On Born's Deformed Reciprocal Complex Gravitational Theory and Noncommutative Gravity

Carlos Castro

August, 2008

Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta. GA. 30314, USA.

Abstract

Born's reciprocal relativity in flat spacetimes is based on the principle of a *maximal* speed limit (speed of light) and a *maximal* proper force (which is also compatible with a *maximal* and *minimal* length duality) and where coordinates and momenta are unified on a single footing. We extend Born's theory to the case of curved spacetimes and construct a *deformed* Born reciprocal general relativity theory in curved spacetimes (without the need to introduce star products) as a local gauge theory of the *deformed* Quaplectic group that is given by the semi-direct product of $U(1,3)$ with the *deformed* (noncommutative) Weyl-Heisenberg group corresponding to *noncommutative* generators $[Z_a, Z_b] \neq 0$. The Hermitian metric is complex-valued with symmetric and nonsymmetric components and there are *two* different complex-valued Hermitian Ricci tensors $\mathcal{R}_{\mu\nu}, \mathcal{S}_{\mu\nu}$. The deformed Born's reciprocal gravitational action linear in the Ricci scalars $\mathcal{R}, \mathcal{S}$ with Torsion-squared terms and BF terms is presented. The plausible interpretation of $Z_\mu = E^a_\mu \ Z_a$ as *noncommuting* p-brane background complex spacetime coordinates is discussed in the conclusion, where E^a_μ is the complex vielbein associated with the Hermitian metric $G_{\mu\nu} = g_{(\mu\nu)} + i g_{[\mu\nu]} = E^a_\mu \ \bar{E}^b_\nu \ \eta_{ab}$. This could be one of the underlying reasons why string-theory involves gravity.

Born's reciprocal ("dual") relativity [1] was proposed long ago based on the idea that coordinates and momenta should be unified on the same footing, and consequently, if there is a limiting speed (temporal derivative of the position coordinates) in Nature there should be a maximal force as well, since force is the temporal derivative of the momentum. A *curved* phase space case scenario has been analyzed by Brandt [2] within the context of the Finsler geometry of the $8D$ tangent bundle of spacetime where there is a limiting value to the proper acceleration and such that generalized $8D$ gravitational equations reduce to ordinary Einstein-Riemannian gravitational equations in the $infinite$ acceleration limit. Other relevant work on the principle of maximal acceleration can be found in [3]. For a recent monograph on Finsler geometry see Vacaru [4].

Born's reciprocal "duality" principle is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i.e. namely, a small/large radius duality, a winding modes/ Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in noncommutative field theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory. The generalized velocity and acceleration boosts (rotations) transformations of the $8D$ Phase space, where X, T, E, P are *all* boosted (rotated) into each-other, were given by [5] based on the group $U(1,3)$ and which is the Born version of the Lorentz group $SO(1,3)$. It was found later on [6] that Planck-Scale Areas are Invariant under pure acceleration boosts which may be relevant to string theory.

Invariant actions for a point-particle in reciprocal Relativity involving Casimir group invariant quantities can be found in [7]. Casimir invariant field equations; unitary irreducible representations based on Mackey's theory of induced representations; the relativistic harmonic oscillator and coherent states can be found in [5]. The granular cellular structure of spacetime, the Schrodinger-Robertson inequality, multi-mode squeezed states, a "non-commutative" relativistic phase space geometry, in which position and momentum are interchangeable and frame-dependent was studied by [8]. Born's reciprocity principle in atomic physics and galactic motion based on $(1/r) + (b/p)$ potentials was studied recently by [9] with little effect on atomic physics but with relevant effects on galactic rotation without invoking dark matter.

In this letter we construct a local gauge theory of the $deformed$ (noncommutative) Quaplectic group given by the semidirect product of $U(1,3)$ with the deformed (noncommutative) Weyl-Heisenberg goup. The $U(1,3)$ arises as the group that leaves invariant the interval in $8D$ phase $dx_\mu dx^\mu + dp_\mu dp^\mu$ space, as well as invariant the symplectic two-form $\omega = \omega_{\mu\nu}\ dx^\mu \wedge dp^\nu$, simultaneously. The novel result in this letter is the $modification$ of the Weyl-Heisenberg algebra, not unlike Yang's noncommutative phase space algebra [10].

The deformed Weyl-Heisenberg algebra involves the generators

$$Z_a = \frac{1}{\sqrt{2}} \left(\frac{X_a}{\lambda_l} - i\frac{P_a}{\lambda_p} \right); \quad \bar{Z}_a = \frac{1}{\sqrt{2}} \left(\frac{X_a}{\lambda_l} + i\frac{P_a}{\lambda_p} \right); \quad a = 1,2,3,4. \quad (1)$$

Notice that we must *not* confuse the *generators* X_a, P_a (associated with the fiber coordinates of the internal space of the fiber bundle) with the ordinary base spacetime coordinates and momenta x_μ, p_μ. The gauge theory is constructed in the fiber bundle over the base manifold which is a $4D$ curved spacetime with *commuting* coordinates $x^\mu = x^0, x^1, x^2, x^3$. The (deformed) Quaplectic group acts as the automorphism group along the internal fiber coordinates. Therefore we must *not* confuse the $deformed$ complex gravity constructed here with the noncommutative gravity work in the literature [11] where the spacetime coordinates x^μ are not commuting.

