

# Polyvector-valued Gauge Field Theories and Quantum Mechanics in Noncommutative Clifford Spaces

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## Abstract

The basic ideas and results behind polyvector-valued gauge field theories and Quantum Mechanics in Noncommutative Clifford spaces are presented. The star products are noncommutative but associative up to *second* order only. The construction of Noncommutative Clifford-space gravity as polyvector-valued gauge theories of twisted diffeomorphisms in Clifford-spaces would require quantum Hopf algebraic deformations of Clifford algebras.

Clifford algebras are deeply related and essential tools in many aspects in Physics. The Extended Relativity theory in Clifford-spaces ( *C*-spaces ) is a natural extension of the ordinary Relativity theory [3] whose generalized polyvector-valued coordinates are Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane, p-brane,... dynamics of p-loops (closed p-branes) in *D*-dimensional target spacetime backgrounds.

It was recently shown [1] how an unification of Conformal Gravity and a  $U(4) \times U(4)$  Yang-Mills theory in four dimensions could be attained from a Clifford Gauge Field Theory in *C*-spaces (Clifford spaces) based on the (complex) Clifford  $Cl(4, C)$  algebra underlying a complexified four dimensional spacetime (8 real dimensions). Clifford-space tensorial-gauge fields generalizations of Yang-Mills theories allows to predict the existence of new particles (bosons, fermions) and tensor-gauge fields of higher-spins in the 10 TeV regime [2]. Tensorial Generalized Yang-Mills in *C*-spaces (Clifford spaces) based on poly-vector valued (anti-symmetric tensor fields) gauge fields  $\mathcal{A}_M(\mathbf{X})$  and field strengths  $\mathcal{F}_{MN}(\mathbf{X})$  have been studied in [2], [3] where  $\mathbf{X} = X_M \Gamma^M$  is a *C*-space poly-vector valued coordinate

$$\mathbf{X} = \sigma \mathbf{1} + x_\mu \gamma^\mu + x_{\mu_1\mu_2} \gamma^{\mu_1} \wedge \gamma^{\mu_2} + x_{\mu_1\mu_2\mu_3} \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} + \dots +$$

$$x_{\mu_1\mu_2\mu_3\dots\mu_d} \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} \dots \wedge \gamma^{\mu_d} \quad (1)$$

In order to match dimensions in each term of (1) a length scale parameter must be suitably introduced. In [3] we introduced the Planck scale as the expansion parameter in (1). The scalar component  $\sigma$  of the  $C$ -space poly-vector valued coordinate  $\mathbf{X}$  was interpreted by [4] as a Stuckelberg time-like parameter that solves the problem of time in Cosmology in a very elegant fashion.

A Clifford gauge field theory in the  $C$ -space associated with the ordinary  $4D$  spacetime requires  $\mathcal{A}_M(\mathbf{X}) = \mathcal{A}_M^A(\mathbf{X}) \Gamma_A$  that is a poly-vector valued gauge field where  $M$  represents the poly-vector index associated with the  $C$ -space, and whose gauge group  $\mathcal{G}$  is itself based on the Clifford algebra  $Cl(3,1)$  of the tangent space spanned by 16 generators  $\Gamma_A$ . The expansion of the poly-vector Clifford-algebra-valued gauge field  $\mathcal{A}_M^A$ , for *fixed* values of  $A$ , is of the form

$$\mathcal{A}_M^A \Gamma^M = \Phi^A + \mathcal{A}_\mu^A \gamma^\mu + \mathcal{A}_{\mu_1\mu_2}^A \gamma^{\mu_1} \wedge \gamma^{\mu_2} + \mathcal{A}_{\mu_1\mu_2\mu_3}^A \gamma^{\mu_1} \wedge \gamma^{\mu_2} \wedge \gamma^{\mu_3} + \dots \quad (2)$$

The index  $A$  spans the 16-dim Clifford algebra  $Cl(3,1)$  of the tangent space such as

$$\Phi^A \Gamma_A = \Phi + \Phi^a \Gamma_a + \Phi^{ab} \Gamma_{ab} + \Phi^{abc} \Gamma_{abc} + \Phi^{abcd} \Gamma_{abcd}. \quad (3a)$$

$$\mathcal{A}_\mu^A \Gamma_A = \mathcal{A}_\mu + \mathcal{A}_\mu^a \Gamma_a + \mathcal{A}_\mu^{ab} \Gamma_{ab} + \mathcal{A}_\mu^{abc} \Gamma_{abc} + \mathcal{A}_\mu^{abcd} \Gamma_{abcd}. \quad (3b)$$

$$\mathcal{A}_{\mu\nu}^A \Gamma_A = \mathcal{A}_{\mu\nu} + \mathcal{A}_{\mu\nu}^a \Gamma_a + \mathcal{A}_{\mu\nu}^{ab} \Gamma_{ab} + \mathcal{A}_{\mu\nu}^{abc} \Gamma_{abc} + \mathcal{A}_{\mu\nu}^{abcd} \Gamma_{abcd}. \quad (3c)$$

etc.....

In order to match dimensions in each term of (2) another length scale parameter must be suitably introduced. For example, since  $\mathcal{A}_{\mu\nu\rho}^A$  has dimensions of  $(length)^{-3}$  and  $\mathcal{A}_\mu^A$  has dimensions of  $(length)^{-1}$  one needs to introduce another length parameter in order to match dimensions. This length parameter does not need to coincide with the Planck scale. The Clifford-algebra-valued gauge field  $\mathcal{A}_\mu^A(x^\mu)\Gamma_A$  in ordinary spacetime is naturally embedded into a far richer object  $\mathcal{A}_M^A(\mathbf{X})\Gamma_A$  in  $C$ -spaces. The advantage of recurring to  $C$ -spaces associated with the  $4D$  spacetime manifold is that one can have a (complex) Conformal Gravity, Maxwell and  $U(4) \times U(4)$  Yang-Mills unification in a very geometric fashion as provided by [1]

