Torsion Fields, Brownian Motions, Quantum and Hadronic Mechanics

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Summary: We review the relation between space-time geometries with torsion fields (the so-called Riemann-Cartan-Weyl (RCW) geometries) and their associated Brownian motions. In this setting, the metric conjugate of the trace-torsion one-form is the drift vector field of the Brownian motions. Thus, in the present approach space-time fluctuations as Brownian motions are -in distinction with Nelson's Stochastic Mechanics- space-time structures. Thus, space and time have a fractal structure. We discuss the relations with Nottale's theory of Scale Relativity, which stems from Nelson's approach. We characterize the Schroedinger equation in terms of the RCW geometries and Brownian motions. In this work, the Schroedinger field is a torsion generating field. The potential functions on Schroedinger equations can be alternatively linear or nonlinear on the wave function, leading to nonlinear and linear creation-annihilation of particles by diffusion systems. We give a brief presentation of the isotopic lift of Quantum Mechanics known as Hadronic Mechanics due to Santilli. We start by giving the isotopic lift (i.e. a non-unitary transformation not identically reducible to the identity) of gauge theories, to show that torsion appears at the basis of gauge theories, and also in the isotopic lift of gauge theories. Using this non-unitary transformations we present the isotopic lift of all the mathematical apparatus and physical aspects of Quantum Mechanics, to present Hadronic Mechanics. This theory lifts a number of fundamental inconsistencies in the conventional approaches to Special and General Relativity, Quantum Mechanics, particle physics, cosmology, superconductivity, biology, etc. We present the relations between Hadronic Mechanics with RCW geometries and Brownian motions in the case of the iso-Schroedinger equation for the strong interactions. This is achieved by either using the diffusion representation for the Schroedinger equation and its isotopic lift, which for the case of trivial noise tensor yields the iso-Heisenberg representation. We derive the iso-Schroedinger equation from this isotopic lift of diffusions representation. We give a Brownian motion model of fusion in Hadronic Mechanics, and extend it to the many body problem. We discuss the possible relation between torsion fields and several anomalous phenomenae. Finally we discuss possible evidence for this theory in the fine structure time periodicity of histograms of arbitrary processes found by Shnoll and collaborators, which show a spacetime anisotropy attributed to space-time fluctuations and a fractal structure of these histograms.

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1 INTRODUCTION

A central problem of contemporary physics is the distinct world views provided by Quantum Mechanics and the theory of Relativity, and more generally of gravitation. In a series of articles [1-4], we have presented an unification between space-time structures, Brownian motions, further extended to fluid-dynamics and Quantum Mechanics (QM, for short in the following). The unification of space-time geometry and classical statistical theory has been possible due to a complementarity of the objects characterizing the Brownian motion, i.e. the noise tensor which produces a metric, and the drift vector field which describes the average velocity of the Brownian, in jointly describing both the space-time geometry and the stochastic processes. These space-time structures can be defined starting from flat Euclidean or Minkowski space-time, and they have in addition to a metric a torsion tensor which is formed from the metric conjugate of the drift vector field. The key to this unification lies in that the laplacian operator defined by this geometrical structure is the differential generator of the Brownian motions; stochastic analysis which deals with the transformation rules of classical observables on diffusion paths ensures that this unification stands in an equal status for both the geometrical and stochastic structures [70]. Thus , in this equivalence, one can choose the Brownian motions as the original structures determining a space-time structure, or conversely, the space-time structures produce a Brownian motion process. Space-time geometries with torsion have lead to an extension of the theory of gravitation which was first explored in joint work by Einstein with Cartan [5], so that the foundations for the gravitational field, for the special case in which the torsion reduces to its trace, can be found in these Brownian motions. Furthermore, in [2] we have shown that the relativistic quantum potential coincides , up to a conformal factor, with the metric scalar curvature. In this setting we are lead to conceive that there is no actual propagation of disturbances but instead an holistic modification of the whole space-time structure due to an initial perturbation which provides for the Brownian process modification of the original configuration. Furthermore, the present theory which has a kinetic Brownian motion generation of the geometries, is related to Le Sage's proposal of a Universe filled with all pervading tiny particles moving in all directions as a pushing (in contrast with Newton's pulling force) source for the gravitational field [52]. Le Sage's perspective was found to be compatible cosmological observation by H. Arp [53]. This analysis stems from the assumption of a non-constant mass in GR which goes back to Hoyle and Narlikar, which in another perspective developed by Wu and Lin generates rotational forces [54]. These rotational forces can be ascribed to the drift trace-torsion vector field of the Brownian processes through the Hodge duality transformation [3], or still to the vorticity generated by this vector field. In our present theory, motions in space *and* time are fractal, they generate the gravitational field of general relativity, and furthermore they generate rotational fields, in contrast with the pulling force of Newton's theory and the pushing force of Le Sage, or in the realm of the neutron, the Coulomb force. Furthermore, in our construction the drift has built-in terms given by the conjugate of electromagnetic-like potential 1-forms, whose associated intensity two-form generate vorticity, i.e. angular momentum. So the present geometries are very different from the metric geometries of General Relativity (GR in the following) and are not in conflict with present cosmological observations.