The four fundamental length, momentum, temporal and energy scales are respectively

$$\lambda_l = \sqrt{\frac{\hbar c}{b}}; \quad \lambda_p = \sqrt{\frac{\hbar b}{c}}; \quad \lambda_t = \sqrt{\frac{\hbar}{bc}}; \quad \lambda_e = \sqrt{\hbar bc}. \tag{2}$$

where b is the *maximal* proper force associated with the Born's reciprocal relativity theory. In the natural units $\hbar = c = b = 1$ all four scales become *unity*. The gravitational coupling is given by

$$G \ = \ \frac{c^4}{\mathcal{F}_{max}} \ = \ \frac{c^4}{b}. \tag{3}$$

and the four scales coincide then with the Planck length, momentum, time and energy, respectively and we can verify that

$$\mathcal{F}_{\max} \ = \ m_P \ \frac{c^2}{L_P} \ \sim \ M_{Universe} \ \frac{c^2}{R_H}. \tag{4}$$

it was proposed in [6] that a certain large (Hubble) /small (Planck) scale *duality* was operating in this Born's reciprocal relativity theory reminiscent of the T-duality in string theory compactifications. The Hermitian generators $Z_{ab}, Z_a, \bar{Z}_a, I$ of the $U(1,3)$ algebra and the *deformed* Weyl-Heisenberg algebra obey the relations

$$(Z_{ab})^\dagger \ = \ Z_{ab}; \quad (Z_a)^\dagger \ = \ \bar{Z}_a; \quad I^\dagger = I; \quad a,b = 1,2,3,4. \tag{5}$$

The standard Quaplectic group [5] is given by the semi-direct product of the $U(1,3)$ group and the unmodified Weyl-Heisenberg $H(1,3)$ group : $\mathcal{Q}(1,3) \equiv U(1,3) \otimes_s H(1,3)$ and is defined in terms of the generators $Z_{ab}, Z_a, \bar{Z}_a, I$ with $a,b = 1,2,3,4$.

A careful analysis reveals that the complex generators $Z_a, \bar{Z}_a$ (with Hermitian *and* anti-Hermitian pieces) of the *deformed* Weyl-Heisenberg algebra can be defined in terms of the Hermitian $U(1,4)$ algebra generators Z_{AB} , where $A,B = 1,2,3,4,5$; $a,b = 1,2,3,4$; $\eta_{AB} = diag\ (+,-,-,-,-)$, as follows

$$Z_a \ = \ (-i\ )^{1/2} \ (\ Z_{a5} \ - \ iZ_{5a}\ ); \ \bar{Z}_a \ = \ (\ i\ )^{1/2} \ (\ Z_{a5} \ + \ iZ_{5a}\ ); \ Z_{55} = \frac{\mathcal{I}}{2} \tag{6}$$

the Hermitian generators are $Z_{AB} \equiv \mathcal{E}_A^B$ and $Z_{BA} \equiv \mathcal{E}_B^A$; notice that the position of the indices is very relevant because $Z_{AB} \neq Z_{BA}$. The commutators are

$$[\mathcal{E}_a^b,\ \mathcal{E}_c^d] \ = \ -i\ \delta_c^b\ \mathcal{E}_a^d + i\ \delta_a^d\ \mathcal{E}_c^b; \quad [\mathcal{E}_c^d,\ \mathcal{E}_a^5] \ = \ -i\ \delta_a^d\ \mathcal{E}_c^5; \quad [\mathcal{E}_c^d,\ \mathcal{E}_5^a] \ = \ i\ \delta_c^a\ \mathcal{E}_5^d. \tag{7}$$

and $[\mathcal{E}_5^5,\ \mathcal{E}_5^a] = -i\ \delta_5^5\ \mathcal{E}_5^a$ .... such that now $\mathcal{I}(= 2Z_{55})$ *no* longer commutes with $Z_a, \bar{Z}_a$. The generators Z_{ab} of the $U(1,3)$ algebra can be decomposed into the Lorentz-subalgebra generators L_{ab} and the "shear"-like generators M_{ab} as

$$Z_{ab} \equiv \frac{1}{2}\,(M_{ab} - iL_{ab}); \quad L_{ab} = L_{[ab]} \ = \ i\,(Z_{ab} - Z_{ba}); \quad M_{ab} = M_{(ab)} \ = \ (Z_{ab} + Z_{ba}), \tag{8}$$

one can see that the "shear"-like generators M_{ab} are $Hermitian$ and the Lorentz generators L_{ab} are $anti - Hermitian$ with respect to the fiber internal space indices. The explicit commutation relations of the Hermitian generators Z_{ab} can be rewritten as

$$[L_{ab},\ L_{cd}] \ = \ (\eta_{bc} L_{ad} - \eta_{ac} L_{bd} - \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \qquad (9a)$$

$$[M_{ab},\ M_{cd}] \ = \ -(\eta_{bc} L_{ad} + \eta_{ac} L_{bd} + \eta_{bd} L_{ac} + \eta_{ad} L_{bc}). \qquad (9b)$$

$$[L_{ab},\ M_{cd}] \ = \ (\eta_{bc} M_{ad} - \eta_{ac} M_{bd} + \eta_{bd} M_{ac} - \eta_{ad} M_{bc}). \qquad (9c)$$

