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ON THE RIEMANN HYPOTHESIS AND TACHYONS IN DUAL STRING SCATTERING AMPLITUDES

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It is the purpose of this work to pursue a novel physical interpretation of the nontrivial Riemann zeta zeros and prove why the *location* of these zeros $z_n = 1/2 + i\lambda_n$ corresponds *physically* to tachyonic-resonances/tachyonic-condensates, originating from the scattering of two on-shell *tachyons* in bosonic string theory. Namely, we prove that if there were nontrivial zeta zeros (violating the Riemann hypothesis) outside the critical line *Real* $z = 1/2$ (but inside the critical strip), these putative zeros *do not* correspond to any *poles* of the bosonic open string scattering (Veneziano) amplitude $A(s, t, u)$. The *physical* relevance of tachyonic-resonances/tachyonic-condensates in bosonic string theory, establishes an important connection between string theory and the Riemann Hypothesis. In addition, one has also a *geometrical* interpretation of the zeta zeros in the critical line in terms of very special (degenerate) triangular configurations in the upper-part of the complex plane.

Keywords: Riemann hypothesis; strings; tachyon; Veneziano amplitude.

1. Introduction

The Riemann's hypothesis (RH) [1,2] states that the nontrivial zeros of the Riemann zeta-function are of the form $z_n = 1/2 + i\lambda_n$. Trivial zeta zeros exist at $z_n = -2n$, for $n =$ *integer*. Most recently a Fractal Supersymmetric Quantum Mechanical (SUSY-QM) model implementing the Hilbert–Polya proposal to prove the RH has been constructed [3]. We provided a Hermitian operator that reproduces all the λ_n for its spectrum based on quantum inverse scattering methods that were related to a fractal potential given by a Weierstrass function (continuous but nowhere differentiable) and applied to the fractal analog of the Comtet–Bandrauk–Campbell (CBC) formula in SUSY QM. It required using suitable fractal derivatives and integrals of irrational order, whose parameter β is one-half the fractal dimension ($D = 1.5$) of the Weierstrass function, which is the fractal dimensions that furnishes a $1/f$ noise in the power spectrum. For previous work related to [3] see [4, 5, 19]. We refer to the number theory website [6] for numerous articles related to the zeta function.

The QM of a particle moving in the hyperbolic plane was studied in 1975 [7], and the scattering matrix s-wave amplitudes for scattering in the Poincaré disk could be expressed in the form [8]:

$$S = \frac{c(k)}{c(-k)} = \frac{\zeta(ik)\zeta(1-ik)}{\zeta(1+ik)\zeta(-ik)} = e^{i2\delta_0(k)} \tag{1}$$

where $c(k)$ are the Harish-Chandra c-functions (Jost functions). The Jost functions are defined whether the space is symmetric or not, and whether a suitable potential is introduced or not. One may notice that when k is real-valued the numerator of Eq. (1) is the complex conjugate of the denominator and for this reason one can write $S(k)$ as a pure phase factor as indicated in the r.h.s. However, when k is complex-valued this is no longer the case and $S(k)$ is no longer given by a pure phase factor. For example, the complex poles of $S(k)$ correspond to the zeros of the zeta functions in the denominator and to their poles in the numerator. s-wave scattering by a potential with a cutoff have been recently studied in [9], where the complex zeros of the Jost functions yield the complex poles of the S-matrix that are located on a horizontal line (below the real axis) and which can be mapped into the critical line of zeros of the Riemann zeta function. They represent resonances. For example, in the case of s-wave scattering in the hyperbolic plane (Poincaré disk) one can show that the complex-poles of the S-matrix correspond to the nontrivial zeros when,

$$k_n = i(1/2 + i\lambda_n). \tag{2}$$

Hence, a Wick rotation of the Riemann critical line yields the complex momenta associated with the double poles of the S-matrix above; i.e. the double zeros of the denominator. If one could find a *physical* reason why the complex double poles of the S-matrix should always occur in complex–conjugate *pairs*:

$$-ik_n = (1 + ik_n)^* = 1 - ik_n^* \Rightarrow k_n = i(1/2 + i\lambda_n). \tag{3}$$

this result (3) would automatically imply that the poles of S are just the Wick-rotations of the nontrivial Riemann zeta zeros. Therefore, one would have found a *physical proof* of the RH by establishing a one-to-one correspondence between the poles of the S matrix and the nontrivial zeta zeros via the Wick-rotation described by Eq. (3): the poles are just i times the nontrivial zeta zeros. Notice that the poles of S in Eq. (1) are $k_n = i(1/2 + \lambda_n)$ and their complex conjugates while the zeros are $k_n = -i(1/2 + \lambda_n)$ and their complex conjugates.