Field theories in Noncommutative spacetimes have been the subject of intense investigation in recent years, see [8] and references therein. Star Product deformations of Clifford Gauge Field Theories based on ordinary Noncommutative spacetimes are straightforward generalizations of the work by [5]. The wedge star product of two Clifford-valued one-forms is defined as

$$\mathbf{A} \wedge_* \mathbf{A} = ( (\mathcal{A}_\mu^A * \mathcal{A}_\nu^B) \Gamma_A \Gamma_B ) dx^\mu \wedge dx^\nu =$$

$$\frac{1}{2} \left( (\mathcal{A}_\mu^A *_s \mathcal{A}_\nu^B) [\Gamma_A, \Gamma_B] + (\mathcal{A}_\mu^A *_a \mathcal{A}_\nu^B) \{\Gamma_A, \Gamma_B\} \right) dx^\mu \wedge dx^\nu. \quad (4)$$

In the case when the coordinates don't commute  $[x^\mu, x^\nu] = \theta^{\mu\nu}$  (constants), the cosine (symmetric) star product is defined by [5]

$$f *_s g \equiv \frac{1}{2} (f * g + g * f) = f g + \left( \frac{i}{2} \right)^2 \theta^{\mu\nu} \theta^{\kappa\lambda} (\partial_\mu \partial_\kappa f) (\partial_\nu \partial_\lambda g) + O(\theta^4). \quad (5)$$

and the sine (anti-symmetric Moyal bracket) star product is

$$f *_a g \equiv \frac{1}{2} (f * g - g * f) = \left( \frac{i}{2} \right) \theta^{\mu\nu} (\partial_\mu f) (\partial_\nu g) + \left( \frac{i}{2} \right)^3 \theta^{\mu\nu} \theta^{\kappa\lambda} \theta^{\alpha\beta} (\partial_\mu \partial_\kappa \partial_\alpha f) (\partial_\nu \partial_\lambda \partial_\beta g) + O(\theta^5). \quad (6)$$

Notice that both commutators *and* anticommutators of the gammas appear in the star deformed products in (4). The star product deformations of the gauge field strengths in the case of the  $U(2, 2)$  gauge group were given by [5] and the expressions for the star product deformed action are very cumbersome .

In this letter we proceed with the construction of Polyvector-valued Gauge Field Theories in *noncommutative* Clifford Spaces (  $C$ -spaces ) which are polyvector-valued *extensions* and *generalizations* of the ordinary *noncommutative* spacetimes. We begin firstly by writing the commutators  $[\Gamma_A, \Gamma_B]$ . For  $pq = \text{odd}$  one has [7]

$$\begin{aligned} & [ \gamma_{b_1 b_2 \dots b_p}, \gamma^{a_1 a_2 \dots a_q} ] = 2 \gamma_{b_1 b_2 \dots b_p}^{a_1 a_2 \dots a_q} - \\ & \frac{2p!q!}{2!(p-2)!(q-2)!} \delta_{[b_1 b_2}^{[a_1 a_2} \gamma_{b_3 \dots b_p]}^{a_3 \dots a_q]} + \frac{2p!q!}{4!(p-4)!(q-4)!} \delta_{[b_1 \dots b_4}^{[a_1 \dots a_4} \gamma_{b_5 \dots b_p]}^{a_5 \dots a_q]} - \dots \end{aligned} \quad (7)$$

for  $pq = \text{even}$  one has

$$\begin{aligned} & [ \gamma_{b_1 b_2 \dots b_p}, \gamma^{a_1 a_2 \dots a_q} ] = - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta_{[b_1}^{[a_1} \gamma_{b_2 b_3 \dots b_p]}^{a_2 a_3 \dots a_q]} - \\ & \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta_{[b_1 \dots b_3}^{[a_1 \dots a_3} \gamma_{b_4 \dots b_p]}^{a_4 \dots a_q]} + \dots \end{aligned} \quad (8)$$

The anti-commutators for  $pq = \text{even}$  are

$$\begin{aligned} & \{ \gamma_{b_1 b_2 \dots b_p}, \gamma^{a_1 a_2 \dots a_q} \} = 2 \gamma_{b_1 b_2 \dots b_p}^{a_1 a_2 \dots a_q} - \\ & \frac{2p!q!}{2!(p-2)!(q-2)!} \delta_{[b_1 b_2}^{[a_1 a_2} \gamma_{b_3 \dots b_p]}^{a_3 \dots a_q]} + \frac{2p!q!}{4!(p-4)!(q-4)!} \delta_{[b_1 \dots b_4}^{[a_1 \dots a_4} \gamma_{b_5 \dots b_p]}^{a_5 \dots a_q]} - \dots \end{aligned} \quad (9)$$

and the anti-commutators for  $pq = \text{odd}$  are

$$\{ \gamma_{b_1 b_2 \dots b_p}, \gamma^{a_1 a_2 \dots a_q} \} = - \frac{(-1)^{p-1} 2p!q!}{1!(p-1)!(q-1)!} \delta_{[b_1}^{[a_1} \gamma_{b_2 b_3 \dots b_p]}^{a_2 a_3 \dots a_q]} - \frac{(-1)^{p-1} 2p!q!}{3!(p-3)!(q-3)!} \delta_{[b_1 \dots b_3}^{[a_1 \dots a_3} \gamma_{b_4 \dots b_p]}^{a_4 \dots a_q]} + \dots \quad (10)$$

For instance,

$$\mathcal{J}_b^a = [\gamma_b, \gamma^a] = 2\gamma_b^a; \quad \mathcal{J}_{b_1 b_2}^{a_1 a_2} = [\gamma_{b_1 b_2}, \gamma^{a_1 a_2}] = -8 \delta_{[b_1}^{[a_1} \gamma_{b_2]}^{a_2]}. \quad (11)$$

$$\mathcal{J}_{b_1 b_2 b_3}^{a_1 a_2 a_3} = [\gamma_{b_1 b_2 b_3}, \gamma^{a_1 a_2 a_3}] = 2 \gamma_{b_1 b_2 b_3}^{a_1 a_2 a_3} - 36 \delta_{[b_1 b_2}^{[a_1 a_2} \gamma_{b_3]}^{a_3]}. \quad (12)$$

$$\mathcal{J}_{b_1 b_2 b_3 b_4}^{a_1 a_2 a_3 a_4} = [\gamma_{b_1 b_2 b_3 b_4}, \gamma^{a_1 a_2 a_3 a_4}] = -32 \delta_{[b_1}^{[a_1} \gamma_{b_2 b_3 b_4]}^{a_2 a_3 a_4]} + 192 \delta_{[b_1 b_2 b_3}^{[a_1 a_2 a_3} \gamma_{b_4]}^{a_4]}. \quad (13)$$

etc...