Defining $Z_{ab} = \frac{1}{2}(M_{ab} - iL_{ab}),\ Z_{cd} = \frac{1}{2}(M_{cd} - iL_{cd})$ after straightforward algebra it leads to the $U(3,1)$ commutators

$$[\ Z_{ab},\ Z_{cd}\ ] \ = \ -i\ (\ \eta_{bc}\ Z_{ad} \ - \ \eta_{ad}\ Z_{cb}\ ). \qquad (9d)$$

as expected, and which requires that the commutators $[M, M] \sim L$ otherwise one would not obtain the $U(3,1)$ commutation relations (9d) nor the Jacobi identities will be satisfied [1] . The commutators of the (anti-Hermitian) Lorentz boosts generators L_{ab} with the $X_c,\ P_c$ generators are

$$[L_{ab},\ X_c] \ = \ (\ \eta_{bc}\ X_a \ - \ \eta_{ac}\ X_b\ ); \quad [L_{ab},\ P_c] \ = \ (\ \eta_{bc}\ P_a \ - \ \eta_{ac}\ P_b\ ). \ (10a)$$

Since the Hetmitian M_{ab} generators are the *reciprocal* boosts transformations which *exchange* X for P, in addition to boosting (rotating) those variables, one has in

$$[M_{ab},\ \frac{X_c}{\lambda_l}] \ = \ -\ \frac{i}{\lambda_p}\ (\ \eta_{bc}\ P_a + \eta_{ac}\ P_b\ ); \ \ [M_{ab},\ \frac{P_c}{\lambda_p}] \ = \ -\frac{i}{\lambda_l}\ (\ \eta_{bc}\ X_a + \eta_{ac}\ X_b\ ) \qquad (10b)$$

such that upon recurring to eqs-(6, 7) and/or eqs-(10) after lowering indices it leads to [2]

$$[\ Z_{ab},\ Z_c\ ] \ = \ -\ \frac{i}{2}\ \eta_{bc}\ Z_a \ + \ \frac{i}{2}\ \eta_{ac}\ Z_b \ - \frac{1}{2}\ \eta_{bc}\ \bar{Z}_a \ - \ \frac{1}{2}\ \eta_{ac}\ \bar{Z}_b$$

$$[\ Z_{ab},\ \bar{Z}_c\ ] \ = \ -\ \frac{i}{2}\ \eta_{bc}\ \bar{Z}_a \ + \ \frac{i}{2}\ \eta_{ac}\ \bar{Z}_b \ + \ \frac{1}{2}\ \eta_{bc}\ Z_a \ + \ \frac{1}{2}\ \eta_{ac}\ Z_b\ . \qquad (10c)$$

In the noncommutative Yang's phase-space algebra case [10], associated with a noncommutative phase space involving noncommuting spacetime coordinates and momentum x^μ, p^μ, the generator $\mathcal{N}$ which appears in the modified $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}\mathcal{N}$ commutator is the *exchange* operator $x \leftrightarrow p$, $\ [p^\mu, \mathcal{N}] = i\hbar x^\mu/R_H^2$ and

[1] This corrects a typo in [12]

[2] These commutators $differ$ from those in [5] because he chose all generators X, P, M, L to be anti-Hermitian so there are no i terms in the commutators in the r.h.s of eq-(10b) and there are also sign changes

$[x^\mu, \mathcal{N}] = iL_P^2 p^\mu/\hbar$. L_P, R_H are taken to be the minimal Planck and maximal Hubble length scales, respectively. The Hubble upper scale R_H corresponds to a *minimal* momentum $\hbar/R_H$, because by "duality" if there is a minimal length there should be a minimal momentum also.

Yang's [10] noncommutative phase space algebra is isomorphic to the conformal algebra $so(4,2) \sim su(2,2)$ after the correspondence $x^\mu \leftrightarrow L^{\mu 5}$, $p^\mu \leftrightarrow L^{\mu 6}$, and $\mathcal{N} \leftrightarrow L^{56}$. In the *deformed* Quaplectic algebra case, it is in addition to the $\mathcal{I}$ generator, the M_{ab} generator which plays the role of the exchange operator of X with P and which also appears in the *deformed* Weyl-Heisenberg algebra leading to a matrix-valued generalized Planck-constant, and noncommutative fiber coordinates, as follows

$$[\frac{X_a}{\lambda_l}, \frac{P_b}{\lambda_p}] = i\,\alpha_\hbar\,(\eta_{ab}\,\mathcal{I} + M_{ab});\ [X_a, X_b] = -\,(\lambda_l)^2\,L_{[ab]};\ [P_a, P_b] = (\lambda_p)^2\,L_{[ab]}; \tag{11}$$