Complex extensions of QM have captured a lot of attention recently [10] in particular PT-pseudo QM involving non-Hermitian Hamiltonians with the most salient feature that the energy eigenvalues are still real or they must appear in complex–conjugate pairs. This latter feature will be the sought-after requirement behind Eq. (3) leading to a physical proof of the RH.

Another interesting feature of the unitarity condition of the S matrix [11] $S(k)[S(k^*)]^* = 1$ is that it must remain invariant under CPT discrete

transformations. Equation (1) clearly obeys the unitarity condition that within the context of the RH would imply that the charge conjugation C operation corresponds to the complex conjugation k goes to k^*. The parity P operation now corresponds to the reflection symmetry with respect to the center of symmetry $1/2 + i0$ of the Riemann fundamental function $Z(s) \equiv \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, which obeys the functional relation $Z(s) = Z(1-s)$. And, finally, the time reversal operation T, corresponds to a mirror symmetry w.r.t. the critical line of nontrivial zeta zeros $1/2 + i\lambda$.

Pigli [12] has discussed why scattering theory on real and p-adic systems involving the Riemann zeta function belong to a wide class of integrable models that can be unified into an Adelic integrable systems whose S-matrix involves the Dirichlet, Langlands, Shimura, and L-functions.

Motivated by the fact that the *trivial* zeta zeros lie in the negative even real axis, $-2n$, and that there are physical poles of the Veneziano amplitude in the negative real axis (at $-n$, twice as many poles than trivial zeta zeros), it raises the question whether the location of the *nontrivial* zeta zeros along the critical line bear a similar physical significance. It is the purpose of this work to prove why the *location* of the nontrivial zeta zeros correspond *physically* to tachyonic resonances (tachyonic condensates) associated with bosonic string scattering amplitudes. We also prove that if there were zeta zeros which violate the RH, outside the critical line *Real* $z = 1/2$ but inside the critical strip, these putative zeros *do not* correspond to any *poles* of the bosonic string scattering amplitude $A(s,t,u)$. In addition, we also have a *geometrical* interpretation of the zeta zeros in the critical line in terms of very special (degenerate) triangular configurations in the upper-part of the complex plane.

2. The Physical Interpretation of the Nontrivial Zeta Zeros in Terms of Tachyonic String Poles

The four-point dual string amplitude obtained by Veneziano [13, 14] was

$$A_4 = A(s,t) + A(t,s) + A(u,s) = \int_R \mathrm{d}x|x|^{\alpha-1}|1-x|^{\beta-1} = B(\alpha,\beta). \tag{4}$$

where the Regge trajectories in the respective s, t, u channels are:

$$-\alpha(s) = 1 + \frac{1}{2}s, \quad -\beta(t) = 1 + \frac{1}{2}t, \quad -\gamma(u) = 1 + \frac{1}{2}u. \tag{5}$$

The conservation of the energy-momentum yields:

$$k_1 + k_2 = k_3 + k_4 \Rightarrow k_1 + k_2 - k_3 - k_4 = 0. \tag{6}$$

In our notation, we define the different channels as:

$$s = (k_1 + k_2)^2, \quad t = (k_2 - k_3)^2, \quad u = (k_1 - k_3)^2. \tag{7}$$

Next we will prove that the sum

$$s+t+u=2(k_1^2+k_2^2+k_3^2)+2(k_1\cdot k_2-k_2\cdot k_3-k_1\cdot k_3)=-8 \tag{8}$$

in mass units of $m_{\rm Planck}=1$, when all the four particles are tachyons one has the on-shell condition:

$$k_1^2=k_2^2=k_3^2=m^2=-2m_{\rm Planck}^2=-2 \tag{9}$$

in the natural units $\hbar=c=G=1\Rightarrow L_{\rm Planck}=1$ such that the string slope parameter in those units is given by $\alpha'=(1/2)L_{\rm Planck}^2=1/2$ and the string mass spectrum is quantized in multiples of the Planck mass $m_{\rm Planck}=1$.

From the conservation of energy-momentum (6) and the tachyon on-shell condition Eq. (9) one can deduce that:

$$(k_1+k_2)^2=(k_3+k_4)^2\Rightarrow k_1\cdot k_2=k_3\cdot k_4. \tag{10}$$

Therefore, from Eqs. (8)–(10) it is straightforward to show:

$$\begin{aligned} s+t+u&=2(-2-2-2)+2(k_1{\cdot}k_2-k_3{\cdot}(k_1+k_2))\\ &=-12+2(k_1{\cdot}k_2-k_3{\cdot}(k_3+k_4)=-12+2(k_1{\cdot}k_2-k_3{\cdot}k_4-k_3{\cdot}k_3)\\ &=-12-2k_3{\cdot}k_3=-12+4=-8. \end{aligned} \tag{11}$$

This relationship among $s+t+u=4m^2=-8$ will be crucial in what follows next. From Eqs. (5), (8), and (11) we learn that:

$$\alpha+\beta+\gamma=1. \tag{12}$$

The relationship given by Eq. (12) can also be understood geometrically as the sums of the angles, in units of π, of an Euclidean triangle found in [15] where new relations among analyticity, Regge trajectories, the Veneziano string amplitudes, and Moebius transformations were studied. Note that the author [15] uses a *different* convention for α, β and γ than ours.