The second step is to write down the *noncommutative* algebra associated with the noncommuting poly-vector-valued coordinates in  $D = 4$  and which can be obtained from the Clifford algebra (7-10) by performing the following replacements (and relabeling indices)

$$\gamma^\mu \leftrightarrow X^\mu, \quad \gamma^{\mu_1 \mu_2} \leftrightarrow X^{\mu_1 \mu_2}, \quad \dots \dots \gamma^{\mu_1 \mu_2 \dots \mu_n} \leftrightarrow X^{\mu_1 \mu_2 \dots \mu_n}. \quad (14)$$

When the spacetime metric components  $g_{\mu\nu}$  are *constant*, from the replacements (14) and the Clifford algebra (7-10) (after one relabels indices), one can then construct the following *noncommutative* algebra among the poly-vector-valued coordinates in  $D = 4$ , and *obeying* the Jacobi identities, given by the relations

$$[X^{\mu_1}, X^{\mu_2}]_* = X^{\mu_1} * X^{\mu_2} - X^{\mu_2} * X^{\mu_1} = 2 X^{\mu_1 \mu_2}. \quad (15)$$

$$[X^{\mu_1 \mu_2}, X^\nu]_* = 4 (g^{\mu_2 \nu} X^{\mu_1} - g^{\mu_1 \nu} X^{\mu_2}). \quad (16)$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^\nu]_* = 2 X^{\mu_1 \mu_2 \mu_3 \nu}, \quad [X^{\mu_1 \mu_2 \mu_3 \mu_4}, X^\nu]_* = -8 g^{\mu_1 \nu} X^{\mu_2 \mu_3 \mu_4} \pm \dots \quad (17)$$

$$[X^{\mu_1 \mu_2}, X^{\nu_1 \nu_2}]_* = -8 g^{\mu_1 \nu_1} X^{\mu_2 \nu_2} + 8 g^{\mu_1 \nu_2} X^{\mu_2 \nu_1} + 8 g^{\mu_2 \nu_1} X^{\mu_1 \nu_2} - 8 g^{\mu_2 \nu_2} X^{\mu_1 \nu_1}. \quad (18)$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^{\nu_1 \nu_2}]_* = 12 g^{\mu_1 \nu_1} X^{\mu_2 \mu_3 \nu_2} \pm \dots \quad (19)$$

$$[X^{\mu_1 \mu_2 \mu_3}, X^{\nu_1 \nu_2 \nu_3}]_* = -36 G^{\mu_1 \mu_2 \nu_1 \nu_2} X^{\mu_3 \nu_3} \pm \dots \quad (20)$$

$$[ X^{\mu_1\mu_2\mu_3\mu_4}, X^{\nu_1\nu_2} ]_* = -16 g^{\mu_1\nu_1} X^{\mu_2\mu_3\mu_4\nu_2} \pm \dots \quad (21)$$

$$[ X^{\mu_1\mu_2\mu_3\mu_4}, X^{\nu_1\nu_2} ]_* = -16 g^{\mu_1\nu_1} X^{\mu_2\mu_3\mu_4\nu_2} + 16 g^{\mu_1\nu_2} X^{\mu_2\mu_3\mu_4\nu_1} - \dots \quad (22)$$

$$[ X^{\mu_1\mu_2\mu_3\mu_4}, X^{\nu_1\nu_2\nu_3} ]_* = 48 G^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} X^{\mu_4} - 48 G^{\mu_1\mu_2\mu_4\nu_1\nu_2\nu_3} X^{\mu_3} + \dots \quad (23)$$

$$[ X^{\mu_1\mu_2\mu_3\mu_4}, X^{\nu_1\nu_2\nu_3\nu_4} ]_* = 192 G^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} X^{\mu_4\nu_4} - \dots \quad (24)$$

etc..... where

$$G^{\mu_1\mu_2\dots\mu_n\nu_1\nu_2\dots\nu_n} = g^{\mu_1\nu_1} g^{\mu_2\nu_2} \dots g^{\mu_n\nu_n} + \text{signed permutations} \quad (25)$$

The metric components  $G^{\mu_1\mu_2\dots\mu_n\nu_1\nu_2\dots\nu_n}$  in  $C$ -space can also be written as a determinant of the  $n \times n$  matrix  $\mathbf{G}$  whose entries are  $g^{\mu_i\nu_j}$

$$\det \mathbf{G}_{n \times n} = \frac{1}{n!} \epsilon_{i_1 i_2 \dots i_n} \epsilon_{j_1 j_2 \dots j_n} g^{\mu_{i_1} \nu_{j_1}} g^{\mu_{i_2} \nu_{j_2}} \dots g^{\mu_{i_n} \nu_{j_n}}. \quad (26)$$

$i_1, i_2, \dots, i_n \subset I = 1, 2, \dots, D$  and  $j_1, j_2, \dots, j_n \subset J = 1, 2, \dots, D$ . One must also include in the  $C$ -space metric  $G^{MN}$  the (Clifford) scalar-scalar component  $G^{00}$  (that could be related to the dilaton field) and the pseudo-scalar/pseudo-scalar component  $G^{\mu_1\mu_2\dots\mu_D\nu_1\nu_2\dots\nu_D}$  (that could be related to the axion field).

One must emphasize that when the spacetime metric components  $g_{\mu\nu}$  are *no longer constant*, the noncommutative algebra among the poly-vector-valued coordinates in  $D = 4$ , does *not* longer *obey* the Jacobi identities. For this reason we restrict our construction to a flat spacetime background  $g_{\mu\nu} = \eta_{\mu\nu}$ .

