A Novel Window Function

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***Abstract*—A novel window function, also known as an apodization or tapering function is proposed. The window is similar in shape, spectral response and interpretation to the Hanning window.**

***Index Terms*—apodization, Hanning, spectral analysis, window**

I. Introduction

In this paper a novel window function is proposed. Such windows, including the Hanning, Hamming, Blackman and Gaussian windows are useful in spectral analysis applications where a sampled signal is multiplied by such a window, usually followed by a discrete Fourier transform (DFT) in order to control spectral roll-off, generally at the expense of spectral resolution.

II. Cosine Window with Quadratic Phase Modulation

A cosine window function that includes a quadratic phase modulation term (linear frequency modulation (LFM)) is defined as follows:

$$w(n) = \frac{1-\cos\left(2\pi\alpha\frac{n}{N} - 2\pi\alpha\left(\frac{n}{N}\right)^2\right)}{1-\cos\left(\frac{\pi}{2}\alpha\right)} \tag{1}$$

or

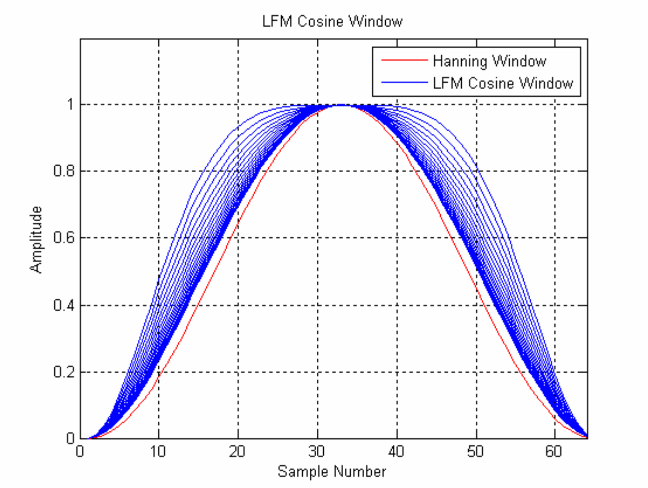
$$w(n) = \frac{1-\cos\left(\beta\frac{n}{N} - \beta\left(\frac{n}{N}\right)^2\right)}{1-\cos\left(\frac{\beta}{4}\right)} \tag{2}$$

where $\beta = 2\pi\alpha$, n is the sample index and N is the number of samples in the window function. The window function is zero outside the interval n = 0 to N-1.

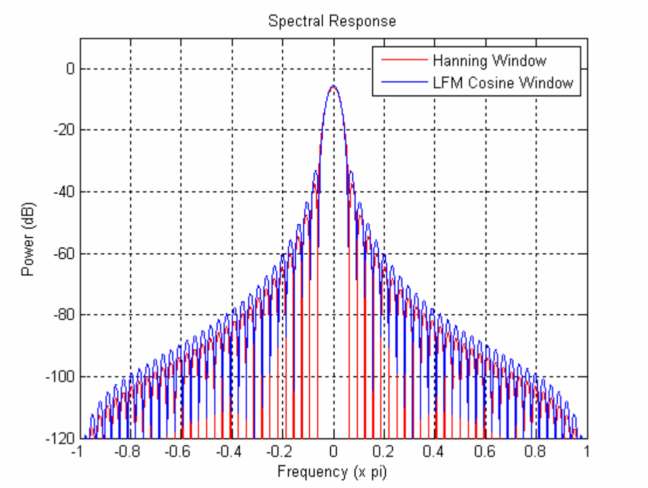
This window is similar in its shape, spectral response and interpretation to the periodic Hanning window function which is defined as

$$w(n) = \frac{1-\cos\left(2\pi\frac{n}{N}\right)}{2} \tag{3}$$

The following figure illustrates the LFM Cosine window for the set of parameters, $0.1 \le \alpha \le 2$ and N = 64. Note that for $\alpha > 2$, the window no longer has a "flat top". A Hanning window is illustrated for comparison.