There exists a well-known relation [13] among the Γ functions in terms of ζ functions appearing in the expression for $A(s,t,u)$ when α, β fall *inside* the critical strip. In this case, the integration region in the real line that defines $A(s,t,u)$ in Eq. (4) can be divided into three parts and leads to the very important identity

$$\begin{aligned} A(s,t,u)=B(\alpha,\beta)&=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}+\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}+\frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}\\ &=\frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(1-\beta)}{\zeta(\beta)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)} \end{aligned} \tag{13}$$

where $\alpha+\beta+\gamma=1$ and α, β are confined to the interior of the critical strip.

The derivation behind Eq. (13) relies on the condition $\alpha+\beta+\gamma=1$ Eq. (12) and the identities

$$\sin\pi(\alpha+\beta)+\sin\pi(\alpha+\gamma)+\sin\pi(\beta+\gamma)=4\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}, \tag{14a}$$

$$\Gamma(\gamma)=\Gamma(1-\alpha-\beta)=\frac{1}{\Gamma(\alpha+\beta)}\frac{\pi}{\sin\pi(\alpha+\beta)} \tag{14b}$$

plus the remaining cyclic permutations from which one can infer

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin\pi(\alpha+\beta)}{\pi} \tag{14c}$$

$$\frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin\pi(\alpha+\gamma)}{\pi} \tag{14d}$$

$$\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\frac{\sin\pi(\beta+\gamma)}{\pi}. \tag{14e}$$

Therefore, Eq. (14) allow us to recast the l.h.s. of (13) as

$$A(s,t,u) = B(\alpha,\beta) = \frac{4}{\pi}\cos\frac{\pi\alpha}{2}\cos\frac{\pi\beta}{2}\cos\frac{\pi\gamma}{2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma). \tag{15a}$$

And, finally, the known functional relation

$$(2\pi)^z\zeta(1-z) = 2\cos\frac{\pi z}{2}\Gamma(z)\zeta(z) \tag{15b}$$

in conjunction with the condition $\alpha+\beta+\gamma = 1$ such that $(2\pi)^{\alpha+\beta+\gamma} = 2\pi$ is what establishes the important identity (13) expressing explicitly the string amplitude $A(s,t,u)$ either in terms of zeta functions or in terms of Γ functions.

We will prove below why the location of the Riemann critical line of zeta zeros given by the complex numbers $\alpha = 1/2+i\lambda$ does correspond to a *real*-valued pole of the scattering amplitude $A(s,t,u)$ if one permits complex-valued energy-momenta and angular-momenta. It is well known that the imaginary parts of the energies in scattering theory corresponds to the inverse lifetime of particle-resonances. The resonance-width is the inverse of the lifetime.

We will show in the appendix that by *cyclic* symmetry one may take the case $\alpha = 0$, which corresponds to a *tachyonic* pole in the s-channel, such that $s = -2$ and $\beta = \gamma^* = 1/2+i\lambda$. One can always choose k_1, k_2 to be *real*-valued and obeying the standard *tachyonic* on-shell condition $(k_1)^2 = (k_2)^2 = -2$ (associated with the ground state of bosonic open strings). Once a *tachyon* is produced in the s-channel, resulting from the scatttering of two incoming *tachyons*, it decays afterwards into the *tachyonic condensate* state made out of two tachyonic-resonances associated with $|k_3>$, $k_4>$, respectively, obeying $k_3 = k_4^*$ (a complex–conjugate *pair* of values).

It is straightforward to show (see appendix) that k_3, k_4 obey indeed the tachyonic-resonance conditions $\mathcal{R}e(k_3)^2 = \mathcal{R}e(k_4)^2 < 0$ and the "particle-antiparticle" tachyonic-condensate condition associated with the imaginary parts $\mathcal{I}m(k_3)^2 = -\mathcal{I}m(k_4)^2$. This *tachyonic condensate* (a "particle-antiparticle" pair of tachyonic resonances) can in fact be produced after the scattering of two on-shell ordinary *tachyons*.

Tachyonic-resonances and/or tachyonic-condensates posses deep physical properties, have profound consequences, and have been studied by numerous authors in the past and today. See, for example, the work in [5, 16, 17]. Similar arguments can be made in the closed bosonic string case since Eq. (13) also applies to the closed bosonic string [18].