The space-time geometrical structures of this theory can be introduced by the Einstein λ transformations on the tetrad fields [2,5], from which the usual Weyl scale transformations on the metric can be derived, but contrarily to Weyl geometries, these structures have torsion and they are integrable in contrast with Weyl's theory [48]. We have called these connections as RCW structures (short for Riemann-Cartan-Weyl); see [1-4] and references therein. We have shown that this approach leads to non-relativistic QM both in configuration space [3] and in the projective Hilbert state-space through the stochastic Schroedinger equation [39] (in the latter case, it was shown that this geometry is related to a theory of the reduction of the wave function through decoherence) , and further to Maxwell's equation and its equivalence with the Dirac-Hestenes equation of relativistic QM [2,21]. The fact that non-relativistic QM can be linked to torsion fields was unveiled recently [3]. In fact, torsion fields have been considered to be as providing deviations of GR outside the reach of present precision measurements [22]. It turns out, as discussed in relation with viscous fluids obeying the Navier-Stokes equations as a universal example of torsion fields [1,4], that quantum wave-functions verifying linear or non-linear Schroedinger equations are another universal, or if wished, mundane examples of torsion fields. In this article we shall show that this extends to the strong interactions. The quantum random ensembles which generate the geometries, or which dually can be seen as generated by them, in the case of the Schroedinger equation can be associated with harmonic oscillators with disordered random phase and amplitude first proposed by Planck, which have the same energy spectrum as the one derived originally by Schroedinger [34,73]. The probabilities of these ensembles are classical since they are associated with classical Brownian motions in the configuration and projective Hilbert-state manifolds, in sharp contrast with the Copenhagen interpretation of QM which is constructed in terms of single system description, and they are related to the scalar amplitude of the spinor field in the case of the Dirac field, and in terms of the modulus of the complex wave function in the non-relativistic case [2,3,21]. We would like to recall at this stage that Khrennikov has proved that Kolmogorov's axiomatics of classical probability theory, in a contextual approach which means an a-priori consideration of a complex of physical conditions, permits the reconstruction of quantum theory [64]. Thus, Khrennikov 's theory places the validity of Quantum Theory in ensembles, in distinction with the Copenhagen interpretation, and is known as the

Vaxho interpretation of QM. In the present approach we obtain both a geometrical characterization of the quantum domain through random ensembles performing Brownian motions which generate the space and time geometries, and additionally a characterization of QM for single systems through the topological Bohr-Sommerfeld invariants associated with the trace-torsion by introducing the concept of Pfaffian system developed by Kiehn in his geometro-topological theory of processes [20], specifically applied to the trace-torsion one-form [81]. Most remarkably, in our setting another relevant example of these space-time geometries is provided by viscous fluids obeying the invariant Navier-Stokes equations of fluid-dynamics, or alternatively the kinematical dynamo equation for the passive transport of magnetic fields on fluids [1,4] . We would like to point out that cosmological observations have registered turbulent large-scale structures which are described in terms of the Navier-Stokes equations [82] so that this model is in principle compatible with that observations.

We must stress the difference between this approach and GR. In the latter, the space-time structure is derived in the sense that it is defined without going through a self-referential characterization, but obtained from the metric. When we introduce torsion, and especially in the case of the trivial metric with null associated curvature tensor, we are introducing a self-referential characterization of the geometry since the definition of the manifold by the torsion, is through the concept of locus of a point (be that temporal or spatial). Indeed, space and time can only be distinguished if we can distinguish inhomogenities, and this is the intent of torsion, to measure the dislocation (in space and time) in the manifold [69]. Thus all these theories stem from a geometrical operation which has a logical background related to the concept of distinction (and more fundamentally, the concept of identity, which is prior to that of distinction) and its implementation through the operation of comparison by parallel transport with the affine connection with non-vanishing torsion; this can be further related with multivalued logics and the appearence of time waves related to paradoxes, which in a cognitive systems approach yield the Schroedinger representation [81]. In comparison, in GR there is also an operation of distinction carried out by the parallel transport of pair of vector fields with the Levi-Civita metric connection yielding a trivial difference, i.e. the torsion is null and infinitesimal parallelograms trivially close, so that it does not lead to the appearence of inhomogenities as resulting from this primitive operation of distinction; these are realized through the curvature derived from the metric.

We shall start this article by introducing non-relativistic QM in terms of diffusion processes in space-time following our work in [3]; the stochastic Schroedinger equation case shall not be dealt with; see [3,39]. Thus in this approach appears that the Schroedinger field can be associated with the field which produces the torsion field. Thus we have recovered a modified characterization that can be traced back to London's treatment of the Weyl geometries which although related with a local change of the metrics, have null torsion [48]. There have been numerous attempts to relate non-relativistic QM to diffusion equations;

the most notable of them is Stochastic Mechanics , due to Nelson [8]. Already Schroedinger proposed in 1930-32 that his equation should be related to the theory of Brownian motions (most probably as a late reaction to his previous acceptance of the single system probabilistic Copenhagen interpretation), and further proposed a scheme he was not able to achieve, the so-called interpolation problem which requires to describe the Brownian motion and the wave functions in terms of interpolating the initial and final densities in a given time-interval [9]. More recently Nagasawa presented a solution to this interpolation problem and further elucidated that the Schroedinger equation is in fact a Boltzmann equation [13], and thus the generation of the space and time structures produced by the Brownian motions has a statistical origin. We have discussed in [3] that the solution of the interpolation problem leads to consider time to be more than a classical parameter (merely a clock), but an active operational variable, as recent experiments have shown [46] which have elicited theoretical studies in [72]; other experiments that suggest an active role of time are further discussed in [3] and in the final section in this article. Neither Nagasawa nor Nelson presented these Brownian motions as space-time structures, but rather as matter fields *on* the vacuum. While Nelson introduced artificially a forward and backward stochastic derivatives to be able to reproduce the Schroedinger equation as a formally time-symmetric equation, Nagasawa was able to solve the interpolation problem in terms of the forward diffusion process and its adjoint backward process, from which without resort to the ad-hoc constructions due to Nelson, he was able to prove in [13] that this was related to the Kolmogorov characterization of time-irreversibility of diffusion processes in terms of the non-exact terms of the drift,that we further related to the trace-torsion [3]. A similar approach to quantization in terms of an initial fractal structure of space-time and the introduction of Nelson's forward and backward stochastic derivatives, was developed by Nottale in his Scale Theory of Relativity [10] [26]. Remarkably, Nottale's approach has promoted the Schroedinger equation to be valid for large scale structures, and predicted the existence of exo-solar planets which were observationally verified to exist [12]. This may further support the idea that the RCW structures introduced in the vacuum by scale transformations, are valid independently of the scale in which the associated Brownian motions and equations of quantum mechanics are posited. Furthermore, Kiehn has proved that the Schroedinger equation in spatial 2D can be exactly transformed into the Navier-Stokes equation for a compressible fluid, if we further take the kinematical viscosity ν to be $\frac{\hbar}{m}$ with m the mass of the electron [11]. We have argued in [3] that the Navier-Stokes equations share with the Schroedinger equation, that both have a RCW geometry at their basis: While in the Navier-Stokes equations the trace-torsion is $\frac{-1}{2\nu}u$ with u the time-dependent velocity one-form of the viscous fluid, in the Schroedinger equation, the trace-torsion one-form incorporates the logarithmic differential of the wave function -just like in Nottale's theory [10]- and further incorporates electromagnetic potential terms in the trace-torsion one-form. This correspondence between trace-torsion one-forms is what lies at the base of Kiehn's correspondance, with an important addendum: While in the approach of the Schroedinger equation the probability density is related to the Schroedinger scale factor (in incorporating the complex phase) and the Born formula turns out to be a formula and not an hypothesis, under the transformation to the Navier-Stokes equations it turns out that the probability density of non-relativistic quantum mechanics, is the enstrophy density of the fluid, i.e. the square of the vorticity, which thus plays a *geometrical* role that substitutes the probability density. Thus, in this approach, while there exist virtual paths sustaining the random behaviour of particles (as is the case also of the Navier-Stokes equations) and interference such as in the two-slit experiments can be interpreted as a superposition of Brownian paths [13], the probability density has a purely geometrical fluid-dynamical meaning. This is of great relevance with regards to the fundamental role that the vorticity, i.e. the fluid's particles angular-momentum has as an organizing structure of the geometry of space and time. In spite that the torsion tensor in this theory is naturally restricted to its trace and thus generates a differential one-form, in the non-propagating torsion theories it is interpreted that the vanishing of the completely skew-symmetric torsion implies the absence of spin and angular momentum densities [22], it is precisely the role of the vorticity to introduce angular momentum into the present theory. We would like to mention the important developments in a beautiful theory of space-time with a Cantorian structure being elaborated in numerous articles by M. El Naschie [67] and a theory of fractals and stochastic processes of QM which has been elaborated by G. Ord [66].