One could interpret the term $\eta_{ab}\,\mathcal{I} + M_{ab}$ as a matrix-valued Planck constant $\hbar_{ab}$ (in units of $\hbar$). The *deformed* (noncommutative) Weyl-Heisenberg algebra can also be rewritten as

$$[Z_a,\ \bar{Z}_b] = -\alpha_\hbar\,(\,\eta_{ab}\,\mathcal{I} + M_{ab}\,);\quad [Z_a,\ Z_b] = [\bar{Z}_a,\ \bar{Z}_b] = -i\,Z_{[ab]} = -L_{ab}.$$

$$[Z_a,\ \mathcal{I}] = 2\,\bar{Z}_a;\quad [\bar{Z}_a,\ \mathcal{I}] = -\,2\,Z_a;\quad [Z_{ab},\ \mathcal{I}] = 0.\quad \mathcal{I} = 2\,Z_{55}. \tag{12}$$

where $[\frac{X_a}{\lambda_l}, \mathcal{I}] = 2i\frac{P_a}{\lambda_p}$; $[\frac{P_a}{\lambda_p}, \mathcal{I}] = 2i\frac{X_a}{\lambda_l}$ and the metric $\eta_{ab} = (+1, -1, -1, -1)$ is used to raise and lower indices . The Planck constant is given in terms of the length and momentum scales of eq-(2) as $\hbar = \alpha_\hbar\ \lambda_l\ \lambda_p \Rightarrow \alpha_\hbar = 1$, since $\lambda_l \lambda_p = \hbar$. The *deformed* Quaplectic algebra given by eqs-(7-12) obeys the Jacobi identities by virtue of the definitions in eq-(6) along with the commutators of eq-(7). No longer $\mathcal{I}$ commutes with $Z_a, \bar{Z}_a$, it *exchanges* them, as one can see from eq-(12) since $Z_{55} = \mathcal{I}/2$. A Matrix-valued generalized Planck-constant of Noncommutative QM in Clifford Spaces have been advanced by [13].

The *complex* tetrad E_μ^a which transforms under the fundamental representation of $U(1,3)$ is defined as

$$E_\mu^a = \frac{1}{\sqrt{2}}\,(\,e_\mu^a + if_\mu^a\,);\quad \bar{E}_\mu^a = \frac{1}{\sqrt{2}}\,(\,e_\mu^a - if_\mu^a\,). \tag{13}$$

The complex Hermitian metric is given by

$$G_{\mu\nu} = \bar{E}_\mu^a\,E_\nu^b\,\eta_{ab} = g_{(\mu\nu)} + ig_{[\mu\nu]} = g_{(\mu\nu)} + iB_{\mu\nu}. \tag{14}$$

such that

$$(G_{\mu\nu})^\dagger = \bar{G}_{\nu\mu} = G_{\mu\nu};\quad \bar{G}_{\mu\nu} = G_{\nu\mu}. \tag{15}$$

where the bar denotes complex conjugation. Despite that the metric is complex the infinitesimal line element is *real*

$$ds^2 = G_{\mu\nu}\,dx^\mu\,dx^\nu = g_{(\mu\nu)}\,dx^\mu\,dx^\nu,\ \ because\ \ i\,g_{[\mu\nu]}\,dx^\mu\,dx^\nu = 0. \tag{16}$$

The (deformed) Quaplectic-algebra-valued anti-Hermitian gauge field $(\mathbf{A}_\mu)^\dagger = - \mathbf{A}_\mu$ is given by

$$\mathbf{A}_\mu = \Omega_\mu^{ab} Z_{ab} + \frac{i}{L_p} (E_\mu^a Z_a + \bar{E}_\mu^a \bar{Z}_a) + i \Omega_\mu I . \quad (17)$$

where a length scale that we chose to coincide with the the Planck length scale L_P has been introduced in the second terms in the r.h.s since the connection $\mathbf{A}_\mu$ must have units of $(length)^{-1}$. In natural units of $\hbar = c = 1$ the gravitational coupling in $4D$ is $G = L_P^2$. Decomposing the anti-Hermitian components of the connection Ω_μ^{ab} into anti-symmetric $[ab]$ and symmetric (ab) pieces with respect to the internal indices

$$\Omega_\mu^{ab} = \Omega_\mu^{[ab]} + i \Omega_\mu^{(ab)}. \quad (18)$$

gives the anti-Hermitian $U(1,3)$-valued connection

$$\Omega_\mu^{ab} Z_{ab} = (\Omega_\mu^{[ab]} + i \Omega_\mu^{(ab)}) \frac{1}{2}(M_{ab} - i L_{ab}) =$$

$$- \frac{i}{2} \Omega_\mu^{[ab]} L_{ab} + \frac{i}{2} \Omega_\mu^{(ab)} M_{ab} \Rightarrow (\Omega_\mu^{ab} Z_{ab})^\dagger = - \Omega_\mu^{ab} Z_{ab}. \quad (19)$$

since $(Z_{ab})^\dagger = Z_{ab}$

The *deformed* Quaplectic algebra-valued (anti-Hermitian) field strength is given by

