NONCOMMUTATIVE (SUPER) p-BRANES AND MOYAL-YANG STAR PRODUCTS WITH A LOWER AND UPPER SCALE

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ABSTRACT

Noncommutative p-brane actions, for even $p+1 = 2n$-dimensional world-volumes, are written explicitly in terms of the *novel* Moyal-Yang (Fedosov-Kontsevich) star product deformations of the Noncommutative Nambu Poisson Brackets (NCNPB) that are associated with the *noncommuting* world-volume coordinates q^A, p^A for $A = 1, 2, 3, ...n$. The latter noncommuting coordinates obey the noncommutative Yang algebra with an ultraviolet L_P (Planck) scale and infrared (R) scale cutoff. It is shown why our p-brane actions in the "classical" limit $\hbar_{eff} = \hbar L_P/R \rightarrow 0$ still acquire nontrivial noncommutative *corrections* that differ from ordinary p-brane actions. Super p-branes actions in the light-cone gauge are also amenable to Moyal-Yang star product deformations as well due to the fact that p-branes moving in flat spacetime backgrounds, in the light-cone gauge, can be recast as gauge theories of volume-preserving diffeomorphisms. The most general construction of noncommutative super p-branes actions based on non (anti) commuting superspaces and quantum group methods remains an open problem.

1. THE YANG'S NONCOMMUTATIVE SPACETIME AGEBRA

Yang's noncommutative space time algebra [1] is a generalization of the Snyder algebra [2] (where now both coordinates and momenta are not commuting) that has received more attention recently [3]. The isomorphism of Yang's algebra [1] to the $4D$ (angular momentum algebra) conformal algebra $SO(4,2)$ was established by Tanaka [3] (within the context of the holographic principle) by using the correspondence $X^\mu \leftrightarrow L_P\Sigma^{\mu 5}$ where L_P is an ultraviolet scale (Planck scale) and $P^\mu \leftrightarrow (\hbar/R)\Sigma^{\mu 6}$ where R is an infrared scale (the throat size of de Sitter, Anti de Sitter space) . $\hbar\Sigma^{\mu\nu}, \hbar\Sigma^{\mu 5}, \hbar\Sigma^{\mu 6}, \hbar\Sigma^{56}$ are the angular momentum operators in $6D$. This construction [2] can be generalized to higher dimensional extensions of Yang's algebra [1] by simply replacing the $SO(4,2)$ algebra with $SO(D,2)$.

Using this correspondence allows to write the exchange commutators of the Yang's spacetime algebra (which exchange X and P in units $c = 1$)

$$[P^\mu, \mathcal{N}] = -i\eta^{66}\frac{\hbar}{R^2}X^\mu. \quad [X^\mu, \mathcal{N}] = i\eta^{55}\frac{L_P^2}{\hbar}P^\mu. \quad \mathcal{N} \equiv \frac{L_P}{R}\Sigma^{56}. \qquad (1-1)$$

The coordinates and momenta are no longer commuting:

$$[X^\mu, X^\nu] = -i\eta^{55}L_P^2\Sigma^{\mu\nu}. \quad [P^\mu, P^\nu] = -i\eta^{66}\frac{\hbar^2}{R^2}\Sigma^{\mu\nu}. \qquad (1-2)$$

where $\hbar\Sigma^{\mu\nu} \equiv \mathcal{M}^{\mu\nu}$ are angular-momentum like operators. The signatures for AdS_5 space are $\eta^{55} = +1$; $\eta^{66} = -1$ and for the *Euclideanized* AdS_5 space are $\eta^{55} = +1$ and $\eta^{66} = +1$. The *modified* Weyl-Heisenberg algebra is :

$$[X^\mu, P^\mu] = -i\hbar\eta^{\mu\nu}\frac{L_P}{R}\Sigma^{56} = -i\hbar\eta^{\mu\nu}\mathcal{N}. \qquad (1-3)$$

The other commutation relations are the standard angular momentum ones

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = i(\eta_{\mu\sigma}\Sigma_{\nu\rho} + \eta_{\nu\rho}\Sigma_{\mu\sigma} - \eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho}). \quad (1-4)$$

and

$$[\Sigma_{\mu\nu}, X_\rho] = i(\eta_{\nu\rho}X_\mu - \eta_{\mu\rho}X_\nu). \quad [\Sigma_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu). \quad (1-5)$$