Secondly, in this article we shall extend the treatment of QM and its relations to torsion fields and Brownian motions, to the Lie-isotopic extension of QM due to Santilli, presently known as Hadronic Mechanics (HM in the following). Before actually introducing HM, we would like to discuss the fundamentals of the relation between HM and the present developments in terms of torsion fields, since torsion bears a close relation with the geometry of Lie symmetry groups. Indeed, if we consider as configuration space a Lie group, there is a canonical connection whose torsion tensor coefficients are non other than the coefficients of the Lie-algebra under the Lie bracket operation [49]. Thus a Lie algebra is characterized by the torsion tensor for the canonical connection. Now let us consider a system that can be described in terms of a Lie group symmetry which is a continuous deformation of the original one. For example, instead of the rotational symmetry we have, say, an ellipsoidal one due to the deformation of the original system (this is the case of a system whose symmetry in the vacuum is no longer valid since it has been embedded in an inhomogeneous anisotropic medium). Then the Lie bracket of the deformed infinitesimal symmetry is a continuous deformation of the original torsion and characterizes completely the symmetry. This is the basic mathematical implicit idea that leads to the present perspective, and allows to present a geometrical structure (the torsion of the infinitesimal symmetry) which is deformed by the isotopic modification (we shall explain below the meaning of the term "isotopic") of Lie group theory due to Santilli, the so-called Lie-Santilli-isotopic theory which started by producing the isotopic lift of the groups of elementary particle physics and special relativity [15-20,65], and encompasses in the physical domain, extensions of classical mechanics (introducing the deformed symmetries) and their associated quantum systems through the isotopic lift of the Schroedinger and Heisenberg representation which incorporates the modifications of the symmetries.

There is a canonical way of producing these isotopic lift, namely through a non-unitary transform acting on all the elements of the standard theory. Here the notion of isotopic lift consists in carrying through the action of non-unitary transformations -the previously referred deformations- on all the mathematical structures with preservation of the basic axioms. Starting by redefining the notion of unit -which stands as the fundamental operation of the theory- achieved by a non-unitary arbitrary operation, then the construction of HM follows by carrying the generalized unit to define the isotopic number fields, functions, configuration and state-space manifolds, differential and integral calculi, metrics, tensors, the Hilbert state-spaces, to resume, all the relevant mathematical structures of quantum mechanics are isotopically lifted through the generalized unit while preserving the whole axiomatic structure of both the mathematical and physical aspects of the theory. Probably the resistence to adopt such a general scheme, is that one tends to confuse the universal character of the concept of unit, which is fixed, and its representation in a certain scale and with a certain object (s) which can be taken arbitrary, as we do in daily life. We must remark at this stage that the introduction of this generalized unit, in contrast with the basic unit of mathematics and physics, establishes a vinculation between these new units and physical processes which is unknown to mathematics, with the exception of the arithmetic of forms which follows from the principle of distinction previously alluded, and the multivalued logics associated to it and self-reference [81]. It is important to remark that the development of HM took more than thirty years to reach the understanding that the isotopic lift could not be restricted to the symmetries, classical and quantum mechanics, but required the lift of number theory, differential and integral calculi, and to the Hilbert space structure as well. For an account of these developments and a complete list of references, the reader is urged to consult [16] in which Santilli gives a complete description of the Lie-isotopic, Lie-admissible, genotopic and hyperstructural theories which extend the present theory; it is most remarkable that these theories as well as HM and the isotopic lift of relativity, admit dual theories in which antimatter can be treated classically and antigravity can be presented in their terms; see [77]). These theories have succeeded in solving several fundamental questions of particle physics, nuclear physics, quantum chemistry cosmology, superconductivity and biology, where the usual approaches have shown unsurmountable inconsistencies [15-20,56,68,74,75,76].