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu] =$$

$$F_{\mu\nu}^{ab} Z_{ab} + i (F_{\mu\nu}^a Z_a + \bar{F}_{\mu\nu}^a \bar{Z}_a) + F_{\mu\nu} I =$$

$$\frac{i}{2} F_{\mu\nu}^{(ab)} M_{ab} - \frac{i}{2} F_{\mu\nu}^{[ab]} L_{ab} + i (F_{\mu\nu}^a Z_a + \bar{F}_{\mu\nu}^a \bar{Z}_a) + F_{\mu\nu} I \quad (20)$$

after decomposing $Z_{ab} = \frac{1}{2}(M_{ab} - iL_{ab})$. The components of the curvature two-form associated with the anti-Hermitian connection $\Omega_\mu^{ab} = \Omega_\mu^{[ab]} + i\Omega_\mu^{(ab)}$ are

$$-i F_{\mu\nu}^{[ab]} = \partial_\mu \Omega_\nu^{[ab]} - \partial_\nu \Omega_\mu^{[ab]} + \Omega_{[\mu}^{[ac]} \Omega_{\nu]}^{[cb]} -$$

$$\Omega_{[\mu}^{(ac)} \Omega_{\nu]}^{(cb)} + \frac{1}{L_P^2} E_{[\mu}^a E_{\nu]}^b + \frac{1}{L_P^2} \bar{E}_{[\mu}^a \bar{E}_{\nu]}^b . \quad (21)$$

$$i F_{\mu\nu}^{(ab)} = \partial_\mu \Omega_\nu^{(ab)} - \partial_\nu \Omega_\mu^{(ab)} + \Omega_{[\mu}^{(ac)} \Omega_{\nu]}^{[cb]} + \Omega_{[\mu}^{(bc)} \Omega_{\nu]}^{[ca]} +$$

$$\frac{1}{L_P^2} E_{[\mu}^a \bar{E}_{\nu]}^b + \frac{1}{L_P^2} E_{[\mu}^b \bar{E}_{\nu]}^a \quad (22)$$

where a summation over the repeated c indices is implied and $[\mu\nu]$ denotes the anti-symmetrization of indices with weight one. Notice the presence of the *extra* terms EE in the above expressions for the *deformed* field strengths due to the *noncommutative* $[Z_a, Z_b] \neq 0$, and which in turn, *modifies* the Weyl-Heisenberg algebra due to the Jacobi identities. In the *undeformed* ordinary Quaplectic-algebra case [12] these terms are *absent* because $[Z_a, Z_b] = 0, ...$ and, furthermore, there is *no* M_{ab} term in the ordinary Weyl-Heisenberg algebra. These extra terms EE in eqs-(21,22) are one of the hallmarks of the *deformed* Quaplectic gauge field theory formulation of the *deformed* Born's Reciprocal Complex Gravity.

The components of the torsion two-form are :

$$F^a_{\mu\nu} \ = \ \partial_\mu E^a_\nu \ - \ \partial_\nu E^a_\mu \ - i \ \Omega^{[ac]}_{[\mu} \ E^c_{\nu]} \ + i \ \Omega^{(ac)}_{[\mu} \ \bar{E}^c_{\nu]} \ - 2i \ \bar{E}^a_{[\mu} \ \Omega_{\nu]}. \qquad (23a)$$

$$\bar{F}^a_{\mu\nu} \ = \ \partial_\mu \bar{E}^a_\nu \ - \ \partial_\nu \bar{E}^a_\mu \ + i \ \Omega^{[ac]}_{[\mu} \ \bar{E}^c_{\nu]} \ - i \ \Omega^{(ac)}_{[\mu} \ E^c_{\nu]} \ + 2i \ E^a_{[\mu} \ \Omega_{\nu]}. \qquad (23b)$$

The remaining field strength has roughly the same form as a $U(1)$ field strength in noncommutative spaces due to the additional contribution of $B_{\mu\nu}$ resulting from the nonabelian nature of the Weyl-Heisenberg algebra in the internal space (fibers) and which is reminiscent of the noncommutativity of the coordinates with the momentum :

$$F_{\mu\nu} \ = \ i \ \partial_\mu \Omega_\nu \ - \ i \ \partial_\nu \Omega_\mu \ + \ \frac{1}{L_P^2} \ E^a_\mu \ \bar{E}^b_\nu \ \eta_{ab} \ - \ \frac{1}{L_P^2} \ \bar{E}^a_\mu \ E^b_\nu \ \eta_{ab} \ =$$

$$i \ \partial_\mu \Omega_\nu \ - i \ \partial_\nu \Omega_\mu \ + \ \frac{1}{L_P^2} \ (\ G_{\mu\nu} \ - \ G_{\nu\mu} \ ) \ = \ i \ \Omega_{[\mu\nu]} \ + i \ \frac{2}{L_P^2} \ G_{[\mu\nu]} \qquad (24)$$

after recurring to the commutation relations (for $\alpha_h = 1$) in eqs-(11, 12) and the Hermitian property of the metric

$$G_{\mu\nu} \ = \ \bar{E}^a_\mu \ E^b_\nu \ \eta_{ab} \ = \ [\ E^a_\mu \ \bar{E}^b_\nu \ \eta_{ab} \ ]^* \ = \ (G_{\nu\mu})^* \ \Rightarrow \ (G_{\mu\nu})^* \ = \ G_{\nu\mu}. \qquad (25)$$

where $*$ stands for (bar) complex conjugation.