These commutators obey the Jacobi identities. When $L_P \to 0$ and $R \to \infty$ one recovers the ordinary *commutative* spacetime algebra. The Snyder algebra [2] is recovered by setting $R \to \infty$ while leaving L_P intact. To recover the ordinary Weyl-Heisenberg algebra is more subtle. Tanaka [3] has shown that the *spectrum* of the operator $\mathcal{N} = (L_P/R)\Sigma^{56}$ is discrete given by $n(L_P/R)$. This is not suprising since the angular momentum generator $\mathcal{M}^{56}$ associated with the *Euclideanized* AdS_5 space is a rotation in the now compact $x^5 - x^6$ directions. This is not the case in AdS_5 space since $\eta^{66} = -1$ and this timelike direction is no longer compact. Rotations involving timelike directions are equivalent to noncompact boosts with a continuous spectrum.

In order to recover the standard Weyl-Heisenberg algebra from Yang's Noncommutative spacetime algebra, and the standard uncertainty relations $\Delta x \Delta p \geq \hbar$ with the ordinary $\hbar$ term , rather than the $n\hbar$ term, one needs to take the limit $n \to \infty$ limit in such a way that the net combination of $n\frac{L_P}{R} \to 1$.

This can be attained when one takes the *double* scaling limit of the quantities as follows :

$$L_P \to 0. \quad R \to \infty. \quad L_P R \to L^2. \quad lim_{n\to\infty}\, n\frac{L_P}{R} = n\frac{L_P^2}{L_P R} = \frac{nL_P^2}{L^2} \to 1. \quad (1-6)$$

From eq-(1-6) one learns then that $nL_P^2 = L_P R = L^2$ where the spectrum n corresponds to the quantization of the angular momentum operator in the $x^5 - x^6$ direction (after embedding the $5D$ hyperboloid of throat size R onto $6D$) . Tanaka [3] has shown why there is a *discrete spectra* for the *spatial* coordinates and *spatial* momenta in Yang's spacetime algebra that yields a *minimum* length L_P (ultraviolet cutoff in energy) and a *minimum* momentum $p = \hbar/R$ (maximal length R , infrared cutoff) . The energy and temporal coordinates had a continous spectrum.

The physical interpretation of the double-scaling limit of eq-(1-6) is that the the area $L^2 = L_P R$ becomes now *quantized* in units of the Planck area L_P^2 as $L^2 = nL_P^2$. Thus the quantization of the area (via the double scaling limit) $L^2 = L_P R = nL_P^2$ is a result of the *discrete* angular momentum spectrum in the $x^5 - x^6$ directions of the Yang's Noncommutative spacetime algebra when it is realized by (angular momentum) differential operators acting on the *Euclideanized* AdS_5 space (two branches of a $5D$ hyperboloid embedded in $6D$). A general interplay between quantum of areas and quantum of angular momentum, for arbitrary values of spin, in terms of the square root of the Casimir $\mathbf{A} \sim L_P^2\sqrt{j(j+1)}$, has been obtained a while ago in Loop Quantum Gravity by using spin-networks techniques and highly technical area-operator regularization procedures [4] .

In [5] we have shown why AdS_4 gravity with a topological term; i.e. an Einstein-Hilbert action with a cosmological constant plus Gauss-Bonnet terms can be obtained from the vacuum state of a BF-Chern-Simons-Higgs theory *without* introducing by *hand* the zero torsion condition imposed in the MacDowell-Mansouri-Chamsedine-West construction. One of the most salient features of [5] was that a *geometric mean* relationship was *derived from first principles*, among the cosmological constant ρ_{vacuum} , the Planck area L_P^2 and the AdS_4 throat size squared R^2 given by $(\rho_v)^{-1} = (L_P)^2(R^2)$. A similar geometric mean relation is also obeyed by the condition $L_P R = L^2 (= nL_P^2)$ in the double scaling limit of Yang's algebra which suggests to identify the cosmological constant as $\rho_{vacuum} = L^{-4}$. Notice that by setting the infrared scale R equal to the Hubble radius horizon (today) R_H and L_P equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density : $\rho \sim L_{Planck}^{-2} R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-120} M_{Planck}^4$.