The purpose of the introduction of the generalized units is to be able to construct a theory in the situations in which QM is no longer applicable, and to recover it in a scale in which the generalized unit recovers the historical unit of mathematics and physics and the associated domains in which GR and QM are applicable. A fundamental example of this new paradigm, is the case of the neutron with its hyperdense structure where interactions are no longer derivable from a potential field; in this situation, the medium is deformable, particles are extended, and non-local non-hamiltonian interactions are the characteristics of the system. Thus, in HM the neutron is considered to be a compressed state of a proton and an electron, revitalizing Rutherford's conception in further elaborations by Animalu and Santilli [56]. This state is produced under the above stated situations and through spin-up-spin-down magnetic couplings [15,56], which plays a crucial role in the Rutherford-Santilli model of the neutron. These spin-spin couplings appear in the definition of the generalized isotopic unit. In this regard, HM has in common with the usual approach to torsion, the relevance of spinor angular momentum densities [22]. Possible relations between torsion as spin or angular momentum densities can be ventured in relation with anomalous spin interactions of the proton, and magnetic resonances as well [43]. Remarkably it has been shown in [56] that completely skew-symmetric torsion can produce a spin flip of high energy fermionic matter at very high densities, and that in this situation helicity can be identified with spin. An intrinsic macroscopic angular momentum would be the evidence of this phenomena. This may be of relevance when taking in consideration the time periodicity of the fine structure of histograms and its relation to macroscopic angular momentum, found by Shnoll and associates [80].

Therefore a geometrical characterization will be possible by introducing the non-linear non-local generalized unit which incorporates the new characteristics of the system under the strong overlapping of the components of the neutron so there is now a deformation of their original symmetry. To close our discussion which started by mentioning the special role of the canonical geometry of a Lie-group and the role of the torsion tensor as the structure coefficients of the Lie-algebra, the modification is produced precisely in these coefficients by multiplying them by the isotopic generalized unit. Thus, at the level of the canonical geometry there exists a modification of the torsion tensor. This will carry out to the whole theory as already explained . However it is pertinent to remark that in the development of HM there is no mention on the relation of the generalized unit in terms of the modification of the torsion as being the fundamental operation in terms of which his theories are constructed. Thus, the present approach, proposes an original *perspective* of HM. It is very important to stress that HM carries to an isotopic modification of Quantum Chemistry, known as Hadronic Chemistry, which allows the computation of the solutions of the iso-Schroedinger equation for molecules in very short times and accuracy in comparison with the results achieved by application of quantum chemistry, while solving its inconsistencies, fundamentally the impossibility of giving a theory for chemical bonds [15,68,84] . This has allowed to produce industrially fluid plasmas with remarkable characteristics, as a first class of clean fuels [15]; furthermore, the theory has proposed the possibility of stimulating nuclear decays as a technological application to deal with nuclear waste in shorter times that those produced by spontaneous decay.

In Santilli's conception with regards to the isotopic lifts of the usual structures of differential geometry, the starting point is that the Minkowski and Euclidean metrics with their associated rotational symmetries are inapplicable for situations of non-point-like particles moving within the inhomogeneous and anisotropic vacuum (this includes the atomic structure as well as the electromagnetic, weak and strong interactions); this is the case of the so-called interior problem; see [14-19]. Thus, in such a medium, the velocity of light depends on the underlying structure of the medium, and no longer coincides with the velocity of light in the vacuum. Consequently, the Lorentz and Poincaré symmetries are no longer applicable in this case, and thus an isotopic lift is constructed to yield a unified approach to the isotopic lifts of GR and Special Relativity; this lead to an axiomatically consistent theory of Relativistic Quantum Mechanics, preserving the fundamental axioms, and resulted in the completion of QM [16]. The reader may note the similarity of this conception with the one we have presented in terms of torsion as producing the basic anisotropies and inhomogenities of the vacuum. The symmetries and geometries of the isotropic and homogeneous vacuum cannot represent matter nor spinor fields by themselves. In the more general case in which the metric is no longer trivial, the usual metric is valid as a representation for the so-called exterior problem in which the astrophysical bodies can be effectively approximated as massive points because their shape does not affect their gravitational trajectories when moving in empty space. In the case of the interior problem, such as the case of ultradense stars in which spin couplings are relevant (a similar approach as the one conceived originally for the introduction of a generalization of general relativity by introducing a spin density tensor associated with the torsion tensor [22]), these couplings are no longer related to a reversible Hamiltonian dynamics. This conception goes back to classical mechanics, in which forces such as friction cannot be described by a Hamiltonian to which they are added as independent forces. These additional terms were incorporated to the foundations of classical mechanics by its founding fathers, and from the physics point of view, was part of the original motivation by Santilli, to attempt a unified formulation of classical and QM through the deformation of the conventional structures and theories [78]. This original approach was posteriorly proved to be inconsistent, due to the need of incorporating into the theory two aspects that had been not perceived originally as imprescindible: Namely, the isotopic lift of the number system and of the Hilbert space defined over these extended fields to be able to achieve a consistent theory of observables, and furthermore, to be able to achieve a consistent isoquantization, the need of extending the notions of differential calculus, fundamentally the differential operator; this was achieved in 1996-1997 ; see [16]. The original attempts by this author to develop the relations between RCW geometries, Brownian motions and the strong interactions were also inconsistent due to the lack of an extension in the full sense previously mentioned [29].

Returning to the discussion of the roles of torsion and the extensions of classical mechanics to account for dissipative forces, we would like to observe that a similar situation is contemplated in classical mechanics from the point of view of considering torsion fields, since friction is a manifestation of anholonomic degrees of freedom which cannot be described by the symmetries and geometry of the frictionless system. Then the equations of classical motion (i.e. differentiable trajectories, as related to a Hamiltonian or Lagrangian approach) have to include additional terms to represent the anholonomic degrees of freedom as exterior forces acting on the system, or alternatively, as internal deformations of the symmetries.