The noncommutative conditions on the polyvector coordinates in condensed notation can be written as

$$[ X^M, X^N ]_* = X^M *_X X^N - X^N *_X X^M = \Omega^{MN}(X) = f^{MN}_L X^L = f^{MNL} X_L \quad (27)$$

the structure constants  $f^{MNL}$  are antisymmetric under the exchange of polyvector valued indices. An immediate consequence of the noncommutativity of coordinates is

$$[ \hat{X}^{\mu_1}, \hat{X}^{\mu_2} ] = 2 \hat{X}^{\mu_1\mu_2} \Rightarrow \Delta X^\mu \Delta X^\nu \geq \frac{1}{2} | \langle \hat{X}^{\mu\nu} \rangle | = X^{\mu\nu} \quad (28)$$

Hence, the bivector area coordinates  $X^{\mu\nu}$  in  $C$ -space can be seen as a measure of the noncommutative nature of the "quantized" spacetime coordinates  $\hat{X}^\mu$ .

The third step is to define the noncommutative star product of functions of  $X$  as

$$(A_1 * A_2)(Z) = \exp\left(\frac{1}{2} \Omega^{MN} \partial_{X^M} \partial_{Y^N}\right) A_1(X) A_2(Y)|_{X=Y=Z} =$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} \Omega^{M_1 N_1} \Omega^{M_2 N_2} \dots \Omega^{M_n N_n} (\partial_{M_1 M_2 \dots M_n}^n A_1) (\partial_{N_1 N_2 \dots N_n}^n A_2) + \dots \quad (29)$$

where the ellipsis in (29) are the terms involving derivatives acting on  $\Omega^{MN}$  and

$$\partial_{M_1 M_2 \dots M_n}^n A_1(Z) \equiv \partial_{M_1} \partial_{M_2} \dots \partial_{M_n} A_1(Z). \quad (30a)$$

$$\partial_{N_1 N_2 \dots N_n}^n A_2(Z) \equiv \partial_{N_1} \partial_{N_2} \dots \partial_{N_n} A_2(Z). \quad (30b)$$

Derivatives on  $\Omega^{mn}$  appear in the ordinary Moyal star product when  $\Omega^{mn}$  depends on the phase space coordinates. For instance, the Moyal star product when the symplectic structure  $\Omega^{mn}(\vec{q}, \vec{p})$  is *not* constant is given by

$$A * B = A \exp\left(\frac{i\hbar}{2} \Omega^{mn} \overleftarrow{\partial}_m \overrightarrow{\partial}_n\right) B =$$

$$A B + i\hbar \Omega^{mn} (\partial_m A \partial_n B) + \frac{(i\hbar)^2}{2} \Omega^{m_1 n_1} \Omega^{m_2 n_2} (\partial_{m_1 m_2}^2 A) (\partial_{n_1 n_2}^2 B) +$$

$$\frac{(i\hbar)^2}{3} [\Omega^{m_1 n_1} (\partial_{n_1} \Omega^{m_2 n_2}) (\partial_{m_1} \partial_{m_2} A \partial_{n_2} B - \partial_{m_2} A \partial_{m_1} \partial_{n_2} B)] + O(\hbar^3). \quad (31)$$

Due to the derivative terms  $\partial_{n_1} \Omega^{m_2 n_2}$  the star product is associative up to second order only [6]  $(f * g) * h = f * (g * h) + O(\hbar^3)$ . Hence, due to the derivatives terms acting on  $\Omega^{MN}(X)$  in (29), the star product will *no* longer be associative beyond second order. When the star product is *associative* and noncommutative, with the fields and their derivatives vanishing fast enough at infinity, one has

$$\int A * B = \int A B + \text{total derivative} = \int A B. \quad (32a)$$

$$\int A * B * C = \int A (B * C) + \text{total derivative} = \int A (B * C) =$$

$$\int (B * C) A = \int (B * C) * A + \text{total derivative} = \int B * C * A \quad (32b)$$

therefore, when the star product is associative and the fields and their derivatives vanishing fast enough at infinity (or there are no boundaries) one has

$$\int A * B * C = \int B * C * A = \int C * A * B. \quad (32c)$$

The relations (32) are essential in order to construct invariant actions under star gauge transformations.

The  $C$ -space differential form associated with the polyvector-valued Clifford gauge field is

$$\mathbf{A} = \mathcal{A}_M dX^M = \Phi d\sigma + \mathcal{A}_\mu dx^\mu + \mathcal{A}_{\mu\nu} dx^{\mu\nu} + \dots + \mathcal{A}_{\mu_1\mu_2\dots\mu_d} dx^{\mu_1\mu_2\dots\mu_d} \quad (33a)$$

where  $\Phi = \Phi^A \Gamma_A$ ,  $\mathcal{A}_\mu = \mathcal{A}_\mu^A \Gamma_A$ ,  $\mathcal{A}_{\mu\nu} = \mathcal{A}_{\mu\nu}^A \Gamma_A$ , ..... The  $C$ -space differential form associated with the polyvector-valued field-strength is

$$\begin{aligned} \mathbf{F} = F_{MN} dX^M \wedge dX^N = & F_{0\ \mu} d\sigma \wedge dx^\mu + F_{0\ \mu_1\mu_2} d\sigma \wedge dx^{\mu_1\mu_2} + \dots \\ & F_{0\ \nu_1\nu_2\dots\nu_d} d\sigma \wedge dx^{\nu_1\nu_2\dots\nu_d} + F_{\mu\nu} dx^\mu \wedge dx^\nu + F_{\mu_1\mu_2\ \nu_1\nu_2} dx^{\mu_1\mu_2} \wedge dx^{\nu_1\nu_2} + \dots \\ & + F_{\mu_1\mu_2\dots\mu_{d-1}\ \nu_1\nu_2\dots\nu_{d-1}} dx^{\mu_1\mu_2\dots\mu_{d-1}} \wedge dx^{\nu_1\nu_2\dots\nu_{d-1}}. \end{aligned} \quad (33b)$$