Non (anti) commuting superspaces have been studied by several authors [6] , however the supersymmetric version (if possible) of the full fledged Yang's algebra, for noncommuting coordinates and momenta, with an upper R and lower scale L_p , is not known, to our knowledge. Having presented this introductory review of Yang's algebra we proceed with the main results of this work.

2. MOYAL-YANG STAR PRODUCTS AND NONCOMMUTATIVE BRANES

Brane actions from quenched $SU(N)$ QCD in the large $N \rightarrow \infty$ limit have been constructed in [7] by means of Moyal deformation quantization methods. Moyal deformations of Gravitational actions via $SU(\infty)$ gauge theories were presented in [8]. Some time ago, Self Dual Gravity from $SU(\infty)$ Self Dual Yang Mills was provided in [9,10] . The area-preserving diffeomorphisms (diffs) algebra of the sphere was shown to be isomorphic to a basis-dependent limit of $SU(\infty)$ by [11] and many important physical applications of membranes and higher spin theories within the context of W_∞ algebras was analyzed by many authors, in particular by [12]. The task now is to construct *novel* Moyal star product deformations of (super) p-brane actions based on the noncommutative spacetime Yang's algebra where the deformation parameter is $\hbar_{eff} = \hbar L_P/R$ for nonzero values of $\hbar$.

The modified Poisson bracket is now given by

$$\{ \mathcal{F} (q^A, p^A) , \mathcal{G} (q^A, p^A) \}_\Omega = (\partial_{Z^A} \mathcal{F}) \Omega^{AB} (\partial_{Z^B} \mathcal{G}) = (\partial_{q^A} \mathcal{F}) \{q^A, q^B\} (\partial_{q^B} \mathcal{G}) + (\partial_{p^A} \mathcal{F}) \{p^A, p^B\} (\partial_{p^B} \mathcal{G}) +$$

$$(\partial_{q^A} \mathcal{F}) \{q^A, p^B\} (\partial_{p^B} \mathcal{G}) + (\partial_{p^A} \mathcal{F}) \{p^A, q^B\} (\partial_{q^B} \mathcal{G}). \quad (2-1)$$

where the entries $\{q^A, q^B\} \neq 0$, $\{p^A, p^B\} \neq 0$, and $\{p^A, q^B\} = -\{p^A, q^B\}$ can be read from the commutators described in the previous section by simply defining the deformation parameter $\hbar_{eff} \equiv \hbar(L_P/R)$.

Denoting the coordinates q^A, p^A by Z^A and when the Poisson structure Ω^{AB} is given in terms of *constant* numerical coefficients, the Moyal star product is defined in terms of the deformation parameter $\hbar_{eff} = \hbar L_P/R$ as

$$(\mathcal{F} * \mathcal{G})(Z) \equiv exp \ [\ (i\hbar_{eff}) \ \Omega^{AB} \ \partial_A^{(Z_1)} \ \partial_B^{(Z_2)} \ ] \ \mathcal{F}(Z_1) \ \mathcal{G}(Z_2)|_{Z_1=Z_2=Z}. \quad (2-2)$$

where the derivatives $\partial_A^{(Z_1)}$ act only on the $\mathcal{F}(Z_1)$ term and $\partial_B^{(Z_2)}$ act only on the $\mathcal{G}(Z_2)$ term.

Because in our case the Poisson structure Ω^{AB} is given in terms of *variable* coefficients, it is a function of the coordinates $\partial\Omega^{AB} \neq 0$, since the Yang's algebra is basically an angular momentum algebra, the suitable Moyal-Yang star product (in R^d) given by Kontsevich [13] will acquire corrections to the ordinary Moyal star product :

$$f * g = fg + i\hbar_{eff} \ \Omega^{ij} \ (\partial_i f \partial_j g) + \frac{(i\hbar_{eff})^2}{2} \ \Omega^{i_1 j_1} \ \Omega^{i_2 j_2} \ (\partial^2_{i_1 i_2} f) \ (\partial^2_{j_1 j_2} g) +$$

$$\frac{(i\hbar_{eff})^2}{3} [\ \Omega^{i_1 j_1} (\partial_{j_1} \Omega^{i_2 j_2}) (\partial_{i_1} \partial_{i_2} f \partial_{j_2} g - \partial_{i_2} f \partial_{i_1} \partial_{j_2} g) \ ] + O(\hbar^3_{eff}). \quad (2-3)$$

The Kontsevich star product is associative up to second order [13] $(f * g) * h = f * (g * h) + O(\hbar^3_{eff})$. The most general expression of the Kontsevich star product in Poisson manifolds is quite elaborate and shall not be given here [13]. Star products in *curved* phase spaces have been constructed by Fedosov [14] . Despite these technical subtlelties it will not affect the final expressions for the "classical" Noncommutative p-brane actions (shown below) when one takes the $\hbar_{eff} \rightarrow 0$ "classical" limit. We will show below that in that limit there are still *nontrivial noncommutative corrections* to the ordinary p-brane actions .