It is well known that the bosonic string spectrum is related to *poles* of $A(s,t,u)$ in the diverse channels. For example, by *duality* one can write the $A(s,t,u)$ as an infinite sum over poles in the s channel or an infinite sum over the poles in the t-channel leading to the ordinary Regge trajectories with real-valued energies and angular-momenta. For example, when $\alpha = -n$ for $n > 0$, the leading Regge trajectory in the s-channel is given by $J = -\alpha(s) = n = 1 + \frac{1}{2}s$. The open-bosonic string ground state is tachyonic since it corresponds to $s = -2$, $J = 0$.

There is however, a very special case when the values of α, β, γ are complex-valued, in particular, the cases $\mathcal{R}e(\alpha+\beta) = 1$ (or any cyclic permutation in α, β, γ) that has not been studied before (to our knowledge) leading to very specific poles of $A(s,t,u)$, which are associated with the nontrivial zeta zeros. It is the most interesting case that we shall study next when α, β are confined to the critical strip: $0 \leq \mathcal{R}e\, z \leq 1$.

Having reviewed the main formulae for the four-point dual-string scattering amplitude associated with four tachyons in terms of ζ functions, we shall find a physical interpretation of the location of the Riemann critical line $1/2 + i\lambda$ of the complex plane in terms of a *real*-valued pole of $A(s,t,u)$ in the tachyonic u-channel $u = (k_1 - k_3)^2 = -2$. In order to show this, one needs to *relax* the condition imposed on each of the individual energy-momentum variables $k_i^2 = -2$, $i = 1, 2, 3, 4$, while still maintaining the crucial relation $s+t+u = -8$ *intact* at the *expense* of performing an analytical continuation in the energy-momentum and angular-momentum; i.e., our energy-momentum and angular-momentum variables are now *complex*-valued.

We will focus next on the very special case $\alpha + \beta = 1$ for α, β complex-valued, and confined to the critical strip, that furnishes a very special class of poles of the string scattering amplitudes $A(s,t,u)$, which are related to the nontrivial zeta zeros. When $\alpha + \beta = 1 \Rightarrow \beta = 1 - \alpha$, it leads to:

$$\alpha + \beta + \gamma = 1 + \gamma = 1 \Rightarrow \gamma = 0 \tag{16}$$

and shall allow to show there is a *real*-valued *pole* of the amplitude $A(s,t,u)$ resulting from an explicit cancellation of certain numerators with the denominators in the products of the ζ's in the r.h.s. of Eq. (13), yielding one remaining ζ factor involving $\gamma = 0$:

$$A(s,t,u) = \frac{\zeta(1-\alpha)}{\zeta(\alpha)}\frac{\zeta(\alpha)}{\zeta(1-\alpha)}\frac{\zeta(1-\gamma)}{\zeta(\gamma)} = \frac{\zeta(1-\gamma)}{\zeta(\gamma)} = \frac{\zeta(1)}{\zeta(0)} = -\infty \tag{17}$$

since $\zeta(0) = -1/2$.

If the $-\infty$ pole of the r.h.s. of $A(s,t,u)$ in Eqs. (13) and (17) corresponds to the poles of the Γ factors in the l.h.s. of Eqs. (13) and (17), we must choose the value of $\Gamma(0)$ to be $\Gamma(0) = -\infty$ and disregard the other value $\Gamma(0) = +\infty$ since the Γ function has a discontinuity at the poles, there is a jump from $+\infty$ to $-\infty$ and vice versa when one hits a pole of the Γ function located at negative integers $-n$.

We shall show next why in the case $\alpha + \beta = 1$, for those values of α, β lying *inside* the critical strip, there is a *real* valued pole of $A(s,t,u)$ in the u-channel *if*

and only *if* $\alpha = 1/2 + i\lambda$. Notice that due to a *cyclic* symmetry of $A(s,t,u)$ in the s, t, u variables, one could have taken instead the condition $\beta + \gamma = 1 \Rightarrow \alpha = 0$ and arrive at a pole in the s-channel instead of a pole in the u-channel. Similarly, we could have imposed $\alpha + \gamma = 1 \Rightarrow \beta = 0$ and arrive at a pole in the t-channel instead. Hence, for example, by choosing $\alpha + \beta = 1$, $\gamma = 0$ in Eq. (13) it leads to

$$A(s,t,u) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} + \Gamma(\gamma) + \Gamma(\gamma) = \frac{\zeta(1-\gamma)}{\zeta(\gamma)}. \tag{18}$$

If there is a *real*-valued pole in the u-channel (when $\gamma = 0$) of the dual string amplitude $A(s,t,u)$ given by $\zeta(1-\gamma)/\zeta(\gamma) = \zeta(1)/\zeta(0) = -\infty$, then we must have:

$$A(s,t,u) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} + 2\Gamma(0) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} - \infty = \frac{\zeta(1)}{\zeta(0)} = -\infty. \tag{19}$$

Due to the fact that the values of α, β are *confined* to live *inside* the critical, one arrives at the conditions

$$\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = \frac{\pi}{\sin(\pi\alpha)} = \textit{real, finite and positive definite} \tag{20a}$$

therefore, Eq. (20a) *cannot* be negative, in particular it *cannot* be equal to $-\infty$. Since the pole in the r.h.s. of (19) is *real*-valued, we can infer that the l.h.s. of (19) must be *real*-valued (and $-\infty$) as well.