The field strength is antisymmetric under the exchange of poly-vector indices  $F_{MN} = -F_{NM}$ . For this reason one has  $F_{00} = 0$  and  $F_{12\dots d\ 12\dots d} = 0$ . Finally, given the noncommutative conditions on the poly-vector coordinates (27), the components of the Clifford-algebra valued field strength  $F_{MN}^C \Gamma_C$  in *Noncommutative C-spaces* are

$$\begin{aligned} F_{[MN]} = F_{[MN]}^C \Gamma_C = & (\partial_M \mathcal{A}_N^C - \partial_N \mathcal{A}_M^C) \Gamma_C + \\ & \frac{1}{2} (\mathcal{A}_M^A * \mathcal{A}_N^B - \mathcal{A}_N^B * \mathcal{A}_M^A) \{ \Gamma_A, \Gamma_B \} + \frac{1}{2} (\mathcal{A}_M^A * \mathcal{A}_N^B + \mathcal{A}_N^B * \mathcal{A}_M^A) [ \Gamma_A, \Gamma_B ]. \end{aligned} \quad (34)$$

The commutators  $[ \Gamma_A, \Gamma_B ]$  and anti-commutators  $\{ \Gamma_A, \Gamma_B \}$  in eqs-(34), where  $A, B$  are polyvector-valued indices, can be read from the relations in eqs-(7-10). Notice that both the standard commutators *and* anticommutators of the gammas appear in the terms containing the star deformed products of (34) and which define the Clifford-algebra valued field strength in noncommutative  $C$ -spaces; i.e. if the products of fields were to commute one would have had only the Lie algebra commutator  $\mathcal{A}_M^A \mathcal{A}_N^B [ \Gamma_A, \Gamma_B ]$  pieces without the anti-commutator  $\{ \Gamma_A, \Gamma_B \}$  contributions in the r.h.s of eq-(34).

We should remark that one is *not* deforming the Clifford algebra involving  $[ \Gamma_A, \Gamma_B ]$  and  $\{ \Gamma_A, \Gamma_B \}$  in eq-(34) but it is the "point" product algebra  $\mathcal{A}_M^A * \mathcal{A}_N^B$  of the fields which is being deformed. (Quantum)  $q$ -Clifford algebras have been studied by [9]. The symmetrized star product is

$$\begin{aligned} \mathcal{A}_M^A *_s \mathcal{A}_N^B \equiv & \frac{1}{2} (\mathcal{A}_M^A * \mathcal{A}_N^B + \mathcal{A}_N^B * \mathcal{A}_M^A) = \mathcal{A}_M^A \mathcal{A}_N^B + \\ & X^{\mu\nu} X^{\kappa\lambda} (\partial_\mu \partial_\kappa \mathcal{A}_M^A) (\partial_\nu \partial_\lambda \mathcal{A}_N^B) + \dots \end{aligned} \quad (35)$$

the antisymmetrized (Moyal bracket) star product is

$$\mathcal{A}_M^A *_a \mathcal{A}_N^B \equiv \frac{1}{2} (\mathcal{A}_M^A * \mathcal{A}_N^B - \mathcal{A}_N^B * \mathcal{A}_M^A) = X^{\mu\nu} (\partial_\mu \mathcal{A}_M^A) (\partial_\nu \mathcal{A}_N^B) +$$

$$X^{\mu\nu} X^{\kappa\lambda} X^{\alpha\beta} (\partial_\mu \partial_\kappa \partial_\alpha \mathcal{A}_M^A) (\partial_\nu \partial_\lambda \partial_\beta \mathcal{A}_N^B) + \dots \quad (36)$$

It is important to emphasize, as it is customary in Moyal star products, that the poly-vector coordinates appearing in the r.h.s of eqs-(35-36) are treated as *c*-numbers (as if they were commuting) while it is the product of functions appearing in the l.h.s of (35-36) which are *noncommutative*.

Star products in noncommutative *C*-space lead to *many more terms* in eqs-(35-36) than in ordinary noncommutative spaces. For example, there are derivatives terms involving polyvectors which do *not* appear in ordinary non-commutative spaces, like

$$-4 g^{\mu_1\nu_1} X^{\mu_2\nu_2} \frac{\partial \mathcal{A}_M^A}{\partial X^{\mu_1\mu_2}} \frac{\partial \mathcal{A}_N^B}{\partial X^{\nu_1\nu_2}} \pm \dots \quad (37a)$$

$$2 (g^{\mu_2\nu} X^{\mu_1} - g^{\mu_1\nu} X^{\mu_2}) \frac{\partial \mathcal{A}_M^A}{\partial X^{\mu_1\mu_2}} \frac{\partial \mathcal{A}_N^B}{\partial X^\nu}. \quad (37b)$$

$$X^{\mu_1\mu_2\mu_3\nu} \frac{\partial \mathcal{A}_M^A}{\partial X^{\mu_1\mu_2\mu_3}} \frac{\partial \mathcal{A}_N^B}{\partial X^\nu}. \quad (37c)$$

$$96 G^{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3} X^{\mu_4\nu_4} \frac{\partial \mathcal{A}_M^A}{\partial X^{\mu_1\mu_2\mu_3\mu_4}} \frac{\partial \mathcal{A}_N^B}{\partial X^{\nu_1\nu_2\nu_3\nu_4}}, \quad \text{etc} \dots \quad (37d)$$

There is a *subalgebra* of the noncommutative polyvector-valued coordinates algebra (27) involving only  $X^\mu$  and the bivector coordinates  $X^{\mu\nu}$  when the space-time metric components  $g_{\mu\nu}$  are *constant*. However, because  $[X^{\mu_1\mu_2}, X^\nu] \neq 0$  one must not confuse the algebra in this case with the ordinary  $\Theta$ -noncommutative algebra  $[X^{\mu_1}, X^{\mu_2}] = \Theta^{\mu_1\mu_2}$  where the components of  $\Theta^{\mu_1\mu_2}$  are comprised of *constants* such that  $[\Theta^{\mu_1\mu_2}, X^\nu] = 0$ .

