics share a unique solution. Astrological observation is then used to test the hypothesis that this theory explains the behavior and magnitude of dark matter and dark energy.

The only Ansatzes used are 1) a spring force that resists motion and 2) dark particles coming as pairs. Given that this simple development requires no major change to quantum mechanics or general relativity and has the potential to explain dark matter or dark energy, more investigation is justified under Occam's razor [Cover & Thomas 1991].

2. Discussion

This is only the beginning as much more investigation is needed. For example, Bertone and Silk [2010] outline a list of questions that a dark matter particle must answer before it becomes a good candidate,"

(i) Does it match the appropriate relic density?
(ii) Is it cold?
(iii) Is it neutral?
(iv) Is it consistent with Big Bang nucleosynthesis (BBN)?
(v) Does it leave stellar evolution unchanged?
(vi) Is it compatible with constraints on self-interactions?
(vii) Is it consistent with direct DM searches?
(viii) Is it compatible with gamma-ray constraints?
(ix) Is it compatible with other astrophysical bounds?
(x) Can it be probed experimentally?"

Some of these questions are already answered, like is it consistent with direct DM searches, while other questions are still to be answered by further research.

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Mountain View, CA – December 2010

APPENDIX A – ENERGY DENSITY AND PROCESS

This analysis hypothesizes that a "dark" particle comes into existence with its pair and the two will independently diffuse in space in each of the three position dimensions. To start with, the magnitude squared of the wavefunction will be the Gaussian distribution

$$x \sim p(x)dx = \frac{1}{\sqrt{2\pi\Delta x^2}} \cdot e^{-\frac{x^2}{2\Delta x^2}} dx$$

We can now calculate the distribution on the momentum p_x, p_y,and p_z. The distribution on p_x will also be a Gaussian with a variance that satisfies the Heisenberg uncertainty principle [Shankar 1994].

$$(\Delta p_x)^2 = \frac{\hbar^2}{4(\Delta x)^2}$$

$$p_x \sim p(p_x)dp_x = \frac{1}{\sqrt{2\pi\Delta {p_x}^2}} \cdot e^{-\frac{{p_x}^2}{2\Delta {p_x}^2}} dp_x$$

Thus the probability distribution on x and p_x (with y & z and p_y & p_z being respectively equal), will be the Gaussian distribution with mean zero and variance $(\Delta x)^2$ and $(\Delta p_x)^2$ respectively.

The total energy of the particle will have both position and momentum components. The additon of the kinetic energy should be familiar $\hbar\omega_{px} = \frac{1}{2m}(p_x)^2$. However there is also a potential energy term $\hbar\omega_x = \frac{m}{2}\left(\frac{x}{\tau}\right)^2$ resulting from the inertial mass. Assuming the particle is at rest relative to a reference frame, the particle will originate at rest and begin to move a total displacement, x in time τ. From Netwon's second Law [Newton 1995] the force on the particle will be

$$F_{particle} = ma = m\frac{\Delta v}{\Delta t} = m\frac{\frac{\Delta x}{\Delta t}}{\Delta t} = \frac{mx}{(\tau)^2}$$

With the particle being at rest on average we can look at Newton's third Law [Newton 1995] or stating Newton's first Law [Newton 1995] backward, i.e. given a particle at rest, the sum of the forces acting on the particle is zero, we have

$$F_{spring} = -F_{particle} = \frac{-mx}{(\tau)^2}$$

Here we see Hook's Law which has an associated potential energy $\hbar\omega_x = \frac{m}{2}\left(\frac{x}{\tau}\right)^2$ leading to the harmonic osscilator with 1-D groud state energy, E_{x0} [Feynman 1965].

$$E_{x0} = \frac{\hbar\omega_0}{2} = \frac{\hbar}{2}\sqrt{\frac{m/\tau^2}{m}} = k_B T$$

Considering the y and z dimensions as well, we are led to the three dimensional harmonic oscillator.