Upon using the real, finite, and positive-definiteness conditions of Eq. (20a), one concludes that the only *nontrivial* values of α, confined to the interior of the critical strip) obeying Eqs. (19) and (20a), must be of the form

$$\alpha = \frac{1}{2} + i\lambda, \quad \beta = 1 - \alpha = \frac{1}{2} - i\lambda, \quad \gamma = 0. \tag{20b}$$

which is the desired sought-after result, and *not* just a plain remark. Concluding, the values of α must live in the Riemman critical line.

In deriving Eq. (20b) from the conditions of Eq. (20a), we have used $\alpha = \alpha_x + i\alpha_y$ in the identity

$$\begin{aligned} \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} &= \frac{\pi}{\sin(\pi\alpha)} = \frac{\pi}{\sin(\pi\alpha_x + i\pi\alpha_y)} \\ &= \frac{\pi}{\sin(\pi\alpha_x)\cosh(\pi\alpha_y) + i\,\cos(\pi\alpha_x)\sinh(\pi\alpha_y)} \end{aligned} \tag{20c}$$

and

$$\cos(i\pi\alpha_y) = \cosh(\pi\alpha_y). \quad \sin(i\pi\alpha_y) = i\,\sinh(\pi\alpha_y). \tag{20d}$$

The *finiteness* condition of Eq. (20a) is due to the fact that α is located *inside* the critical strip and this will exclude solutions like $\alpha = \pm n$, $n > 1$ generating unwanted poles in Eq. (20a). Naturally, there are *trivial* solutions to Eq. (20c) given by $\alpha_y = 0 \Rightarrow \alpha = \textit{real}$, which can be disregarded since we know that there are *no* real-valued zeta zeros *inside* the critical strip. There are trivial real-valued zeta zeros only at $-2n$, for $n = \textit{integer}$. Furthermore, the Hadamard-Valle de la

Poussin theorem excludes zeros in the boundaries of the critical strip $\alpha = 0 + i\lambda$ and $\alpha = 1 + i\lambda$.

The value

$$-\gamma = 0 = \left(1 + \frac{1}{2}u\right) = 1 + \frac{1}{2}(k_1 - k_3)^2 = 0 \tag{21}$$

is associated with a unique real-valued pole of $A(s,t,u)$ that corresponds to a *tachyon* exchanged in the u-channel:

$$u = (k_1 - k_3)^2 = -2m^2_{\text{Planck}} = -2 \tag{22}$$

with a *zero* angular momentum $-J = \gamma = 0$.

Concluding, in the very special case $\alpha + \beta = 1$, for those values of α, β lying inside the critical strip, we have found that there is *a* single *real*-valued pole $A(s,t,u) = -\infty$ of the four-point open bosonic string amplitude $A(s,t,u)$ that is associated with a *tachyon* exchanged in the u-channel $u = (k_1 - k_3)^2 = -2m^2_{\text{Planck}} = -2$, with zero angular-momentum $J = -\gamma = 0$, if, and only if, the values of α are $\alpha = 1/2 + i\lambda$ (the location of the Riemann critical line of nontrivial zeta zeros). Therefore, one has found a *physical* interpretation of the *location* of the nontrivial zeta zeros as those values of α inside the critical strip that generate poles of the open bosonic four-point string scattering amplitude $A(s,t,u)$. By cyclic symmetry, the same results hold, had we set $\beta + \gamma = 1$, $\alpha = 0$ (pole in the s-channel), or $\alpha + \gamma = 1$, $\beta = 0$ (pole in the t-channel).

Are these solutions *unique*? We shall see next that other values of α lying in the critical strip (hypothetical nontrivial zeta zeros) do *not* correspond to poles of $A(s,t,u)$. Hence, the values $\alpha = 1/2 + i\lambda$ are indeed very special since these are the only values *inside* the critical strip, which yield poles of $A(s,t,u)$.