The curvature tensor is defined in terms of the anti-Hermitian connection $\Omega^{[ab]}_\mu + i \ \Omega^{(ab)}_\mu$ as

$$\mathcal{R}^\rho_{\mu\nu\lambda} \ \equiv \ (\ F^{[ab]}_{\mu\nu} \ + \ i \ F^{(ab)}_{\mu\nu} \ ) \ (E^\rho_a \ E_{b\lambda} \ + \ \bar{E}^\rho_a \ \bar{E}_{b\lambda} \ ). \qquad (26)$$

where the explicit components $F^{[ab]}_{\mu\nu}$ and $F^{(ab)}_{\mu\nu}$ can be read from the defining relations (21,22). Note that both values of values of $F^{[ab]}_{\mu\nu}$ and $F^{(ab)}_{\mu\nu}$ are purely *imaginary* such that one may rewrite the complex-valued $F^{ab}_{\mu\nu}$ field strength as $(\mathcal{F}^{(ab)}_{\mu\nu} + i\mathcal{F}^{[ab]}_{\mu\nu})$ for real valued $\mathcal{F}^{(ab)}_{\mu\nu}, \ \mathcal{F}^{[ab]}_{\mu\nu}$ expressions. The contraction of indices yields *two* different complex-valued (Hermitian) Ricci tensors [3] given by

$$\mathcal{R}_{\mu\lambda} \ = \ g^{\sigma\nu} \ g_{\rho\sigma} \ R^\rho_{\mu\nu\lambda} \ = \ \delta^\nu_\rho \ R^\rho_{\mu\nu\lambda} \ = \ R_{(\mu\lambda)} \ + \ i \ R_{[\mu\lambda]}; \quad (\mathcal{R}_{\mu\lambda})^* \ = \ \mathcal{R}_{\lambda\mu} \qquad (27)$$

[3] There is a third Ricci tensor $Q_{[\mu\nu]} = \mathcal{R}^\rho_{\mu\nu\lambda}\delta^\lambda_\rho$ related to the curl of the *nonmetricity* Weyl vector Q_μ [14] and which we set to zero

and

$$\mathcal{S}_{\mu\lambda} \ = \ g^{\sigma\nu} \ g_{\sigma\rho} \ R^{\rho}_{\mu\nu\lambda} \ = \ \mathcal{S}_{(\mu\lambda)} \ + \ i \ \mathcal{S}_{[\mu\lambda]}; \quad (\mathcal{S}_{\mu\lambda})^* \ = \ \mathcal{S}_{\lambda\mu} \tag{28}$$

due to the fact that

$$g^{\sigma\nu} \ g_{\rho\sigma} \ = \ \delta^{\nu}_{\rho} \ \ and \ \ g^{\sigma\nu} \ g_{\sigma\rho} \ \neq \ \delta^{\nu}_{\rho}. \tag{29}$$

because $g_{\sigma\rho} \neq g_{\rho\sigma}$. The position of the indices is crucial.

A further contraction yields the generalized (real-valued) Ricci scalars

$$\mathcal{R} \ = \ (g^{(\mu\lambda)} \ + i \ g^{[\mu\lambda]}) \ (\ R_{(\mu\lambda)} \ + \ i \ R_{[\mu\lambda]} \ ) =$$

$$\mathcal{R} \ = \ g^{(\mu\lambda)} \ R_{(\mu\lambda)} \ - \ B^{\mu\lambda} \ R_{[\mu\lambda]}; \quad g^{[\mu\lambda]} \equiv B^{\mu\lambda}. \tag{30a}$$

$$\mathcal{S} \ = \ (g^{(\mu\lambda)} \ + i \ g^{[\mu\lambda]}) \ (\ S_{(\mu\lambda)} \ + \ i \ S_{[\mu\lambda]} \ ) =$$

$$\mathcal{S} \ = \ g^{(\mu\lambda)} \ \mathcal{S}_{(\mu\lambda)} \ - \ B^{\mu\lambda} \ \mathcal{S}_{[\mu\lambda]}. \tag{30b}$$

The first term $g^{(\mu\lambda)} \ R_{(\mu\lambda)}$ corresponds to the usual scalar curvature of the ordinary Riemannian geometry. The presence of the extra terms $B^{\mu\lambda} \ R_{[\mu\lambda]}$ and $B^{\mu\lambda} \ \mathcal{S}_{[\mu\lambda]}$ due to the anti-symmetric components of the metric and the two different types of Ricci tensors are one of the hallmarks of the deformed Born complex gravity. We should notice that the inverse complex metric is

$$g^{(\mu\lambda)} \ + \ ig^{[\mu\lambda]} \ = \ [\ g_{(\mu\nu)} \ + \ ig_{[\mu\nu]} \ ]^{-1} \ \neq \ (g_{(\mu\nu)})^{-1} \ + \ (ig_{[\mu\nu]})^{-1}. \tag{31}$$

so $g^{(\mu\nu)}$ is now a complicated expression of both $g_{\mu\nu}$ *and* $g_{[\mu\nu]} = B_{\mu\nu}$. The same occurs with $g^{[\mu\nu]} = B^{\mu\nu}$. Rigorously we should have used a different notation for the inverse metric $\tilde{g}^{(\mu\lambda)} \ + \ i\tilde{B}^{[\mu\lambda]}$, but for notational simplicity we chose to drop the tilde symbol.