Our final expressions below, in the $\hbar_{eff} \rightarrow 0$ limit, already encode the $Noncommutative$ structures inherent in the *noncommuting* world volume coordinates. We shall display as well the Kontsevich star products corrections . The Noncommutative Moyal-Yang Bracket defined in terms of the Kontsevich star product is : $\{\{\mathcal{F} \ , \ \mathcal{G}\}\} \equiv \ \mathcal{F} * \mathcal{G} \ - \mathcal{G} * \mathcal{F}$. In particular, when one relates the (in the even-dimensional world-volume case , $p + 1 = 2n$) world-volume coordinates $\sigma^1, \sigma^2, ..., \sigma^{p+1}$ of p-branes to the $2n$ phase space variables q^A, p^A as shown in [7,8] , one has

$$\{\{ \ X_\mu(q^A, p^A) \ , X_\nu(q^A, p^A) \ \}\} = X_\mu \ * \ X_\nu \ - \ X_\nu \ * \ X_\mu. \quad (2-4)$$

where one has rewritten $X^\mu(\sigma^1, \sigma^2, ...)$ by $X^\mu(q^A, p^B)$. A Moyal-Yang star-product deformation of the Nambu-Poisson Brackets (MYNPB) can be defined when $p + 1 = 2n = even$ [15] :

$$\{\{ \ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}} \}\}_{MYNPB} = \{\{ \ X_{\mu_1}, X_{\mu_2} \ \}\} \ * \ \{\{ \ X_{\mu_3}, X_{\mu_4} \ \}\} \ * \ ..... * \ \{\{ \ X_{\mu_p}, X_{\mu_{p+1}} \ \}\} \pm .... \quad (2-5)$$

where the ellipsis denotes signed permutations; i.e. the Moyal-Yang star-product deformations of the Nambu-Poisson-Brackets (MYNPB) can be decomposed as suitable antisymmetrized sums of Moyal-Yang star products of the Moyal-Yang brackets (MYB) among *pairs* of variables.

When $p+1 = odd$, attempts have bee made to introduce deformations based on the Zariski star product deformations of the Nambu Poisson Brackets (NPB), but unfortunately these deformed brackets failed to obey all the required algebraic properties of a (quantum) bracket [15]. Therefore, to our knowledge, only when $p+1 = 2n$ is even one can perform a suitable star product deformations of the NPB. The Dirac-Nambu p-brane action is

$$S \;=\; T \int [d^{p+1}\sigma] \;\sqrt{|det\;(G_{ab})|} = \;T \int [d^{p+1}\sigma] \;\sqrt{|det\;[G_{\mu\nu}\;(\partial_a X^\mu)\;(\partial_b X^\nu)]|}. \qquad (2-6)$$

where T is the p-brane tension. When the target spacetime background is $flat$, $G_{\mu\nu} = \eta_{\mu\nu}$, the determinant can be rewritten in terms of Nambu Poisson Brackets (NPB) as

$$det\;(G_{ab}) \;=\; \{\; X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}}\;\}\;\{\; X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}}\}_{NPB}. \qquad (2-7)$$

However, when the target spacetime background is *curved*, $G_{\mu\nu} = G_{\mu\nu}(X^\rho(\sigma))$, the determinant is :

$$det\;(G_{ab}) = \;\{\; X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}}\;\}\;\{\; X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p+1}}\}_{NPB}\; G_{\mu_1\nu_1} G_{\mu_2\nu_2}............G_{\mu_{p+1}\nu_{p+1}}. \quad (2-8)$$

and one cannot naively pull the metric factors $G_{\mu\nu}$ inside the brackets and perform the index contractions inside the brackets. The Noncommutative branes action is simply obtained in a two step process. Firstly, we construct the Moyal-Yang action S^*_{MY} by using Moyal-Yang star products and brackets in the special case $p+1 = 2n = even$