Now to understand the need of carrying the extensions produced by the isotopic lifts, it is founded in the fact that the isotopic lift of Relativity due to Santilli (see [18]) is applicable for the electromagnetic and weak interactions but not applicable for the case of hadrons. These have a charge radius of 1 fm ($10^{-13}cm$) which is the radius of the strong interactions. Unlike the electromagnetic and weak interactions a necessary condition to activate the strong interaction is that hadrons enter into a condition of mutual interpenetration. In view of the developments below, we would like to stress that the modification of the symmetries of particles under conditions of possible fusion, is the first step for the usual developments of fusion theories which have been represented in terms of diffusion processes that overcome the Coulomb repulsive potential which impedes the fusion [37]; Brownian motions and other stochastic processes also appear in a phenomenological approach to the many body problem in particle and nuclear physics, but with no hint as to the possibility of an underlying space-time structure [83]. The basic idea goes back to the foundational works of Smoluchowski (independently of A. Einstein's work in the subject) in Brownian motion [38]. In the case of fusion theories, we have a gas of neutrons (which have an internal structure) and electrons, or an hadron gas; in these cases the fused particles are considered to be alike a compressible fluid with an unstable neck in its fused drops which have to be stabilized to achieve effective fusion; we can see here the figure of deformed symmetries. Thus, the situation for the application of Brownian motion to fusion is a natural extension to the subatomic scale of the original theory. We finally notice that the models for fusion in terms of diffusion do not require QM nor quantum chromodynamics [37]. In contrast, HM stems from symmetry group transformations that describe the contact fusion processes that deform the neutron structure, and lead to the isotopic Schroedinger equation which in this article, together with the isotopic Heisenberg representation, will be applied to establish a link between the RCW geometries, fusion processes and diffusions. The reason for the use of the iso-Heisenberg representation, is that in Santilli's theory, the isotopic lift of the symmetries in carried out in terms of the iso-Heisenberg representation, where its connection with classical mechanics under the quantization rules including the isotopic lift is transparent. In this article we shall present a quantization method for both QM and HM in terms of diffusion processes, and in terms of a diffusion-Heisenberg and isodiffusion-isoHeisenberg representations which we shall present below.

2 TORSION AND THE NON-CLOSURE OF PARALLELOGRAMS

We want to introduce torsion in terms of the self-referential definition of the manifold structure in terms of the concept of difference or distinction derived from the operation of comparison. We shall assume that there are two observers on a manifold (of dimension n), say observer 1 and observer 2, which may not be moving inertially. To compare measurements and to establish thus a sense of objectivity (identity of their results), they need to compare their measurements which take place in the tangent space at different points of the n-dimensional manifold M in which they are placed, so they have to establish the difference between their reference frames, i.e. the difference between the set of orthogonal (or pseudo-orthogonal) vectors at their locations, the so-called n-beins . Let $e_i(\mathcal{P}_0) = e_i^\alpha(\mathcal{P}_0)\partial_\alpha, i = 1, \ldots, n$ be the basis for observer 1 at point $\mathcal{P}_0$, and similarly $e_i(\mathcal{P}_1) = e_i^\alpha(\mathcal{P}_1)\partial_\alpha$ the reference frame for observer 2 at $\mathcal{P}_1$; let us denote the reference frame at the tangent space to the point $\mathcal{P}_1$ when parallely transported (without changing its length and angle) from $\mathcal{P}_0$ to $\mathcal{P}_1$ by $e_i(\mathcal{P}_0 \to \mathcal{P}_1)$ along a curve joining $\mathcal{P}_0$ to $\mathcal{P}_1$ with an affine connection, whose covariant derivative operator we denote as ∇ [44]. Then, ∇e_1 is the difference between $e_i(\mathcal{P}_0 \to \mathcal{P}_1)$ and $e_i(\mathcal{P}_1)$. This gap defect originates either to the deformation of $e_i(\mathcal{P}_0)$ along its path to $\mathcal{P}_1$, which cannot be transformed away by a change of coordinates, or by a change of coordinates from $\mathcal{P}_0$ to $\mathcal{P}_1$, which is not intrinsic and thus can be transformed away, or finally, by a combination of both. Let us move observer's one frame over two different paths. Parallel displacing an incremental vector $dx^b e_b$ from the point $\mathcal{P}_0$ along the basis vector e_a over an infinitesimal distance dx^a to the point $\mathcal{P}_1 = \mathcal{P}_0 + dx^a$ gives the vector

$$e_b dx^b(\mathcal{P}_0 \to \mathcal{P}_1) = dx^b e_b(\mathcal{P}_0) + \Gamma^c_{ba} dx^b \wedge dx^b e_c. \tag{1}$$

Similarly, the parallel transport of the incremental vector $dx^a e_a$ from the point $\mathcal{P}_0$ to $\mathcal{P}_2$ along the frame e_b over an infinitesimal distance dx^b to the point $\mathcal{P}_2 = \mathcal{P}_0 + dx^b$ gives the vector $e_a(\mathcal{P}_1 \to \mathcal{P}_2) = dx^a e_a(\mathcal{P}_1) + \Gamma^c_{ab} dx^a \wedge dx^b e_c$. The gap defect between $e_a(\mathcal{P}_0 \to \mathcal{P}_1)$ and the value of $dx^b e_b(\mathcal{P}_1)$ is

$$dx^b \nabla e_b(\mathcal{P}_1) = dx^b(\frac{\partial e_b}{\partial x^a}) \wedge dx^a - \Gamma^c_{ba} dx^a \wedge dx^b e_c, \tag{2}$$

and the gap defect between the vector $e_b(\mathrm{P}_1 \to \mathcal{P}_2)$ and $e_b dx^b(\mathcal{P}_2)$ is

$$dx^a De_a(\mathcal{P}_2) = dx^a(\frac{\partial e_a}{\partial x^b}) \wedge dx^a - \Gamma^c_{ab} dx^a \wedge dx^b e_c. \tag{3}$$

Therefore, the total gap defect between the two vectors is the comparison already alluded to)

$$dx^b \nabla e_b(\mathcal{P}_1) - dx^a De_a(\mathcal{P}_2) = (\frac{\partial e_b}{\partial x^a} - \frac{\partial e_a}{\partial x^b}) dx^a \wedge dx^b + [\Gamma^c_{ab} - \Gamma^c_{ba}] dx^a \wedge dx^b e_c, \quad (4)$$

where we recognize in the first term the Lie-bracket

$$[e_a, e_b] = (\frac{\partial e_b}{\partial x^a} - \frac{\partial e_a}{\partial x^b}) dx^a \wedge dx^b, \quad (5)$$

which we can write still as

$$[e_a, e_b] = C^c_{ab} e_c, \quad (6)$$

where C^c_{ab} are the coefficients of the anholonomity tensor, and then finally we can write the difference in eq.(4) as

$$dx^b e_b(\mathcal{P}_1) - dx^a \nabla e_a(\mathcal{P}_2) = (C^c_{ab} + [\Gamma^c_{ab} - \Gamma^c_{ba}]) dx^a \wedge dx^b e_c. \quad (7)$$