One could add an *extra* contribution to the complex-gravity real-valued action stemming from the terms $iB^{\mu\nu}F_{\mu\nu}$ which is very reminiscent of the BF terms in Schwarz Topological field theory and in Plebanksi's formulation of gravity. In the most general case, one must include both the contributions from the torsion and the $i \ B^{\mu\nu}F_{\mu\nu}$ terms. The contractions involving $G^{\mu\nu} = g^{(\mu\nu)} + iB^{\mu\nu}$ with the components $F_{\mu\nu}$ (due to the antisymmetry property of $F_{\mu\nu} = -F_{\nu\mu}$) lead to

$$i \, B^{\mu\nu} \ F_{\mu\nu} \ = \ - \ B^{\mu\nu} \ (\ \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu \ ) - 2 \ B^{\mu\nu} \ B_{\mu\nu} \ = \ - \ B^{\mu\nu} \ \Omega_{\mu\nu} - 2 \ B^{\mu\nu} \ B_{\mu\nu} \ . \tag{32}$$

where we have set the length scale $L_P = 1$ for convenience. These BF terms contain a mass-like term for the $B_{\mu\nu}$ field. Mass terms for the $B_{\mu\nu}$ and a *massive* graviton formulation of bi-gravity (in addition to a massles graviton) based on a $SL(2, C)$ gauge formulation have been studied by [14], [15], [16],. When the torsion is not constrained to vanish one must include those contributions as well. The real-valued torsion two-form is $(F^a_{\mu\nu} Z_a + \bar{F}^a_{\mu\nu} \bar{Z}_a) dx^\mu \wedge dx^\nu$ and the torsion tensor and torsion vector are

$$T^{\rho}_{\mu\nu} \ \equiv \ (F^a_{\mu\nu} \ E^{\rho}_a + \bar{F}^a_{\mu\nu} \ \bar{E}^{\rho}_a); \quad T_{\mu\nu\rho} \ = \ g_{\rho\sigma} \ T^{\sigma}_{\mu\nu}; \quad T_\mu \ = \ \delta^{\nu}_{\rho} \ T^{\rho}_{\mu\nu}. \tag{33}$$

The (real-valued) action, linear in the two (real-valued) Ricci curvature scalars and quadratic in the (real-valued) torsion is of the form

$$\frac{1}{2\kappa^2}\int_{M_4} d^4x \sqrt{\mid det\ (g_{(\mu\nu)} + iB_{\mu\nu})\mid}\,(\,a_1\,\mathcal{R} + a_2\,\mathcal{S} + a_3\,T_{\mu\nu\rho}\,T^{\mu\nu\rho} + a_4\,T_\mu\,T^\mu + complex\ conjugate). \tag{34}$$

where $\kappa^2 = 8\pi G$ is the gravitational coupling and in natural units $\hbar = c = 1$ one has $G = L^2_{Planck}$. We may add the BF terms (32) to the action (34) as well [4]. The action (34) is invariant under infinitesimal $U(1,3)$ gauge transformations of the complex tetrad $\delta E^a_\mu = (\xi^a_{b(1)} + i\xi^a_{b(2)})E^b_\mu$ where the real $\xi^{(1)}_{[ab]}$ and imaginary $\xi^{(2)}_{(ab)}$ components of the complex parameter are anti-symmetric and symmetric, respectively, with respect to the indices a, b for anti-Hermitian infinitesimal $U(1,3)$ gauge transformations.

The a_1, a_2, a_3, a_4 are suitable numerical coefficients that will be constrained to have certain values if one wishes to avoid the presence of ghosts, tachyons and higher order poles in the propagator, not unlike it occurs in Moffat's nonsymmetric gravity theory [14]. The instabilities of Moffat's nonsymmetric gravity found by [15] are bypassed when one extends the theory to spacetimes with *complex* coordinates [16]. The action (34) defined in $4D$ can be extended to a $4D$ *complex* spacetime; i.e. an action in $8D$ real-dimensional Phase Space associated with the cotangent bundle of spacetime. The geometry of curved Phase spaces and bounded complex homogeneous domains has been studied by [17]. The presence of matter sources can be incorporated, for example, by recurring to the invariant action for a point-particle in Born's Reciprocal Relativity involving Casimir group invariant quantities associated with the world-line of the particle. The quantization of a point-particle corresponding to the undeformed Quaplectic group is far richer than the ordinary Poincare case since acceleration boosts can change the spin of the particle. The spectrum contains towers of integer massive spin states, as well as unconventional massless representations [7].