$$Energy = \hbar\omega = \hbar\omega_x + \hbar\omega_{px} + \hbar\omega_y + \hbar\omega_{py} + \hbar\omega_z + \hbar\omega_{pz}$$

$$Energy = \hbar\omega = \frac{m}{2}\left(\frac{x}{\tau}\right)^2 + \frac{1}{2m}(p_x)^2 + \frac{m}{2}\left(\frac{y}{\tau}\right)^2 + \frac{1}{2m}(p_y)^2 + \frac{m}{2}\left(\frac{z}{\tau}\right)^2 + \frac{1}{2m}(p_z)^2$$

If we assume the process is ergodic and that the particle is at a equilibrium temperature T the equipartition theorem can be evoked where, the average energy is equal to $k_B T/2$ for each quadratic dimension [Feynman 1965].

$$\overline{\frac{m}{2}\left(\frac{x}{\tau}\right)^2} = \overline{\frac{1}{2m}(p_x)^2} = \frac{k_B T}{2}$$

Since the average of x and p_x is equal to zero, the average of x^2 and ${p_x}^2$ is equal to the variance $(\Delta x)^2$ and $(\Delta p_x)^2$

$$(\Delta p_x)^2 = m \cdot k_B T$$

$$(\Delta x)^2 = \frac{\hbar^2}{4m \cdot k_B T}$$

$$\tau = \frac{\hbar}{2k_B T}$$

$$\overline{\hbar\omega} = \frac{6 \cdot k_B T}{2}$$

After τ seconds, the particle will be found in phase space in each of the three dual dimensions: (x, p_x), (y, p_y), and (z, p_z).

Given the relationship $\omega_x = ax^2\ (a > 0)$ and with $x \sim p(x)$) as above, the distribution on ω_x is [Pan 2007],

$$\omega_x \sim p(\omega_x)d\omega_x = \frac{1}{\sqrt{2\pi\Delta x^2 a\omega_x}} e^{\frac{-\omega_x}{2a\Delta x^2}} d\omega_x$$

Thus from the probability distributions on x, y, and z given above the distribution on ω_x is derived

$$\omega_x \sim p(\omega_x)d\omega_x = \sqrt{\frac{\hbar}{\pi k_B T \omega_x}} e^{\frac{-\hbar\omega_x}{k_B T}} d\omega_x$$

Fortunately the distribution on ω_{px}, given p_x, p_y, and p_z above, is the same

$$\omega_{px} \sim p(\omega_{px})d\omega_{px} = \sqrt{\frac{\hbar}{\pi k_B T \omega_{px}}} e^{\frac{-\hbar\omega_{px}}{k_B T}} d\omega_{px}$$

Since ω is simply the sum of the independent variables ω_x, ω_y, ω_z and ω_{px}, ω_{py}, ω_{pz} we can solve for the distribution on ω as the convolution of the other distributions [Bracewell 1986].

$$\omega \sim p(\omega) = p(\omega_x) * p(\omega_{px}) * p(\omega_y) * p(\omega_{py}) * p(\omega_z) * p(\omega_{pz})$$

$$\omega \sim p(\omega)d\omega = \frac{\hbar^3\omega^2}{2(k_B T)^3} e^{\frac{-\hbar\omega}{k_B T}} d\omega$$

To find the average energy density, integrate over frequency and divide by the volume of space that the particle and it's pair could possibly fall within. The volumes should be the space a photon can traverse in time τ in either direction, i.e. $(2c\tau)^3$

$$\rho = \frac{\int_0^\infty \frac{\hbar\omega}{c^2} p(\omega)d\omega}{V} = \frac{\int_0^\infty \frac{\hbar\omega}{c^2} \frac{\hbar^3\omega^2}{2(k_B T)^3} e^{\frac{-\hbar\omega}{k_B T}} d\omega}{c^2(2c\tau)^3} = \frac{3(k_B T)^4}{\hbar^3 c^5}$$

APPENDIX B – SOLUTIONS

1. Equation of state $w = 1/3$

a. Quantum Mechanical Solution

Solving for $\ell(t)$ begins with the Heisenberg uncertainty relation [Shankar 1994].

$$\Delta p \Delta x = m\Delta(\dot{x})_{Linear}(t)\Delta x_{Linear}(t) = \frac{\hbar}{2}$$