Let us identify the *four* hypothetical nontrivial zeta zeros lying inside the critical strip $(0 < \mathcal{R}e\, z < 1)$ at

$$\alpha_n, \quad \beta_n = \alpha_n^*. \quad 1 - \alpha_n, \quad 1 - \beta_n = 1 - \alpha_n^* \tag{23}$$

respectively, such that

$$\zeta(\alpha_n) = \zeta(\beta_n) = \zeta(1 - \alpha_n) = \zeta(1 - \beta_n) = 0. \quad 2 > \alpha_n + \beta_n > 1, \ -1 < \gamma_n < 0. \tag{24}$$

The r.h.s. of (13) does *not* have poles by inspection:

$$\begin{aligned} A(s,t,u) &= \frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)}\frac{\zeta(1-\beta_n)}{\zeta(\beta_n)}\frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} = \frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)}\frac{\zeta(1-\alpha_n^*)}{\zeta(\alpha_n^*)}\frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} \\ &= \left\|\frac{\zeta(1-\alpha_n)}{\zeta(\alpha_n)}\right\|^2 \frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} = C_n \frac{\zeta(1-\gamma_n)}{\zeta(\gamma_n)} = \text{real and finite} \end{aligned} \tag{25}$$

when $-1 < \gamma_n < 0$. The real constants $C_n = \|\zeta(1-\alpha_n)/\zeta(\alpha_n)\|^2 = 0/0$ are *finite* because there are *no* poles in the l.h.s. of Eq. (13) by inspection of the arguments

of the Γ functions when the parameters α, β, γ are restricted by the conditions described above in Eq. (24).

What we have shown in Eq. (25) is that if there were nontrivial zeta zeros outside the critical Riemann line these zeros *do not correspond* to *poles* of $A(s,t,u)$. However, this fact alone does not necessarily mean that these zeros do not exist, but only that if they existed they do not have a *physical* interpretation in terms of the *poles* of $A(s,t,u)$.

Identical conclusions follow if one had the values of $\alpha = \beta^*$ lying to the *left* of the Riemman critical line *Real* $(z) < 1/2$ since in this case we have

$$0 < \alpha + \beta < 1; \quad 1 > \gamma > 0 \tag{26}$$

in this case, α, β, γ are *all* confined to the critical strip, and there is no pole in $\zeta(1-\gamma_n)/\zeta(\gamma_n)$.

Once again we reiterate that there are *no* poles in the l.h.s. of Eq. (13) by inspection of the arguments of the Γ functions when $\alpha = \beta^*$ and α, β are confined to the interior of the critical strip obeying $2 > \alpha + \beta > 1$ and $-1 < \gamma < 0$, or the conditions $0 < \alpha + \beta < 1$ and $1 > \gamma > 0$, respectively. The latter conditions imply that α, β, γ are *all* confined to the interior of the critical strip and validates the use of Eq. (13) as indicated by [13]. In this case, the fact that there are no poles in the l.h.s. of Eq. (13) ensures us that the real constants C_n in Eq. (25) are *finite*.

Furthemore, we will see that the solutions $\alpha = 1/2 + i\lambda$ have also a clear definite *geometrical* interpretation when the Euclidean triangle with three vertices degenerates into a vertical strip in the upper complex plane comprising one vertex located at infinity (with zero angle) and the other two vertices (with angle $\pi/2$) located on the real-axis and separated by a distance, see [15]:

$$d = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)} = \frac{\pi}{\sin(\pi\alpha)} = \frac{\pi}{\sin(\pi/2 + i\pi\lambda)} = \frac{\pi}{\cos(i\pi\lambda)} = \frac{\pi}{\cosh\,\pi\lambda}. \tag{27}$$

Once again we must remind the reader that our notation for α, β, γ differs from [15].

Despite the fact that α, $\beta = \alpha^*$ are complex-valued their sum $\alpha + \beta = 1 = \textit{real}$, thus the sum of the three angles of the triangle is still $\pi(\alpha + \beta + \gamma) = \pi$. Therefore, the discrete number of the imaginary parts of the nontrivial zeta zeros λ_n are associated with a discrete number of possible distances between the two variable vertices of the triangles situated in the real-axis of the complex plane and given by $d_n = \pi/\cosh(\pi\lambda_n)$. Physical systems with this type of hyperbolic spectrum of scales d_n are interesting in their own right, because the values d_n in Eq. (27) can be thought of the *regularized* values of $A(s,t,u)$ by extracting their finite parts in Eq. (19) by means of Eq. (20a). This deserves further investigation.

Concluding, motivated by the fact that the trivial zeta zeros lie in the negative even real-axis, $-2n$, and that there are physical poles of the Veneziano amplitude in the negative real-axis (at $-n$, twice as many poles than trivial zeta zeros), it raises the question whether the location of the nontrivial zeta zeros along the critical line bear a similar physical significance. The answer is yes: we

have provided a novel physical interpretation (to our knowledge) of the RH based on the existence of a special type of poles of the open bosonic string scattering (Veneziano) amplitude $A(s,t,u)$; namely, that the *location* of the nontrivial zeta zeros, $\mathcal{R}e\, z = 1/2$, have a unique correspondence to a tachyonic pole in the u-channel of mass-squared $-2m^2_{\rm Planck} = -2$, with zero angular-momentum $J = 0$, if and only if, $\alpha = \beta^* = 1/2 + i\lambda$ and $\gamma = 0$.