$$S^*_{MY} \;=\; T \int [d^{2n}\sigma] \;\sqrt{\;|\;det\;[\;G_{ab}(X^\rho(\sigma))\;]\;|_*}. \qquad (2-9)$$

where the Moyal-Yang deformations of the determinant $det\;[\;G_{ab}(X^\rho(\sigma))\;]_*$ are :

$$\frac{1}{(i\hbar_{eff})^{2n}}\{\{\; X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{p+1}}\;\}\}\; * \{\{\; X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p+1}}\}\}\; * \; G_{\mu_1\nu_1} \; * \; G_{\mu_2\nu_2} \; * \; .... \; * \; G_{\mu_{p+1}\nu_{p+1}}. \qquad (2-10)$$

The correct Moyal-Yang deformed action S^*_{MY}, for p-branes (such that $p+1 = 2n$) moving in *curved* backgrounds, must involve naturally the Moyal-Yang deformations of the determinant $det\;(G_{ab})_*$ as shown in eq-(2-10) . However, when the target spacetime is $flat$ one could use the other form of the action given by

$$S^*_{Moyal} \;=\; T \int \; d^{2n}\sigma \;\sqrt{\frac{1}{(i\hbar_{eff})^{2n}(2n)!}|\;\{\{\; X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{2n}}\;\}\}\; * \;\{\{\; X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{2n}}\}\}\;|}. \quad (2-11)$$

The second step after eq-(2-10) is to take the $\hbar_{eff} \rightarrow 0$ limit such that the star products of functions reduce to ordinary pointwise products and the Moyal-Yang Brackets (MYB) reduce to Noncommutative Poisson Brackets (NCPB)

$$lim_{\hbar_{eff}\rightarrow 0}\;\frac{1}{i\hbar_{eff}}\{\{X^\mu, X^\nu\}\} \rightarrow \{X^\mu, X^\nu\}_{NCPB} = (\partial_{q^A} X^\mu)\;\{q^A, q^B\}\;(\partial_{q^B} X^\nu)\; +$$

$$(\partial_{p^A} X^\mu)\;\{p^A, p^B\}\;(\partial_{p^B} X^\nu)\; + \;(\partial_{q^A} X^\mu)\;\{q^A, p^B\}\;(\partial_{p^B} X^\nu)\; + \;(\partial_{p^A} X^\mu)\;\{p^A, q^B\}\;(\partial_{q^B} X^\nu). \qquad (2-12)$$

where the entries $\{q^A, q^B\} \neq 0$, $\{p^A, p^B\} \neq 0$, and $\{p^A, q^B\} = -\{p^A, q^B\}$ can be read from the $4D$ Yang's algebra, in the particular case $\eta^{55} = \eta^{66} = 1$ (which is associated with an Euclideanized AdS space)

$$\{q^A, q^B\}_{NCPB} = lim_{\hbar_{eff}\rightarrow 0}\;\frac{1}{i\hbar_{eff}}[q^A, q^B] = -\frac{L^2}{\hbar}\Sigma^{AB}. \qquad (2-13a)$$

$$\{p^A, p^B\}_{NCPB} = lim_{\hbar_{eff}\to 0} \ \frac{1}{i\hbar_{eff}}[p^A, p^B] = -\frac{\hbar}{L^2}\Sigma^{AB} \qquad (2-13b)$$

$$\{q^A, p^B\}_{NCPB} = lim_{\hbar_{eff}\to 0} \ \frac{1}{i\hbar_{eff}}[q^A, p^B] = -\eta^{AB}. \qquad (2-13c)$$

with

$$\Sigma^{AB} \equiv \frac{1}{\hbar}(q^A p^B - q^B p^A). \quad q^A = q^1, q^2. \quad p^A = p^1, p^2 \qquad (2-13d)$$

one can generalize Yang's original 4-dim algebra [1] to Noncommutative $2n$-dim world-volumes and/or space-times by working with the $2n+2$-dim angular-momentum algebra $SO(d,2) = SO(p+1,2) = SO(2n,2)$. Therefore, the S_{NC} action may now be written in terms of the Noncommutative Nambu Poisson Brackets (NCNPB)