Solving the differential equation gives the known result of quantum diffusion as studied in [Belavkin 2005, Nelson 1966].

$$\int 2\Delta x_{Linear}(t) d(\Delta x_{Linear}(t)) = \int \frac{\hbar}{m} dt$$

$$\Delta x_{linear}(t) = \left(\frac{\hbar}{m} t\right)^{\frac{1}{2}}$$

Leading to the quantum mechanical length scale,

$$\ell(t) = \sqrt{2}\Delta x_{linear}(t) = \left(\frac{2\hbar}{m} t\right)^{\frac{1}{2}}$$

This diffusion is also called imaginary diffusion as it can be derived by taking the quantum mechanical kinetic energy Hamiltonian and making a Minkowski transformation to imaginary time.

$$H = \frac{p^2}{2m}$$

$$i\hbar \frac{\partial}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

$$it \to t$$

$$\frac{\partial}{\partial t} = \frac{\hbar}{2m} \frac{\partial^2}{\partial x^2}$$

$$D = \frac{\hbar}{2m}$$

D is also be derived from Einstein's kinetic theory [Kubo 1966].

$$D = \mu k_B T = \frac{v}{F} k_B T = \frac{v}{m(v/\tau)} k_B T = \frac{\hbar}{2m}$$

Plugging into the variance we have

$$(\Delta x)^2_{Linear} = 2Dt = \frac{\hbar}{m} \cdot t$$

b. General Relativistic Solution

Solving for L when the temperature is variable and inversely proportional to the length scale we can write the density as,

$$\rho(L) = \frac{3\hbar}{c(L(t))^4}$$

Simple calculus solves the Freidmann equation when the density is dominated by this equation of state, w = 1/3.

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho(L) = \frac{8\pi G\hbar}{c(L(t))^4}$$

Solving for $L(t)$

$$L(t) = \left(\frac{32\pi G\hbar}{c}\right)^{1/4}\sqrt{t}$$

The above is the general relativistic solution for the scale factor of one particle pair.

Equating the general relativistic length scale, $L(t)$, to the quantum mechanical length scale, $\ell(t)$, solves for the square of the mass of the particle.

$$m^2 = \frac{\hbar c}{8\pi G}$$

The obvious solution is the positive reduced Planck mass. This solution is interesting as a fundamental particle with this mass simplifies many of the expressions in general relativity [Einstein 1956].

2. Equation of state $w = -1$

a. Quantum Mechanical Solution

In this regime the temperature is constant and we can write

$$\tau_0 = \frac{\hbar}{2k_B T_0}$$

In this case we have,

$$\ell(t) = \left(\frac{2\hbar}{m}\tau_0\right)^{\frac{1}{2}} f(t)$$

And

$$(\dot{\ell(t)}) = \frac{1}{2}\left(\frac{2\hbar}{m\tau_0}\right)^{\frac{1}{2}} f(t)$$

Where $f(t)$ is a function to be determined. Thus we have

$$\ell(t) = 2\tau_0(\dot{\ell(t)})$$

Solving the differential equation we have,

$$\ell(t) = \left(\frac{2\hbar}{m}\tau_0\right)^{\frac{1}{2}} e^{\frac{t}{2\tau_0}} = \frac{\hbar}{\sqrt{mk_B T_0}} e^{\frac{k_B T_0}{\hbar}t}$$

b. General Relativistic Solution

To solve the dual solution to Friedmann's equation we need to solve for the value of the constant density. We do so by directly inserting L_0 which can be evaluated by taking the solution of L from the $w = \frac{1}{3}$ solution evaluated at τ_0.

$$L_0 = L_{w=\frac{1}{3}}(\tau_0) = \frac{\hbar}{\sqrt{mk_B T_0}}$$

$$\rho(L = L_0) = \frac{3\hbar}{c(L_0)^4}$$

With this replacement and substituting in the reduced Planck mass, the Friedmann equation becomes

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho = \frac{(k_B T_0)^2}{\hbar^2}$$

Solving for $L(t)$ and inserting L_0 at $t = 0$ yields,

$$L(t) = \frac{\hbar}{\sqrt{mk_B T_0}} e^{\frac{k_B T_0}{\hbar}t}$$

Which is equal to $\ell(t)$.