The analog of a Yang-Mills action in *C*-spaces when the background *C*-space metric  $G^{MN}$  is *X*-independent is given by

$$S = \int [DX] \langle F_{MN}^A \Gamma_A * F_{PQ}^B \Gamma_B \rangle G^{MP} G^{NQ}. \quad (38)$$

where  $\langle \Gamma_A \Gamma_B \rangle$  denotes the Clifford-scalar part of the Clifford geometric product of two generators. As mentioned in the introduction suitable powers of a length scale must be included in the expansion of the terms inside the integrand in order to have consistent dimensions (the action is dimensionless). The action (38) becomes

$$\int [DX] ( F_{MN} * F^{MN} + F_{MN}^a * F_a^{MN} + F_{MN}^{a_1 a_2} * F_{a_1 a_2}^{MN} + \dots + F_{MN}^{a_1 a_2 \dots a_d} * F_{a_1 a_2 \dots a_d}^{MN} ). \quad (39)$$

the measure in *C*-space is given by

$$DX = d\sigma \prod dx^\mu \prod dx^{\mu_1\mu_2} \prod dx^{\mu_1\mu_2\mu_3} \dots dx^{\mu_1\mu_2\dots\mu_d}. \quad (40a)$$

The Clifford-valued gauge field  $\mathcal{A}_M$  transforms under star gauge transformations according to  $\mathcal{A}'_M = U_*^{-1} * \mathcal{A}_M * U_* + U_*^{-1} * \partial_M U_*$ . The field strength  $F$  transforms covariantly  $F'_{MN} = U_*^{-1} * F_{MN} * U_*$  such that the action (39) is star gauge invariant.  $U_* = \exp_*(\xi(X)) = \exp_*(\xi^A(X)\Gamma_A)$  is defined via a star power series expansion  $U_* = \sum_n \frac{1}{n!} (\xi(X))^n_*$  where  $(\xi(X))^n_* = \xi(X) * \xi(X) * \dots * \xi(X)$ . The integral  $\int F * F = \int F F + \text{total derivatives}$ . If the fields vanish fast enough at infinity and/or there are no boundaries, the contribution of the total derivative terms are zero.

If, and only if, the star product were *associative* one would have had star gauge invariance of the action (39) under infinitesimal  $\delta F = [F, \xi]_*$  transformations

$$\delta_\xi S = \int \langle F * [F, \xi]_* \rangle = \int \langle F * F * \xi \rangle - \int \langle F * \xi * F \rangle. \quad (40b)$$

If the star product were associative due to the relations in eqs-(32) one could write eq-(40b) in the form

$$\begin{aligned} \int F^A * F^B * \xi^C \langle \Gamma_A \Gamma_B \Gamma_C \rangle - \int F^A * \xi^C * F^B \langle \Gamma_A \Gamma_C \Gamma_B \rangle = \\ \int F^B * \xi^C * F^A \langle \Gamma_B \Gamma_C \Gamma_A \rangle - \int F^A * \xi^C * F^B \langle \Gamma_A \Gamma_C \Gamma_B \rangle = 0 \end{aligned} \quad (40c)$$

such that in order to arrive at the zero result in (40c), assuring  $\delta S = 0$ , one needs to use the cyclic property of the scalar part of the geometric product

$$\langle \Gamma_A \Gamma_B \Gamma_C \rangle = \langle \Gamma_B \Gamma_C \Gamma_A \rangle = \langle \Gamma_C \Gamma_A \Gamma_B \rangle \quad (40d)$$

and *relabel* the indices  $B \leftrightarrow A$  in the third term of (40c). However, since the star product (29) is associative up to *second* order only, one has star gauge invariance of the action up to *second* order only. Nevertheless, one still has ordinary gauge invariance under infinitesimal transformations  $\delta F = [F, \xi]$  of the action because

$$S = \int \langle F * F \rangle = \int \langle F F \rangle + \text{total derivative} = \int \langle F F \rangle \quad (40e)$$

and the last integral is gauge invariant. In ordinary commutative  $C$ -spaces one can perform the mode expansion in integer powers of the poly-vector coordinates

$$\begin{aligned} \mathcal{A}_M(X) = \mathcal{A}_M(\sigma, \mathbf{x}^\mu, x^{\mu_1\mu_2}, \dots, x^{\mu_1\mu_2\dots\mu_d}) = \\ \sum_{n_I} \mathcal{A}_{M,n_I}(\mathbf{x}^\mu) \sigma^{n_o} (x^{12})^{n_{12}} \dots (x^{123})^{n_{123}} \dots (x^{12\dots d})^{n_{123\dots d}}. \end{aligned} \quad (41)$$

The sum over the spacetime dependent fields  $\mathcal{A}_{M,n_I}(\mathbf{x}^\mu)$  is taken over the infinite number of integer-valued modes associated with the collection set of integers

$$n_I = n_o, n_{12}, \dots, n_{123}, \dots, n_{1234}, \dots, n_{12\dots d}. \quad (42)$$

In Noncommutative  $C$ -spaces we may replace the ordinary products of the poly-vector valued coordinates in eq-(41) for their star products.

To finalize we provide a description of QM in Noncommutative  $C$ -spaces based on Yang's Noncommutative phase space algebra [10]. There is a *subalgebra* of the  $C$ -space operator-valued coordinates which is *isomorphic* to the Noncommutative Yang's  $4D$  spacetime algebra [10]. This can be seen after establishing the following correspondence between the  $C$ -space vector/bivector (area-coordinates) algebra, associated to the  $6D$  angular momentum (Lorentz) algebra, and the Yang's spacetime algebra via the  $SO(6)$  generators  $\Sigma^{ij}$  in  $6D$  ( $i, j = 1, 2, 3, \dots, 6$ ) as follows [11]

$$i \hbar \Sigma^{\mu\nu} \leftrightarrow i \frac{\hbar}{\lambda^2} \hat{X}^{\mu\nu}, \quad i \Sigma^{56} \leftrightarrow i \frac{R}{\lambda} \mathcal{N}. \quad (43a)$$

$$i \lambda \Sigma^{\mu 5} \leftrightarrow i \hat{X}^\mu, \quad i \Sigma^{\mu 6} \leftrightarrow i \frac{R}{\hbar} \hat{P}^\mu \quad (43b)$$