By cyclic symmetry, the same argument applies to the s, t channels. The case $\alpha = 0$ (corresponding to a tachyonic pole $s = -2$ in the s-channel, studied in detail in the appendix below) yields the solutions $\beta = \gamma^* = 1/2 + i\lambda$ (which agree with the location of the zeta zeros in the critical line) and are associated with a pair of tachyonic-resonances (tachyonic-condensates) originating from the scattering of two incoming *tachyons*, obeying the on-shell condition $(k_1)^2 = (k_2)^2 = -2$. Therefore, an important connection between string theory and the RH exists within the context of establishing the correspondence among the zeta zeros and poles of the string scattering amplitudes $A(s,t,u)$. We found that *if* there were zeta zeros that *violated* the RH, these zeros do *not* correspond to any *poles* of the Veneziano amplitudes $A(s,t,u)$; i.e., these zeros would be very *anomalous* in this respect.

Appendix. On Tachyonic Resonances/Tachyonic Condensates and the Riemann Hypothesis

We shall derive the tachyonic resonances/tachyonic condensate conditions that describe the location of the nontrivial zeta zeros in the Riemann critical line. When $\alpha = 0$, one has that $\beta = \gamma^* = \frac{1}{2} + i\lambda$ so there is a tachyon in the s-channel $s = (k_1 + k_2)^2 = -2$ and $J = 1 + \frac{1}{2}s = 0$ that decays into the tachyonic-condensate comprising of a pair of tachyonic-resonances ("particle-antiparticle" pair) obeying $k_3 = k_4^*$ and $\mathcal{R}e\, k_3^2 = \mathcal{R}e\, k_4^2 < 0$; along with $\mathcal{I}m\, k_3^2 = -\mathcal{I}m\, k_4^2$.

From the energy-momentum conservation law, $k_1 + k_2 = k_3 + k_4$, one can rewrite

$$s = (k_1 + k_2)^2, \quad t = (k_2 - k_3)^2 = (k_1 - k_4)^2, \quad u = (k_1 - k_3)^2. \tag{A.1}$$

Given $\beta = \gamma^* = 1/2 + i\lambda$ it yields

$$-\beta = 1 + \frac{1}{2}t, \quad -\gamma = 1 + \frac{1}{2}u. \Rightarrow t = u^* = -3 - 2i\lambda. \tag{A.2}$$

The conservation of energy-momentum and the complex–conjugate pair condition $k_3 = k_4^*$ combined with the conservation of angular-momentum and the Regge trajectory conditions

$$J(s = -2) = 1 + \frac{1}{2}s = 0. \quad J_{k_3} = 1 + \frac{1}{2}(k_3)^2. \quad J_{k_4} = J^*_{k_3}. \quad J = J_{k_3} + J_{k_4} = 0 \tag{A.3}$$

leads to

$$k_3^2 = -2 + 2i\xi(\lambda), \quad k_4^2 = -2 - 2i\xi(\lambda) \tag{A.4}$$

that satisfy the conditions

$$J = 1 + \frac{1}{2}k^2 \Rightarrow J_{k_3} = 0 + i\xi, \quad J_{k_4} = J^*_{k_3} = 0 - i\xi \tag{A.5}$$

$$J = J_{k_3} + J_{k_4} = 0; \quad |k_1 + k_2; J = 0\rangle \rightarrow |k_3; J = 0 + i\xi\rangle \oplus |k_4; J = 0 - i\xi\rangle \tag{A.6}$$

From $s + t + u = -8$ and the above equations, one learns

$$\begin{aligned} -1 &= k_1 \cdot (k_3 + k_4) \text{ and } 1 + 2i(\xi - \lambda) = -2k_1 \cdot k_4 \Rightarrow 2i(\xi - \lambda) \\ &= k_1 \cdot (k_3 - k_4) = k_1 \cdot (k_3 - k_3^*) \Rightarrow \xi - \lambda \\ &= k_1 \cdot \mathcal{I}m(k_3) \Rightarrow \xi - \lambda = E_1\mathcal{E}_3 - p_1\pi_3 = -E_1\mathcal{E}_4 + p_1\pi_4 \end{aligned} \tag{A.7}$$

where we have defined

$$\begin{aligned} &k_3 = (E_3 + i\mathcal{E}_3, p_3 + i\pi_3), \quad k_4 = (E_4 + i\mathcal{E}_4, p_4 + i\pi_4), \quad k_3 = k_4^* \\ &\Rightarrow E_3 = E_4, \quad \mathcal{E}_3 = -\mathcal{E}_4, \quad p_3 = p_4, \quad \pi_3 = -\pi_4. \end{aligned} \tag{A.8}$$