3. Equation of state $w = -\frac{1}{3}$

a. Quantum Mechanical Solution

The quantum diffusion described above in the first term has a variance that is linear in time; A second type of quantum diffusion occurs due to multiple frequencies evolving at different rates. While this derivation will follow the method outlined by Bohm [1951], the precise result is found in Shankar 1994].

$$\Psi(p_x) = \frac{1}{\sqrt{\sqrt{2\pi\Delta p_x{}^2}}} \cdot e^{-\frac{p_x{}^2}{4\Delta p_x{}^2}}$$

$$\psi(x) = \int_{-\infty}^{\infty} e^{i(p_x x - E(p_x)t)/\hbar}\Psi(p_x)\, dp_x$$

Where

$$E(p_x) = \frac{p_x{}^2}{2m}$$

Solving the integral and taking the magnitude squared results in a probability distribution with a variance that is grows in quadratic time.

$$\psi^*(x)\psi(x) = \frac{1}{\sqrt{2\pi\left(\Delta x^2+\frac{k_B T_0}{m}t^2\right)}} \cdot e^{-\frac{x^2}{2\left(\Delta x^2+\frac{k_B T_0}{m}t^2\right)}}dx$$

Considering only the quadratic term $\Delta x^2_{Quadratic}$ we have

$$\Delta x^2_{Quadratic}(t) = \frac{k_B T_0}{m}t^2$$

b. General Relativistic Solution

Using a variation of the derivation of Friedmann's equation as given by Liddle [2003], we equate the average gravitational potential energy of a sphere with radius r to the three dimensional potential energy of the harmonic oscillator with $k_B T_0/2$ per degree of freedom hypothesized in appendix A

$$\overline{PE_{gravity}} = \overline{\frac{-GMm}{r}} = \overline{PE_{Hooks\ law}} = \frac{3k_B T_0}{2}$$

When $M = 4\pi r^3 \rho_{curve}/3$

$$\overline{\frac{-GMm}{r}} = \frac{-4\pi G\rho_{curve} m\overline{r^2}}{3}$$

We can re-write $\overline{r^2}$ in terms of our length scale where $L^2 = 2\overline{x^2}$

$$\overline{r^2} = \overline{x^2} + \overline{y^2} + \overline{z^2} = 3\overline{x^2} = \frac{3L^2}{2}$$

If the temperature is kept constant at T_0 we can re-write this in the form of the Freidmann equation.

$$\left(\frac{(\dot{L})}{L}\right)^2 = \frac{8\pi G}{3}\rho_{curve} = -\frac{2k_B T_0}{m}\left(\frac{1}{L}\right)^2$$

It is not yet clear how to handle the negative sign (however later when ρ_{curve} is added to the other densities it all balances out and the solution is real).

$$L = \sqrt{\frac{-2k_B T_0}{m}}\,t$$

Looking at the absolute value we have as we should,

$$|L^2(t)| = |\ell^2(t)| = 2\left|\Delta x^2_{Quadratic}(t)\right|$$

This regime is interesting as it provides the explanation for the hypothesized spring force.

$$F = -\frac{d}{dr}PE(r) = -\frac{d}{dr}\frac{-4\pi G\rho_{curve} m r^2}{3}$$
$$= \frac{8\pi G\rho_{curve}}{3}mr = -\frac{2k_B T_0}{m}\left(\frac{1}{L}\right)^2 mr$$

Plugging in for the magnitude of $|L^2(t)|$ evaluated at $t = \tau = \hbar/2k_B T$ we derive F_{spring} from appendix A and thus validate our 1st Ansatz.

$$F_{spring} = -\frac{m}{\tau^2}r$$

The thermal energy curves the space around the particle which provides the resistive spring force which places the "dark" particle in the ground (vacuum) state.