where the indices  $\mu, \nu = 1, 2, 3, 4$ . The scales  $\lambda$  and  $R$  are a lower and upper scale respectively, like the Planck and Hubble scale. The  $SO(6)$  algebra  $[\Sigma^{ij}, \Sigma^{kl}] = -g^{ik}\Sigma^{jl} + \dots$  can be recast in terms of a *noncommutative* phase space algebra as

$$[\hat{P}^\mu, \mathcal{N}] = -i \eta^{66} \frac{\hbar}{R^2} \hat{X}^\mu, \quad [\hat{X}^\mu, \mathcal{N}] = i \eta^{55} \frac{\lambda^2}{\hbar} \hat{P}^\mu. \quad (44)$$

$$[\hat{X}^\mu, \hat{X}^\nu] = -i \eta^{55} \hat{X}^{\mu\nu}, \quad [\hat{P}^\mu, \hat{P}^\nu] = -i \eta^{66} \frac{\hbar^2}{R^2 \lambda^2} \hat{X}^{\mu\nu}, \quad \hat{X}^{\mu\nu} = \lambda^2 \Sigma^{\mu\nu}. \quad (45)$$

$$[\hat{X}^\mu, \hat{P}^\mu] = i \hbar \eta^{\mu\nu} \frac{\lambda}{R} \Sigma^{56} = i \hbar \eta^{\mu\nu} \mathcal{N}, \quad [\hat{X}^{\mu\nu}, \mathcal{N}] = 0. \quad (46)$$

The last relation is the *modified* Heisenberg algebra in  $4D$  since  $\mathcal{N}$  does *not* commute with  $X^\mu$  nor  $P^\mu$ . The remaining *nonvanishing* commutation relations are

$$[\Sigma^{\mu\nu}, \hat{X}^\rho] = -i \eta^{\mu\rho} \hat{X}^\nu + i \eta^{\nu\rho} \hat{X}^\mu \quad (47a)$$

$$[\Sigma^{\mu\nu}, \hat{P}^\rho] = -i \eta^{\mu\rho} \hat{P}^\nu + i \eta^{\nu\rho} \hat{P}^\mu. \quad (47b)$$

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\tau}] = -i \eta^{\mu\rho} \Sigma^{\nu\tau} + i \eta^{\nu\rho} \Sigma^{\mu\tau} - \dots \quad (47c)$$

the last relation is the same as that in eq-(18) after reabsorbing factors of 2 in the definition of  $\Sigma^{\mu\nu}$ . Eqs-(44-47) are the defining relations of the Yang's Noncommutative  $4D$  spacetime algebra involving the  $8D$  phase-space variables  $X^\mu, P^\mu$  and the angular momentum (Lorentz) generators  $\Sigma^{\mu\nu}$  in  $4D$ . The above

commutators obey the Jacobi identities. An immediate consequence of Yang's noncommutative algebra is that now one has a modified products of uncertainties

$$\begin{aligned}\Delta X^\mu \Delta P^\nu &\geq \frac{\hbar}{2} \eta^{\mu\nu} \|\langle \Sigma^{56} \rangle\|; \quad \Delta X^\mu \Delta X^\nu \geq \frac{\lambda^2}{2} \|\langle \Sigma^{\mu\nu} \rangle\| \\ \Delta P^\mu \Delta P^\nu &\geq \frac{1}{2} \left(\frac{\hbar}{R}\right)^2 \|\langle \Sigma^{\mu\nu} \rangle\|.\end{aligned}\quad (48)$$

The Noncommutative phase space Yang's algebra in  $4D$  can be generalized to the Noncommutative Clifford phase space algebra associated to the  $4D$  spacetime after following the same prescription as in eqs-(43) by invoking higher dimensions (  $12D$  in this case instead of  $6D$  ) as follows

$$X^\mu \leftrightarrow \lambda \Gamma^\mu \wedge \Gamma^5, \quad P^\mu \leftrightarrow \frac{\hbar}{R} \Gamma^\mu \wedge \Gamma^6. \quad (49)$$

$$\begin{aligned}X^{\mu_1\mu_2} &\leftrightarrow \Upsilon^{[\mu_1\mu_2]} \text{ [57]} \neq \lambda^2 \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^5 \wedge \Gamma^7 \\ P^{\mu_1\mu_2} &\leftrightarrow \Upsilon^{[\mu_1\mu_2]} \text{ [68]} \neq \left(\frac{\hbar}{R}\right)^2 \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^6 \wedge \Gamma^8.\end{aligned}\quad (50)$$

$$\begin{aligned}X^{\mu_1\mu_2\mu_3} &\leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3]} \text{ [579]} \neq \lambda^3 \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9 \\ P^{\mu_1\mu_2\mu_3} &\leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3]} \text{ [6810]} \neq \left(\frac{\hbar}{R}\right)^3 \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^6 \wedge \Gamma^8 \wedge \Gamma^{10}.\end{aligned}\quad (51)$$

$$\begin{aligned}X^{\mu_1\mu_2\mu_3\mu_4} &\leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4]} \text{ [57911]} \neq \lambda^4 \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^{\mu_4} \wedge \Gamma^5 \wedge \Gamma^7 \wedge \Gamma^9 \wedge \Gamma^{11} \\ P^{\mu_1\mu_2\mu_3\mu_4} &\leftrightarrow \Upsilon^{[\mu_1\mu_2\mu_3\mu_4]} \text{ [681012]} \neq \left(\frac{\hbar}{R}\right)^4 \Gamma^{\mu_1} \wedge \Gamma^{\mu_2} \wedge \Gamma^{\mu_3} \wedge \Gamma^{\mu_4} \wedge \Gamma^6 \wedge \Gamma^8 \wedge \Gamma^{10} \wedge \Gamma^{12}.\end{aligned}\quad (52)$$

The indices  $\mu_1, \mu_2, \mu_3, \mu_4$  range from 1, 2, 3, 4. The extra indices span 8 additional directions (dimensions) leaving a total dimension of  $4 + 8 = 12$ . The *noncommutative* Clifford phase space algebra commutators are defined in terms of the algebra

$$[\Upsilon^{MN}, \Upsilon^{PQ}] = -i G^{MP} \Upsilon^{NQ} + i G^{MQ} \Upsilon^{NP} + i G^{NP} \Upsilon^{MQ} - i G^{NQ} \Upsilon^{MP} \quad (53)$$

The generators obey  $\Upsilon^{MN} = -\Upsilon^{NM}$ , and  $G^{MN} = G^{NM}$  under an exchange of multi-indices  $M \leftrightarrow N$ .