If we further introduce the vector-valued torsion two form

$$T = \frac{1}{2} T^c_{ab} dx^a \wedge dx^b e_c := \nabla e_b(e_a) - \nabla e_a(e_b) - [e_a, e_b]^c e_c \quad (8)$$

we find that the components T^c_{ab} are given by the so-called torsion tensor

$$T^c_{ab} = C^c_{ab} + [\Gamma^c_{ab} - \Gamma^c_{ba}] \quad (9)$$

Thus, we have two possibilities for the non-closure of infinitesimal paralblelograms. Either by anholonomity, or due to the non-symmetricity of the Christoffel coefficients. These are radically different. The latter can in some instances be set to be equal to zero, while the other term cannot. Say we have a coordinate transformation continuously differentiable $(x^1, \ldots, x^n) \rightarrow (y^1, \ldots, y^n)$ so we have that an holonomous transformation, i.e. we have that each dy^i is exact of the form

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j. \quad (10)$$

Then, if we take an holonomous basis $e_j = (\frac{\partial y^i}{\partial x^j}) \frac{\partial}{\partial y_i}$, then the anholonomity vanishes, $[e_i, e_j] = 0$ identically on M, and we are left for the expression for the torsion tensor

$$T^c_{ab} = \Gamma^c_{ab} - \Gamma^c_{ba}. \quad (11)$$

Anholonomity is very important. It is related to the existence of a time density, and is related to the Sagnac effect, to the Thomas precession, etc. [41]). Nowadays, relativistic rotation has become an issue of great interest, and the interest lays in rotating anholomous frames, in distinction with non-rotating holonomous frames. The torsion tensor evidences how the manifold is folded or dislocated, and the latter situation can be produced by tearing the manifold of by the addition of matter or fields to it. These are the well known Volterra operations of condensed matter physics (initially, in metalurgy [42]), and was the first engineering example of the fundamental role of torsion. The second example was elaborated in the pioneering work by Gabriel Kron in the geometrical representation of electric networks, and lead to the concept of negative resistance [27]. Contemporarily, negative resistance has become an important issue, after the discovery of its existence in some materials, with an accompanying apparent phenomenon of superconductivity [30]

3 RIEMANN-CARTAN-WEYL GEOMETRIES

In this section we follow [1,2]. In this article M denotes a smooth connected compact orientable n-dimensional manifold (without boundary). While in our initial works, we took for M to be space-time, there is no intrinsic reason for this limitation, in fact if can be an arbitrary configuration manifold and still a phase-space associated to a dynamical system. The paradigmatical example of the latter, is the projective space associated to a finite-dimensional Hilbert-space of a quantum mechanical system [3,39]. We shall further provide M with a linear connection as already described in the previous section, by a covariant derivative operator ∇ which we assume to be compatible with a given metric g on M, i.e. $\nabla g = 0$. Here, the metric can be the Minkowski degenerate metric, or an arbitrary positive-definite (i.e. Riemannian) metric. Given a coordinate chart (x^α) $(\alpha = 1, \ldots, n)$ of M, a system of functions on M (the Christoffel symbols of ∇) are defined by $\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma(x)^\alpha_{\beta\gamma} \frac{\partial}{\partial x^\alpha}$. The Christoffel coefficients of ∇ can be decomposed as:

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \frac{1}{2} K^\alpha_{\beta\gamma}. \tag{12}$$

The first term in (12) stands for the metric Christoffel coefficients of the Levi-Civita connection ∇^g associated to g, i.e. $\left\{ {\alpha \atop \beta\gamma} \right\} = \frac{1}{2}(\frac{\partial}{\partial x^\beta} g_{\nu\gamma} + \frac{\partial}{\partial x^\gamma} g_{\beta\nu} - \frac{\partial}{\partial x^\nu} g_{\beta\gamma}) g^{\alpha\nu}$, and

$$K^\alpha_{\beta\gamma} = T^\alpha_{\beta\gamma} + S^\alpha_{\beta\gamma} + S^\alpha_{\gamma\beta}, \tag{13}$$

is the cotorsion tensor, with $S^\alpha_{\beta\gamma} = g^{\alpha\nu} g_{\beta\kappa} T^\kappa_{\nu\gamma}$, and from eqs. (4) and (9) it follows that $T^\alpha_{\beta\gamma} = (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$ the skew-symmetric torsion tensor. We are interested in (one-half) the Laplacian operator associated to ∇, i.e. the operator acting on smooth functions on M defined as

$$H(\nabla) := 1/2\nabla^2 = 1/2 g^{\alpha\beta} \nabla_\alpha \nabla_\beta. \tag{14}$$

A straightforward computation shows that $H(\nabla)$ only depends in the trace of the torsion tensor and g, since it is

$$H(\nabla) = 1/2\triangle_g + \hat{Q} \equiv H(g, Q), \tag{15}$$

with $Q := Q_\beta dx^\beta = T^\nu_{\nu\beta} dx^\beta$ the trace-torsion one-form and where $\hat{Q}$ is the vector field associated to Q via g: $\hat{Q}(f) = g(Q, df)$, for any smooth function f defined on M. Finally, $\triangle_g$ is the Laplace-Beltrami operator of g: $\triangle_g f = \mathrm{div_g}\ \mathrm{grad} f$, $f \in C^\infty(M)$, with $\mathrm{div_g}$ the Riemannian divergence. Thus for any smooth function, we have $\triangle_g f = 1/[det(g)]^{\frac{1}{2}} g^{\alpha\beta} \frac{\partial}{\partial x^\beta}([det(g)]^{\frac{1}{2}} \frac{\partial}{\partial x^\alpha} f)$. Furthermore, the second term in (15), $\hat{Q}$, coincides with the Lie-derivative with respect to the vectorfield $\hat{Q}$: $L_{\hat{Q}} = i_{\hat{Q}} d + d i_{\hat{Q}}$, where $i_{\hat{Q}}$ is the interior product with respect to $\hat{Q}$: for arbitrary vectorfields $X_1, \ldots, X_{k-1}$ and ϕ a k-form defined on M, we have $(i_{\hat{Q}}\phi)(X_1, \ldots, X_{k-1}) = \phi(\hat{Q}, X_1, \ldots, X_{k-1})$. Then, for f a scalar field, $i_{\hat{Q}} f = 0$ and

$$L_{\hat{Q}} f = (i_{\hat{Q}} d + d i_{\hat{Q}}) f = i_{\hat{Q}} df = g(Q, df) = \hat{Q}(f). \tag{16}$$