The spectral response of the window for $\alpha = 0.1$ and a Hanning window for comparison is illustrated below



In the limit, for $\alpha = 0$, one can show that the window can be approximated by

$$w(n) = 16\left(\left(\frac{n}{N}\right)^2 - 2\left(\frac{n}{N}\right)^3 + \left(\frac{n}{N}\right)^4\right) \tag{4}$$

(a window function in its own right)

III. Interpretation

The multiplication of a sampled signal by a Hanning window can be thought of as adding the original signal, scaled by ½ to two scaled, frequency shifted versions of itself as indicated by the following equation

$$x(n)w(n) = \frac{x(n)}{2} - \frac{x(n)}{4}e^{\left(j2\pi\frac{n}{N}\right)} - \frac{x(n)}{4}e^{\left(-j2\pi\frac{n}{N}\right)} \tag{5}$$

since

$$\cos(\phi) = \frac{e^{j\phi} + e^{-j\phi}}{2}. \tag{6}$$

The frequency shift is equivalent to +1 and -1 frequency bins respectively as seen at the output of a DFT of equation 5. The application of the window in the frequency domain is equivalent to a circular convolution of the DFT of the original signal by the sequence [-¼ ½ -¼].

The proposed window is very similar in the sense that a scaled version of the original signal is added to two scaled, frequency shifted, linear frequency modulated versions of itself. In this case, the signal that is frequency shifted by α bins is effectively swept over -2α bins and conversely, the signal that is frequency shifted by $-\alpha$ bins is effectively swept over 2α bins. The denominator of the window function normalizes the window to a maximum of one. This is can be seen in the following equation

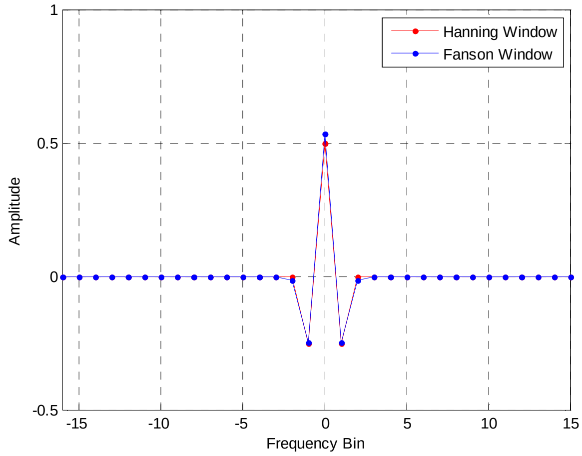
$$x(n)w(n) = \frac{1}{1-\cos\left(\frac{\pi}{2}\alpha\right)}\left\{\begin{array}{l} x(n) - \frac{x(n)}{2}e^{\left(2\pi\alpha\frac{n}{N}\right)}e^{\left(-2\pi\alpha\left(\frac{n}{N}\right)^2\right)} \\ -\frac{x(n)}{2}e^{\left(-2\pi\alpha\frac{n}{N}\right)}e^{\left(2\pi\alpha\left(\frac{n}{N}\right)^2\right)} \end{array}\right\} \tag{7}$$

where

$$e^{\left(+/-2\pi\alpha\left(\frac{n}{N}\right)^2\right)}$$

represents effectively a linear frequency modulation over $+/-2\alpha$ bins respectively.

The frequency domain coefficients of the Hanning window and LFM Cosine window at the output of a DFT for $\alpha = .1$ and N = 32 are illustrated below.



IV. Sine window with quadratic phase modulation

A similar window can be defined using a sine function as

$$w(n) = \frac{\sin\left(2\pi\alpha\frac{n}{N} - 2\pi\alpha\left(\frac{n}{N}\right)^2\right)}{\sin\left(\frac{\pi}{2}\alpha\right)} \tag{8}$$

which has a flat top for α in the range of 0 to 1 and a narrower main lobe and much higher side lobes than the LFM cosine window.

V. Conclusion

The proposed LFM cosine window is useful for signal processing applications like spectral estimation or filter design. It has a similar response to the Hanning window but with more flexible design control through the parameter α.

Acknowledgment

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References

[1] Nuttall, Albert H. (February 1981). "Some Windows with Very Good Sidelobe Behavior". IEEE Transactions on Acoustics, Speech, and Signal Processing 29 (1): 84-91. Extends Harris' paper, covering all the window functions known at the time, along with key metric comparisons.

[2] Harris, Fredric J. (January 1978). ""On the use of Windows for Harmonic Analysis with the Discrete Fourier Transform"". Proceedings of the IEEE 66 (1): 51–83. Article on FFT windows which introduced many of the key metrics used to compare windows.