4. Holistic Solution

a. Langevin's Statistical Mechanics Solution

The Langevan equation begins with the force equation with both a resistive force that opposes and is proportional to the velocity and a noise driving force [Kubo 1966].

$$m(\ddot{x}) = -\frac{m}{\tau}(\dot{x}) + F_{Noise}$$

Einstein's relation can be used to show that $\tau = \hbar/2k_B T$ as we defined above [Kubo 1966]. Multiplying by x and using the product rule of differentiation yields,

$$mx(\ddot{x}) = m\frac{d}{dt}\left(x(\dot{x})\right) - m\left((\dot{x})\right)^2 = -\frac{m}{\tau}x(\dot{x}) + xF_{Noise}$$

With the substitution of the spring force,

$$F_{Noise} = F_{spring} = \frac{-mx}{(\tau)^2}$$

$$m\frac{d}{dt}\left(x(\dot{x})\right) = -\frac{m}{\tau}x(\dot{x}) + m(\dot{x})^2 - \frac{mx^2}{(\tau)^2}$$

Taking the time average and using the equipartition theorem [Feynman 1965] on the kinetic and potential energy leaves,

$$m\frac{d}{dt}\left(\overline{x(\dot{x})}\right) = -\frac{m}{\tau}\overline{x(\dot{x})} + k_BT - k_BT$$

$$\overline{x(\dot{x})} = Ce^{-t/\tau}$$

Setting the initial condition to obey the Heisenberg Uncertainty equation,

$$\overline{x(\dot{x})}(t=0) = \frac{\hbar}{2m}$$

$$\overline{x(\dot{x})} = \frac{\hbar}{2m}e^{-t/\tau}$$

From here the rest of the derivation is the same as the classical Langevin derivation

$$\frac{d}{dt}\left(\overline{x^2}\right) = 2\overline{x(\dot{x})} = \frac{\hbar}{m}e^{-t/\tau}$$

$$\overline{x^2} = \Delta x^2_{Langevin} = \int_0^t \frac{\hbar}{m}e^{-t'/\tau}dt = \frac{\hbar}{m}\tau\left(1 - e^{-t/\tau}\right)$$
$$= 2D\tau\left(1 - e^{-t/\tau}\right)$$

b. General Relativistic Solution

Solution provided in main body.

APPENDIX C – BLACK BODY RADIATION

The energy density in appendix A was related to fermions. In the fermions' case, there can be only one particle per quantum state. However radiation can have more than one particle per quantum state. The energy will be radiated by a photon of energy $\hbar\omega$; since there can actually be M photons per mode (as a photon is a boson) the total energy is $M\hbar\omega$. Again enlisting the help of the equipartition theorem, we solve for M [Feynman 1965].

$$Energy = M \cdot \hbar\omega$$
$$= \frac{m}{2}\left(\frac{x}{\tau}\right)^2 + \frac{1}{2m}(p_x)^2 + \frac{m}{2}\left(\frac{y}{\tau}\right)^2$$
$$+ \frac{1}{2m}\left(p_y\right)^2 + \frac{m}{2}\left(\frac{z}{\tau}\right)^2 + \frac{1}{2m}(p_z)^2$$

$$M \cdot \overline{\hbar\omega} = \frac{6 \cdot k_BT}{2}$$

$$M \cdot \overline{\frac{m}{2}\left(\frac{x}{\tau}\right)^2} = M \cdot \overline{\frac{1}{2m}(p_x)^2} = \frac{k_BT}{2}$$

$$(\Delta p_x)^2 = \frac{m \cdot k_BT}{M}$$

$$(\Delta x)^2 = \frac{M\hbar^2}{4m \cdot k_BT}$$

$$\tau = \frac{M\hbar}{2k_BT}$$

Plugging these into our equation for the probability the energy is ω, $p_{radiation}(\omega)d\omega$

$$p_{radiation}(\omega)d\omega = \frac{M^3\hbar^3\omega^2}{2(k_BT)^3}e^{\frac{-M\hbar\omega}{k_BT}}d\omega$$

Next to find the density we need Poynting's theorem. The energy $\hbar\omega$ built up by the particle's diffusion is transferred to the photon field every τ seconds. As the radiation moves away from the particle, general relativistic effects red-shifts the energy and dilates the cycle duration. Using Poynting's theorem as a form of conservation of energy [Bittencourt 1995], the delta in radiated power between when the photon is emitted in the high gravitational fields near the particle and the radiation power a distance $c\tau$ away from the particle is found. This delta can be thought of as the energy density trapped by the gravitational fields.