This last condition (A.8) in conjunction with

$$k_3^2 = -2 + 2i\xi = (E_3 + i\mathcal{E}_3)^2 - (p_3 + i\pi_3)^2 \tag{A.9}$$

implies

$$-2 = E_3^2 - \mathcal{E}_3^2 - p_3^2 + \pi_3^2 \tag{A.10}$$

$$2\xi = 2E_3\mathcal{E}_3 - 2p_3\pi_3. \tag{A.11}$$

where

$$k_1 + k_2 = k_3 + k_4 \Rightarrow 2E_3 = E_1 + E_2. \quad 2p_3 = p_1 + p_2. \tag{A.12}$$

The five equations present in (A.7), (A.10), (A.11), and (A.12) with five unknowns E_3, $\mathcal{E}_3$, p_3, π_3, ξ have explicit solutions in terms of λ, E_1, p_1, E_2, p_2 given by

$$\mathcal{E}_3 = \frac{-2\xi p_1 + (\xi - \lambda)(p_1 + p_2)}{E_1(p_1 + p_2) - (E_1 + E_2)p_1} \tag{A.13}$$

$$\pi_3 = \frac{-2\xi E_1 + (\xi - \lambda)(E_1 + E_2)}{E_1(p_1 + p_2) - (E_1 + E_2)p_1} \tag{A.14}$$

where ξ is explicitly given by

$$\xi(\lambda, E_1, E_2, p_1, p_2) = \pm\frac{1}{2}\sqrt{\frac{3(E_1p_2 - E_2p_1)^2 + 4\lambda^2(E_1E_2 - p_1p_2 - 2)}{2 + E_1E_2 - p_1p_2}} \tag{A.15}$$

which is symmetric under the exchange $1 \leftrightarrow 2$. The positive sign of the square root $\xi > 0$ corresponds to $k_3^2 = -2 + 2i\xi$ and the negative sign to the complex conjugate solution $k_4^2 = -2 - 2i\xi$.

Given

$$s = (k_1 + k_2)^2 = -2 = k_1^2 + k_2^2 + 2k_1 \cdot k_2 = -2 - 2 + 2k_1 \cdot k_2 \Rightarrow k_1 \cdot k_2 = 1 \quad \text{(A.16)}$$

so that the denominator in (A.15) is

$$2 + E_1 E_2 - p_1 p_2 = 2 + k_1 \cdot k_2 = 2 + 1 = 3 \quad \text{(A.17)}$$

and Eq. (A.15) is finally

$$\xi(\lambda, E_1, E_2, p_1, p_2) = \pm \frac{1}{2} \sqrt{\frac{3(E_1 p_2 - E_2 p_1)^2 - 4\lambda^2}{3}} \quad \text{(A.18)}$$

where the terms inside the square root must be positive-definite to ensure that the solutions are real-valued and not imaginary. The geometrical interpretation of Eq. (A.8) is such that the area in energy-momentum space is bounded by those values determined by λ. When $\xi = 0$ one recovers the ordinary tachyonic solutions in Eq. (A.4).

It is straightforward to verify that the term $E_1 p_2 - E_2 p_1$ in (A.18) is Lorentz invariant, as it should. This can easily be checked by writing the Lorentz boosts transformations in term of the boost parameter ω

$$E' = E \cosh(\omega) + p \sinh(\omega), \quad p' = p \cosh(\omega) + E \sinh(\omega),$$
$$E^2 - p^2 = \textit{invariant} \quad \text{(A.19)}$$

and using them in

$$\begin{aligned} E_1' p_2' - E_2' p_1' &= (E_1 \cosh(\omega) + p_1 \sinh(\omega))(p_2 \cosh(\omega) + E_2 \sinh(\omega)) \\ &\quad - (E_2 \cosh(\omega) + p_2 \sinh(\omega))(p_1 \cosh(\omega) + E_1 \sinh(\omega)) \\ &= (E_1 p_2 - E_2 p_1)(\cosh^2(\omega) - \sinh^2(\omega)) = (E_1 p_2 - E_2 p_1) = \textit{invariant} \end{aligned} \quad \text{(A.20)}$$

as a result of the identity $\cosh^2(\omega) - \sinh^2(\omega) = 1$.

Concluding, in this appendix we have shown that the system of four relations given in Eqs. (A.12)–(A.14) and the relation in Eq. (A.18) furnishes a one-parameter family of solutions (parametrized by λ) for the five unknowns E_3, $\mathcal{E}_3$, p_3, π_3, ξ, corresponding to the production of a pair of tachyonic-resonances (tachyonic-condensates) resulting from the scattering of two incoming on-shell tachyons that are associated with the location of the nontrivial Riemann zeta zeros in the critical line.

Therefore, we have found a physical interpretation of the location of these nontrivial zeros, in the same vein that the location of the trivial zeta zeros (negative even integers) along the negative real-axis also correspond to poles of the Veneziano amplitude. If there were zeros that violated the RH these zeros would be very anomalous in the sense they do not correspond to any poles, whatsoever, of the string scattering amplitude!

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