Thus, our laplacian operator admits being written as

$$H(g, Q) = \frac{1}{2}\triangle_g + L_{\hat{Q}}. \tag{17}$$

Therefore, assuming that g is non-degenerate, we have defined a one-to-one mapping

$$\nabla \rightsquigarrow H(g, Q) = 1/2\triangle_k + L_{\hat{Q}}$$

between the space of g-compatible linear connections ∇ with Christoffel coefficients of the form

$$\Gamma^\alpha_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \ + \ \frac{2}{(n-1)} \left\{ \delta^\alpha_\beta \ Q_\gamma \ - \ g_{\beta\gamma} \ Q^\alpha \right\}, n \neq 1 \tag{18}$$

and the space of elliptic second order differential operators on functions.

4 RIEMANN-CARTAN-WEYL DIFFUSIONS

In this section we shall recall the correspondence between RCW connections defined by (18) and diffusion processes of scalar fields having $H(g, Q)$ as its diffrential generator. For this, we shall see this correspondence in the case of scalars. Thus, naturally we have called these processes as *RCW diffusion processes.*. For the extensions to describe the diffusion processes of differential forms, see [1], [4].

For the sake of generality, in the following we shall further assume that $Q = Q(\tau, x)$ is a time-dependent 1-form. In this setting τ is the universal time variable due to Stuckelberg [7]. The stochastic flow associated to the diffusion generated by $H(g, Q)$ has for sample paths the continuous curves $\tau \mapsto x(\tau) \in M$

satisfying the Itô invariant non-degenerate s.d.e. (stochastic differential equation)

$$dx(\tau) = \sigma(x(\tau))dW(\tau) + \hat{Q}(\tau, x(\tau))d\tau. \tag{19}$$

In this expression, $\sigma : M \times R^m \rightarrow TM$ is such that $\sigma(x) : R^m \rightarrow TM$ is linear for any $x \in M$, the noise tensor, so that we write $\sigma(x) = (\sigma_i^\alpha(x))$ $(1 \leq \alpha \leq n$, $1 \leq i \leq m)$ which satisfies

$$\sigma_i^\alpha \sigma_i^\beta = g^{\alpha\beta}, \tag{20}$$

where $g = (g^{\alpha\beta})$ is the expression for the metric in covariant form, and $\{W(\tau), \tau \geq 0\}$ is a standard Wiener process on R^m with zero mean with respect to the standard centered Gaussian function, and covariance given by $\mathrm{diag}(\tau, \ldots, \tau)$; finally, $dW(\tau) = W(\tau + d\tau) - W(\tau)$ is an increment. Now , it is important to remark that m can be arbitrary, i.e. we can take noise tensors defined on different spaces, and obtain the space diffusion process. In regards to the equivalence between the stochastic and the geometric picture, this enhances the fact that there is a freedom in the stochastic picture, which if chosen as the originator of the equivalence, points out to a more fundamental basis of the stochastic description. This is satisfactory, since it is impossible to identify all the sources for noise, and in particular those coming from the vacuum, which we take as the source for the randomness. Note that in taking the drift and the diffusion tensor as the original objects to build the geometry, the latter is derived from objects which are associated to *collective* phenomena. Note that if we start with eq. (19), we can reconstruct the associated RCW connection by using eq.(20) and the fact that the trace-torsion is the g-conjugate of the drift, i.e., in simple words, by lowering indexes of $\hat{Q}$ to obtain Q. We shall not go into the details of these constructions, which relies heavily on Stochastic Analysis on smooth manifolds [55,70], but yet we shall apply them to give a derivation of the noise term of the diffusion processes corresponding to the iso-Schroedinger equation.

5 THE HODGE DECOMPOSITION OF THE TRACE-TORSION FIELD

To obtain the most general form of the RCW laplacian in the non-degenerate case, we only need to know the most general decomposition of 1-forms. In this section, the metric g is positive-definite. We consider the Hilbert space given by the completion of the pre-Hilbert space of square-integrable smooth differential forms of degree k $(0 \leq k \leq n)$ on M, with respect to the Riemannian volume $\mathrm{vol_g}$, which we denote as $L^2(\sec(\Lambda^k(T^*M))$. We shall focus on the decomposition of 1-forms, so let $\omega \in L^2(\sec(T^*M))$; then we have the Hilbert space decomposition

$$\omega = df + A_{\mathrm{coex}} + A_{\mathrm{harm}}, \tag{21}$$

where f is a smooth real valued function on M, $A_{\rm coex}$ is a smooth coexact 1-form, i.e. there exists a smooth 2-form, β such that $\delta\beta_2 = A_{\rm coex}$ [2], so that $A_{\rm coex}$ is coclosed, i.e.

$$\delta A_{\rm coex} = \delta(\delta\beta_2) = 0, \tag{22}$$

and $A_{\rm harm}$ is a closed and coclosed smooth 1-form, then

$$\delta A_{\rm harm} = 0, dA_{\rm harm} = 0, \tag{23}$$

or equivalently, $A_{\rm harm}$ is harmonic, i.e.

$$\triangle_1 A_{\rm harm} \equiv {\rm trace}(\nabla^g)^2 A_{\rm harm} - R^{\alpha}_{\beta}(g)(A_{\rm harm})_\alpha \gamma^\alpha = 0, \tag{24}$$

with $R^{\alpha}_{\beta}(g) = R^{\mu\beta}_{\mu\alpha}(g)$ the Ricci metric curvature tensor. Eq. (24) is the sourceless Maxwell-de Rham equation. An extremely important fact is that this is a Hilbert space decomposition, so that it has unique terms, which are furthermore orthogonal in Hilbert space, i.e.