When there is no current we have,

$$P = \frac{d}{dt}E = -S \cdot A$$

In figure 4 below, the inner circle has a radius equal the particle's spatial step as defined by Heisenberg uncertainty principle.

$$dx = cdt = \frac{\hbar}{2mc}$$

When the particle radiates the power flowing through this hypothetical inner circle is equal to the energy of the photon divided by the cycle time of the diffusion process, $\hbar\omega/\tau$. The outer circle has a radius equal to the extent of a photon traveling for τ seconds which if $mc^2 \gg k_BT$ is beyond the range of the particle's general relativistic effects.

The power flux at this point in space is reduced by the factor $(1 - r_s/dx)/(1 - r_s/c\tau) \cong (1 - r_s/dx)$, where r_s is the Schwarzchild radius. This reduction in power flux is due to the energy being gravitationally red-shifted by the amount $\sqrt{1 - r_s/dx}/\sqrt{1 - r_s/c\tau}$, and the time being dilated by the factor, $\sqrt{1-r_s/c\tau}/\sqrt{1-r_s/dx}$ [Schutz 1985]. If $mc^2 \gg k_B T$

$$S_1 \cdot A_1 = \frac{\hbar\omega}{\tau}$$

$$S_2 \cdot A_2 = \frac{\hbar\omega}{\tau}(1 - r_s/dx)$$

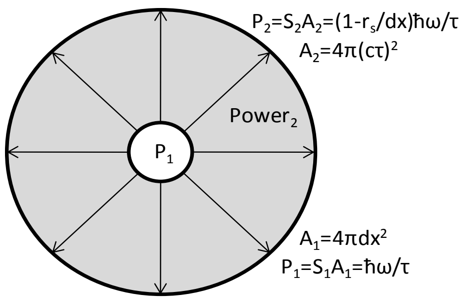

Figure 3 Poynting Vector between inner cutoff and distant outer boundary

The power between the inner radius and the outer radius (the grey area in figure 4 above) is

$$P_2 - P_1 = S_1 \cdot A_1 - S_2 \cdot A_2 = \frac{\hbar\omega}{\tau}\frac{r_s}{dx} = \frac{\hbar\omega}{2\pi\tau}$$

$P_2 - P_1$ divided by the area of the outer radius, represents the power flux. Thus $(P_2 - P_1)/A_2$ divided by the speed of light and multiplied by the probability distribution on the radiated energy is the average energy density of the radiation field gravitationally trapped by the mass of the particle.

$$\rho_{radiation-M}(\omega)d\omega = \frac{\hbar\omega}{2\pi\tau}\frac{p_{radiation}(\omega)d\omega}{c \cdot 4\pi(c\tau)^2}$$

$$= \frac{\hbar\omega^3}{2\pi^2 c^3} e^{\frac{-M\hbar\omega}{k_B T}} d\omega$$

Lastly sum over all states M from 1 to ∞, and both degrees of polarization [8] since all are possible.

$$\rho_{radiation}(\omega)d\omega = \sum_{pol=1,-1} \sum_{M=1}^{\infty} \rho_{radiation-M}(\omega)d\omega$$

$$= \frac{\hbar\omega^3 d\omega}{\pi^2 c^3} \sum_{M=1}^{\infty} e^{\frac{-M\hbar\omega}{k_B T}} = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\left(e^{\hbar\omega/kT} - 1\right)} d\omega$$

One will recognize the energy density of Black Body radiation, and of course integrating over ω gives the total average energy density [Reif 1965]

$$\rho_{radiation} = \int_0^{\infty} \rho_{radiation}(\omega)d\omega = \frac{\pi^2 (k_B T)^4}{15\hbar^3 c^3}$$

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