$$S_{NC} \ = \ T \int \ d^{2n}\sigma \ \sqrt{\frac{1}{(2n)!}|\ \{\ X^{\mu_1}, X^{\mu_2}, ..., X^{\mu_{2n}}\ \}_{NC}\ \{\ X^{\nu_1}, X^{\nu_2}, ..., X^{\nu_{p+1}}\}_{NC}\ G_{\mu_1\nu_1}G_{\mu_2\nu_2}......|}. \qquad (2-14)$$

defined as

$$\{\ X_{\mu_1}, X_{\mu_2}, ..., X_{\mu_{p+1}}\}_{NCNPB} \equiv$$
$$\{\ X_{\mu_1}, X_{\mu_2}\ \}_{NCPB}\ \{\ X_{\mu_3}, X_{\mu_4}\ \}_{NCPB}............\{\ X_{\mu_p}, X_{\mu_{p+1}}\ \}_{NCPB}\ +\ permutations. \qquad (2-15)$$

where the $\{\ X_{\mu_1}, X_{\mu_2}\ \}_{NCPB}, ...$ brackets are defined by eqs-(2-12, 2-13). Notice that the measure :

$$d^{2n}\sigma \equiv dq^1 \wedge dp^1 \wedge dq^2 \wedge dp^2 \wedge ..... \wedge dq^n \wedge dp^n. \qquad (2-16)$$

has world-volume dimensions $\hbar^n$ that compensates for the dimensions $\hbar^{-n}$ of the square root expression of (2-14) stemming from the brackets.

To illustrate the *corrections* to the ordinary p-brane actions due to the inherent noncommutative world volume coordinates we will present the explicit corrections to the p-brane action described by (2-17a) whose world volume is $p+1 = 2n$-dimensional spanned by the $q^1, q^2, ...q^n$ and $p^1, p^2, ...p^n$ coordinates. The NCPB are :

$$\{X^{\mu_1}, X^{\mu_2}\}_{NCPB} = \sum_{i=1}^{i=n} \ \partial_{[p^i} X^{\mu_1} \partial_{q^i]} X^{\mu_2} \ - \frac{L^2}{\hbar^2} \ \sum_{i,j \ i\neq j} (q^i p^j - q^j p^i) \ \partial_{[q^i} X^{\mu_1} \partial_{q^j]} X^{\mu_2} \ -$$
$$\frac{1}{L^2} \ \sum_{i,j \ i\neq j} (q^i p^j - q^j p^i) \ \partial_{[p^i} X^{\mu_1} \partial_{p^j]} X^{\mu_2}. \qquad (2-17)$$

these NCPB are the ones that define the NCNPB

$$\{\ X^{\mu_1}, X^{\mu_2}, X^{\mu_3}, ... X^{\mu_{2n}}\}_{NCNPB} \equiv$$

$$\{\ X^{\mu_1}, X^{\mu_2}\ \}_{NCPB}\ \{\ X^{\mu_3}, X^{\mu_4}\ \}_{NCPB}........\ \{\ X^{\mu_{2n-1}}, X^{\mu_{2n}}\ \}_{NCPB}.\ +\ signed\ permutations. \qquad (2-18)$$

and which are inserted into the Noncommutative p-brane action (2-14) when $p+1 = 2n$. The last two terms in the r.h.s of (2-17) explicitly furnish the *corrections* to the ordinary p-brane actions (2-14) due to the inherent noncommutative world-volume coordinates expressed in eqs-(2-13a, 2-13b, 2-13c, 2-13d). Notice that the limits $\hbar = 0$ and/or $L = 0, \infty$ in eq-(2-17) are *singular* even if one were to take take $L^2/\hbar^2 \to 1$. As it was stated earlier, in the "classical " $\hbar_{eff} \to 0$ limit, there are still *nontrivial noncommutative* corrections to the ordinary classical p-brane actions, and for this reason our p-brane actions described in (2-14) differ from the standard p-brane actions.