The algebra (53) has the same structure as a *generalized spin algebra* and satisfies the Jacobi identities. We must stress that

$$[\Upsilon^{MN}, \Upsilon^{PQ}] \neq [ [\Gamma^M, \Gamma^N], [\Gamma^P, \Gamma^Q] ]. \quad (54)$$

except in the special case when  $M, N, P, Q$  are all *bivector* indices : hence we must *emphasize* that the generalized spin algebra (53) is *not isomorphic* to the noncommutative algebra of eqs-(15-24) ! For example, from the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]}]^{[579]}, \Upsilon^{[\nu_1\nu_2\nu_3]}]^{[6810]} = -i G^{[\mu_1\mu_2\mu_3]}]^{[\nu_1\nu_2\nu_3]} \Upsilon^{[579]}]^{[6810]}. \quad (55a)$$

one can infer the Weyl-Heisenberg algebra commutator

$$[X^{\mu_1\mu_2\mu_3}, P^{\nu_1\nu_2\nu_3}] = -i \hbar^3 G^{[\mu_1\mu_2\mu_3]}]^{[\nu_1\nu_2\nu_3]} \Upsilon^{[579]}]^{[6810]}. \quad (55b)$$

From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]}]^{[579]}, \Upsilon^{[\nu_1\nu_2\nu_3]}]^{[579]} = -i G^{[579]}]^{[579]} \Upsilon^{[\mu_1\mu_2\mu_3]}]^{[\nu_1\nu_2\nu_3]}. \quad (56a)$$

one can infer the commutator among the tri-vector coordinates

$$[X^{\mu_1\mu_2\mu_3}, X^{\nu_1\nu_2\nu_3}] = -i \lambda^6 G^{[579]}]^{[579]} \Upsilon^{[\mu_1\mu_2\mu_3]}]^{[\nu_1\nu_2\nu_3]}. \quad (56b)$$

where  $\Upsilon^{[\mu_1\mu_2\mu_3]}]^{[\nu_1\nu_2\nu_3]}$  is a generalized angular momentum (spin) generator. From the commutator

$$[\Upsilon^{[\mu_1\mu_2\mu_3]}]^{[579]}, \Upsilon^{[579]}]^{[6810]} = i G^{[579]}]^{[579]} \Upsilon^{[\mu_1\mu_2\mu_3]}]^{[6810]}. \quad (57a)$$

one can infer the commutator

$$[X^{\mu_1\mu_2\mu_3}, \Upsilon^{[579]}]^{[6810]} = i \lambda^6 \frac{1}{\hbar^3} G^{[579]}]^{[579]} P^{\mu_1\mu_2\mu_3}. \quad (57b)$$

which *exchanges* the  $X^{\mu_1\mu_2\mu_3}$  for  $P^{\mu_1\mu_2\mu_3}$ , etc ..... Therefore, eqs-(55,56,57) are the suitable tri-vector analog of eqs-(44,45,46). Clearly, the above non-vanishing commutators *differ* from those in eqs-(15-24) and will modify the QM wave equations when one introduces potential terms like  $V(X) = g(X * X * \dots * X)$ . QM in ordinary (commutative)  $C$ -spaces can be found in [11].

Having provided the basic ideas and results behind polyvector gauge field theories in Noncommutative Clifford spaces, the construction of Noncommutative Clifford-space gravity as polyvector valued gauge theories of twisted diffeomorphisms in  $C$ -spaces will be the subject of future investigations. It would require quantum Hopf algebraic deformations of Clifford algebras [9]. Such theory is far richer than gravity in Noncommutative spacetimes [12].

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## References

- [1] C. Castro, "The Clifford Space Geometry of Conformal Gravity and  $U(4) \times U(4)$  Yang-Mills Unification" to appear in the IJMPA.
- [2] C. Castro, Annals of Physics **321**, no.4 (2006) 813.  
S. Konitopoulos, R. Fazio and G. Savvidy, Europhys. Lett. **85** (2009) 51001.  
G. Savvidy, Fortsch. Phys. **54** (2006) 472.
- [3] C. Castro, M. Pavsic, *Progress in Physics* **1** (2005) 31; *Phys. Letts B* **559** (2003) 74; *Int. J. Theor. Phys* **42** (2003) 1693.
- [4] M. Pavsic, *The Landscape of Theoretical Physics: A Global View, From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle* (Kluwer Academic Publishers, Dordrecht-Boston-London, 2001).
- [5] A. Chamseddine, "An invariant action for Noncommutative Gravity in four dimensions" hep-th/0202137. *Comm. Math. Phys* **218**, 283 (2001). "Gravity in Complex Hermitian Spacetime" arXiv : hep-th/0610099.
- [6] M. Kontsevich, *Lett. Math. Phys.* **66** (2003) 157.
- [7] K. Becker, M. Becker and J. Schwarz, *String Theory and M-Theory : A Modern Introduction* (Cambridge University Press, 2007, pp. 543-545).
- [8] R. Szabo, "Quantum Gravity, Field Theory and Signatures of Noncommutative Spacetime" arXiv : 0906.2913.
- [9] B. Fauser, "A treatise on Quantum Clifford Algebras" math.QA/0202059; Z. Osiewicz, "Clifford Hopf Algebra and bi-universal Hopf algebra" q-qlg/9709016; C. Blochmann "Spin representations of the Poincare Algebra" Ph. D Thesis, math.QA/0110029.
- [10] C.N Yang, *Phys. Rev* **72** (1947) 874; Proceedings of the International Conference on Elementary Particles, ( 1965 ) Kyoto, pp. 322-323.
- [11] C. Castro, *Journal of Physics A : Math. Gen* **39** (2006) 14205. *Progress in Physics* **2** April (2006) 86.
- [12] P. Aschieri, M. Dimitrijevic, F. Meyer and J. Wess, *Class. Quant. Grav.* **23** (2006) 1883.