To conclude, we should emphasize that the complex deformed Born Reciprocal Gravitational theory advanced here $differs$ from the modified gravitational theories in the literature [14], [16], [18], and it is mainly due to the fact that we have constructed a deformed complex Born's reciprocal gravitational theory in $4D$ as a gauge theory of the $deformed$ Quaplectic group given by the semidirect product of $U(1,3)$ with the $deformed$ (noncommutative) Weyl-Heisenberg algebra of eqs-(11, 12a, 12b). The deformed Weyl-Heisenberg algebra already encodes the noncommutativity of the fiber coordinates such that $Z_\mu(w^i) = E^a_\mu(w^i)\ Z_a$ and $\bar{Z}_\mu(w^i) = \bar{E}^a_\mu(w^i)\ \bar{Z}_a$ could be interpreted as the p-brane *noncommutative* target complex-spacetime background embedding functions $Z_\mu(w^i), \bar{Z}_\mu(w^i)$ in terms of the $p+1$ world-volume coordinates w^i ($i = 1, 2, ..., p+1$). Since the vielbein E^a_μ is required in the definition of the embedding coordinates $Z_\mu, \bar{Z}_\mu$, it is not surprising to see why string-theory (p-

[4] Yang-Mills types of actions $F \wedge^* F$ can also be included

branes) encodes gravity. For plausible relations between nonsymmetric gravity and string theory see [14], [16], [19] and references therein. Noncommutative p-branes actions based on Moyal-Yang-Kontsevich star products with a lower and upper length scales were constructed in [20]. Finally, gravitational theories based on Born's reciprocal relativity principle [12], involving a maximal speed limit and a maximal proper force, is a very promising avenue to quantize gravity that does *not* rely in *breaking* the Lorentz symmetry at the Planck scale, in contrast to other approaches based on deformations of the Poincare algebra, Hopf algebras, quantum groups, etc...

Acknowledgments

We are indebted to M. Bowers for assistance, to Howard Brandt for providing further references in [1] pertaining Born's Reciprocal Relativity and Stephen Low and Robert Delbourgo for discussions.

References

[1] M. Born, Proc. Royal Society **A 165**, 291 (1938). Rev. Mod. Physics **21**, 463 (1949). *The Born-Einstein Letters* (Walker, NY, 1971) pp. 131-135.

[2] H. Brandt, Found. Phys. Lett **2**, 39 (1989). Contemporary Mathematics **196**, 273 (1996). Chaos, Solitons and Fractals **10**, No. 2-3, 267 (1999).

[3] E. Caianiello, "Is there a maximal acceleration?", Lett. Nuovo Cimento **32**, 65 (1981).

[4] S. Vacaru, P. Stavrinos, E. Gaburov, and D. Gonta, *"Clifford and Riemann-Finsler Structures in Geometric Mechanics and Gravity"*, Selected Works Differential Geometry-Dynamical Systems Monograph (Geometry Balkan Press, 2006).

[5] S. Low: Jour. Phys **A** Math. Gen **35**, 5711 (2002). J. Math. Phys. **38**, 2197 (1997). S. Low, "Hamilton Relativity Group for Noninertial States In Quantum Mechanics" [arXiv : math-ph/0710.3599]. S. Low, "Reciprocally Relativity of Noninertial Frames : Quantum Mechanics" [arXiv : math-ph/0606015].

[6] C. Castro, Foundations of Physics **35**, No.6, 971 (2005). Int J. Mod Phys **A 18**, No. 29, 5445 (2003).

[7] J. Govaerts, P. Jarvis, S. Morgan and S. Low, " Worldline Quantization of a Reciprocally Invariant System" arXiv : 0706.3736. P. Jarvis and S. Morgan, "Constraint Quantization of a worldline system invariant under reciprocal Relativity, II " arXiv.org : 0806.4794.

[8] P. Jarvis and S. Morgan, "Born reciprocity and the granularity of spacetime" [arXiv : math-ph/0508041].

[9] R. Delbourgo and D. Lashmar, "Born reciprocity and the $1/r$ potential" hep-th/0709.0776.

[10] C Yang, Phys. Rev **72** (1947) 874.

[11] P. Achieri, M. Dimitrijevic, F. Meyer and J. Wess, "Noncommutative Geometry and Gravity" [arXiv : hep-th/0510059] and references therein.

[12] C. Castro, Int. J. Mod. Phys **A 23**, (2008) 1487.

[13] C. Castro, J. Phys **A** : Math. Gen **39** (2006) 14205.

[14] J. Moffat, J. Math. Phys **36**, no. 10 (1995) 5897. J. Moffat and D. Boal, Phys. Rev **D 11**, 1375 (1975). J. Moffat, "Noncommutative Quantum Gravity" arXiv : hep-th/0007181. N. Poplawski, " On Nonsymmetric purely affine Gravity" gr-qc/0610132,

[15] T. Damour, S. Deser and J. McCarthy, Phys. Rev **D 47**, 1541 (1993). T. Jansen and T. Prokopec, " Problems and Hopes in Nonsymmetric Gravity" arXiv : gr-qc/0611005. C. Mantz and T. Prokopec, " Resolving Curvature Singularities in Holomorphic Gravity" arXiv : 0805.4721.

[16] A. Chamseddine, Comm. Math. Phys **218**, 283 (2001). " Gravity in Complex Hermitian Spacetime" arXiv : hep-th/0610099.

[17] E. Coquereaux and A. Jadczyk, Rev. Math. Phys **2**, No. 1, 1 (1990).

[18] A. Einstein, Ann. Math **46**, 578 (1945). A. Einstein and E. Strauss, Ann. Math **47**, 731 (1946).

[19] J. A. Nieto and J. Socorro, "Is nonsymmetric gravity related to string theory?", arXiv: hep-th/9610160.

[20] C. Castro, Phys. Letts **B 626** (2005) 209.