Concluding, the action (2-14) written explicitly in terms of NCNPB given by eqs-(2-17, 2-18) is the sought-after Noncommutative p-brane action associated with the *noncommuting* world-volume coordinates

q^A, p^A given by the Yang's algebra (2-12, 2-13) after one identifies the $p+1 = 2n$ world-volume coordinates σ's with the $2n$-dim phase space variables q^A, p^B . Finally, the Moyal-Yang (Kontsevich) deformed p-brane action (2-11), for non-zero values of $\hbar_{eff}$, requires to write

$$\frac{1}{i\hbar_{eff}}\{\{X^{\mu_1}, X^{\mu_2}\}\}_* = \{X^{\mu_1}, X^{\mu_2}\}_{NCPB} + \frac{(i\hbar_{eff})}{2}\ \Omega^{i_1 j_1}\ \Omega^{i_2 j_2}\ (\partial^2_{i_1 i_2} X^{[\mu_1})\ (\partial^2_{j_1 j_2} X^{\mu_2]}) +$$

$$\frac{(i\hbar_{eff})}{3}[\ \Omega^{i_1 j_1}(\partial_{j_1}\Omega^{i_2 j_2})(\partial_{i_1}\partial_{i_2}X^{[\mu_1}\partial_{j_2}X^{\mu_2]} - \partial_{i_2}X^{[\mu_1}\partial_{i_1}\partial_{j_2}X^{\mu_2]})\ ] + O(\hbar^2_{eff}). \qquad (2-19)$$

that are introduced in the expression (2-11) for the Moyal-Yang deformed p-brane action S^* after using eqs-(2-3, 2-5, 2-13) for the Kontsevich star product, the MYNPB and the symplectic matrix $\Omega^{ij}_{2n\times 2n}$ respectively.

3. NONCOMMUTATIVE SUPER p-BRANES,

p-branes as composite antisymmetric tensor field theories with a volume preserving diffs invariance were studied in [16,17]. In this section we will precisely show how light-cone gauge super p-branes actions are amenable for star product deformations as well due to the fact that p-branes can be recast as gauge theories of volume-preserving diffs in the light-cone gauge [18]. Super p-branes actions exist only for certain values of (p, D), and for certain values of the world-volume and target spacetime background signatures, due to constraints which originate by marching the number of physical bosonic and fermionic degrees of freedom. For example, supermembranes ($p = 2$) of Minkowski signatures can only be constructed in $D = 4, 5, 7, 11$ dimensions. The number of physical bosonic degrees of freedom is $D - 3 = 1, 2, 4, 8$ which matches the fermionic physical degrees of freedom.

The lightcone gauge is obtained after imposing the conditions [18] :

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \quad X^+ = x^+ + p^+\tau. \quad g_{00} = -h = -det\ (g_{ik}). \quad g_{0i} = 0. \quad \Gamma^+\Theta = 0. \qquad (3-1)$$

The lightcone gauge action for super p-branes of *spherical* topology, moving in $flat$ target spacetime backgrounds, can be rewritten in terms of Nambu-Poisson brackets of the physical lightcone variables $X^I(\tau, \sigma^a), \Psi(\tau, \sigma^a)$ as follows [18]

$$S = -\frac{1}{2}T\int d^{p+1}\sigma\ \ [\ (D_0 X^I)^2\ -\frac{1}{p!}\{\ X_{i_1}, X_{i_2}, ..., X_{i_p}\ \}^2 +$$

$$\frac{i}{2}\bar{\Psi} D_0\Psi + \frac{1}{(p-1)!}\ \bar{\Psi}\ \gamma^{i_1 i_2 ... i_{p-1}}\ \{\ X_{i_1}, X_{i_2}, ..., X_{i_{p-1}}, \Psi\ \}\ ]. \qquad (3-2)$$

where $I = 1, 2, 3, ...D - 2$ and $\sigma^a = \sigma^1, \sigma^2, ..., \sigma^p$ are the spatial p-degrees of freedom of the super p-brane. Since the action (3-2) involves taking the Nambu Poisson Brackets w.r.t to the p spatial variables of the p-brane, the only even p values for the super p-branes correspond now to $p = 2$ (membranes) in $D = 4, 5, 7, 11$ dimensions and $p = 4$ in $D = 9$ dimensions. The Moyal-Yang (Kontsevich) star products deformations of (3-2) are displayed below in eq-(3-11, 3-12).

The covariant world-volume temporal derivative is defined

$$D_0 X^I = \frac{\partial X^I}{\partial\tau} + u^a\partial_a X^I = \frac{\partial X^I}{\partial\tau} + \{\ A^1, A^2, ..., A^{p-1}, X^I\ \}. \qquad (3-3)$$

in terms of the world-volume gauge field u^a satisfying the divergence free condition $\partial_a u^a = 0$.

For p-branes with spherical topology the world-volume gauge field u^a can be expressed as the multi-symplectic gradient of the $p-1$ functions $A^1, A^2, ...A^{p-1}$ as follows

$$u^a = \epsilon^{a_1 a_2 ....a_{p-1} a}\ \frac{\partial A^1}{\partial\sigma^a_1}\frac{\partial A^2}{\partial\sigma^a_2}........\frac{\partial A^{p-1}}{\partial\sigma^a_{p-1}} = \partial_b W^{ab} \equiv \partial_b\ [\ \epsilon^{a_1 a_2 ....a_{p-2} ba}\frac{\partial A^1}{\partial\sigma^{a_1}}\frac{\partial A^2}{\partial\sigma^{a_2}}........A^{p-1}\ ]. \qquad (3-4)$$

The action displays a gauge invariance under p-volume preserving diffs

$$\delta X^I = \lambda^a \frac{\partial X^I}{\partial \sigma^a}. \quad \frac{\partial \lambda^a}{\partial \sigma^a} = 0. \qquad (3-5)$$

The divergence free condition of the parameters is required to preserve the p-volumes and

$$\delta u^a = -\frac{\partial \lambda^a}{\partial \tau} - \ u^b \partial_b \lambda^a + \lambda^b \partial_b u^a. \qquad (3-6)$$

These transformations can be rewritten in terms of Nambu-Poisson brackets by rewriting the parameters λ^a as the multi-symplectic gradient involving another set of parameters (functions) $\Lambda^a(\sigma^1, \sigma^2, ..., \sigma^p)$:

$$\lambda^a = \epsilon^{a_1 a_2 ....a_{p-1} a} \ \frac{\partial \Lambda^1}{\partial \sigma^{a_1}} \frac{\partial \Lambda^2}{\partial \sigma^{a_2}} .... \frac{\partial \Lambda^{p-1}}{\partial \sigma^{a_{p-1}}}. \qquad (3-7)$$

$$\delta X^I = \{ \ \Lambda^1, \Lambda^2, ..., \Lambda^{p-1}, X^I \ \}. \qquad (3-8)$$

$$\delta u^a = -\frac{\partial \lambda^a}{\partial \tau} - \{ \ A^1, A^2, ..., A^{p-1}, \lambda^a \ \} + \{ \ \Lambda^1, \Lambda^2, ..., \Lambda^{p-1}, u^a \ \}. \qquad (3-9)$$

There is the residual Abelian gauge invariance $W^{ab} \rightarrow W^{ab} + \partial_c \Theta^{abc}$.

The Moyal-Yang (Kontsevich) star products deformations require to replace the term $(D_0 X^I)^2$ in (3-2) by

$$(D_0 X^I)^2_* = (\ \frac{\partial X^I}{\partial \tau} + \{\{ \ A^1, A^2, ..., A^{p-1}, X^I \ \}\}_* \ ) \ * \ (\ \frac{\partial X^I}{\partial \tau} + \{\{ \ A^1, A^2, ..., A^{p-1}, X^I \ \}\}_* \ ) \qquad (3-10)$$

The term

$$\{ \ X_{i_1}, X_{i_2}, ..., X_{i_p} \ \}^2 \rightarrow \{\{ \ X_{i_1}, X_{i_2}, ..., X_{i_p} \ \}\}_* \ * \ \{\{ \ X_{i_1}, X_{i_2}, ..., X_{i_p} \ \}\}_* \quad etc...... \qquad (3-11)$$

and, naturally, the symmetry transformations (3–8, 3-9) require the use of Moyal-Nambu brackets

We conclude by adding that it is desirable to recur to superspace methods to build covariant actions beyond the light cone gauge and to deform these super p-brane actions in superspace by deforming the superspace measure associated with the super p-branes. In order to achieve this one would need the proper formulation of Non (anti) commuting superspace algebras. This remains an open problem. Another interesting avenue to pursue the study of noncommutative branes is based on the fact that Yang's algebra can also be realized in terms of the holographic area-coordinates algebra of the Clifford-space associated with the $6D$ Clifford algebra. Namely, the holographic-area coordinates algebra is isomorphic to the Yang's algebra by recurring to the coordinates/angular momentum correspondence found by Tanaka [3]. A current review of the Extended Relativity program in Clifford spaces can be found in [19].

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