$$((df, A_{\rm coex})) = 0, ((df, A_{\rm harm})) = 0, ((A_{\rm coex}, A_{\rm harm})) = 0, \tag{25}$$

so that the decomposition of 1-forms (as we said before, this is also valid for k-forms, with the difference that f is a $k-1$-form, β_2 is really a $k+1$-form and $A_{\rm harm}$ is a k-form) has unique terms, and a fortiori, this is also valid for the Cartan-Weyl 1-form. We have proved that $A_{\rm coex}$ and $A_{\rm harm}$ are further linked with Maxwell's equations, both for Riemannian and Lorentzian metrics. For the stationary state which we shall describe in the next section, they lead to the equivalence of the Maxwell equation and the relativistic quantum mechanics equation of Dirac-Hestenes in a Clifford bundle setting [2,21] whenever the coclosed (Hertz potential) term and the (Aharonov-Bohm) harmonic term are both dependent on all the 4D variables while they are infinitesimal rotations defined on the spin-plane.

5.1 The Decomposition Of The Cartan-Weyl Form And The Stationary State

We wish to elaborate further on the decomposition of Q in the particular state in which the diffusion process generated by $H(g,Q)$, in the case M has a Riemannian metric g, and has a τ-invariant state corresponding to the asymptotic stationary state. Thus, we shall concentrate on the diffusion processes of scalar fields generated by

$$H(g,Q) = \frac{1}{2}(\triangle + L_{\hat{Q}}), {\rm with}\ Q = d{\rm ln}\psi^2 + A_{\rm coex} + A_{\rm harm}. \tag{26}$$

[2]Here δ denotes the codifferential operator, the adjoint of d, introduced above; see [1,2, 44].

This is the invariant form of the (forward) Fokker-Planck operator of this theory (and furthermore of the Schroedinger operator when introducing the phase function to the exact term of Q). Through this identification, we note that ψ is the scale field in the Einstein λ transformations from which in the vacuum, the RCW geometry can be obtained ; see [2]. We are interested now in the $\mathrm{vol_g}$-adjoint operator defined in $L^2(\sec(\Lambda^n(T^*M)))$, which we can think as an operator on densities, ϕ. Thus,

$$H(g,Q)^\dagger\phi = \frac{1}{2}(\triangle_g\phi - \mathrm{div_g}(\phi\mathrm{grad}\ \ln\ \phi) - \mathrm{div_g}(\phi\hat{A})). \tag{27}$$

The operator described by eq. (27) is the backward Fokker-Planck operator. The transition density $p^\nabla(\tau,x,y)$ is determined by the fundamental solution (i.e. $p^\nabla(\tau,x,-)\to\delta_x(-)$ as $\tau\to 0^+$) of the equation on the first variable

$$\frac{\partial u}{\partial\tau} = H(g,Q)(x)u(\tau,x,-). \tag{28}$$

Then , the diffusion process $\{x(\tau):\tau\geq 0\}$, gives rise to the Markovian semigroup $\{P_\tau = \exp(\tau H(g,Q)) : \tau\geq 0\}$ defined as

$$(P_\tau f)(x) = \int p^\nabla(\tau,x,y)f(y)\mathrm{vol_g}(y). \tag{29}$$

It has a unique τ-independant-invariant state described by a probability density ρ independant of τ determined as the fundamental weak solution (in the sense of the theory of generalized functions) of the τ-independent Fokker-Planck equation:

$$H(g,Q)^\dagger\rho \equiv \frac{1}{2}(-\delta d\rho + \delta(\rho Q)) = 0. \tag{30}$$

Let us determine the corresponding form of Q, say $Q_{\mathrm{stat}} = d\mathrm{ln}\psi^2 + A_{\mathrm{stat}}$. We choose a smooth real function U defined on M such that

$$H(g,Q_{\mathrm{stat}})^\dagger(e^{-U}) = 0, \tag{31}$$

so that

$$-de^{-U} + e^{-U}Q = \delta(-\delta\Pi + A_{\mathrm{harm}}), \tag{32}$$

for a 2-form Π and harmonic 1-form A_{harm}; thus, if we set the invariant density to be given by $\rho = e^{-U}\mathrm{vol_g}$, then

$$Q_{\mathrm{stat}} = d\mathrm{ln}\psi^2 + \frac{A}{\psi^2},\ \text{with}\ A = -\delta\Pi_2 + A_{\mathrm{harm}}. \tag{33}$$

Now we project $\frac{A}{\psi^2}$ into the Hilbert-subspaces of coexact and harmonic 1-forms, to complete thus the decomposition of Q_{stat} obtaining thus Hertz and Aharonov-Bohm potential 1-forms for the stationary state respectively. Yet these potentials have now a built-in dependence on the invariant distribution, and although they give rise to Maxwell's theory, the interpretation is now different. [3] Indeed, we have an inhomogeneous random media, and these potentials depend on the τ-invariant distribution of the media. We can use further the Hodge-decomposition of $Q_{\rm stat}$ to manifest the quantum potential as built-in. Indeed, if we multiply it by ψ and apply ∂, then we get that $d\ln\psi$, and the coexact and harmonic terms of $Q_{\rm stat}$ decouple in the resultant field equation which turns out to be

$$\triangle_g \psi = [g^{-1}(d\ln\psi, d\ln\psi) - \delta d\ln\psi]\psi, \tag{34}$$

with nonlinear potential $V := g^{-1}(d\ln\psi, d\ln\psi) - \delta d\ln\psi$, which has the form of (twice) a relativistic quantum potential extending Bohm's potential in non-relativistic Quantum Mechanics [35]. We have seen in [2], that from scale-invariance it follows that the quantum potential coincides up to a conformal factor with the metric scalar curvature. Thus in this setting it turns out that the random motions with noise tensor satisfying eq. (20) and drift vector field given by grad$\ln\psi$, the Riemannian gradient of the logarithm of the wave function, generate the gravitational field. We can see in this identity the relation with the ideas due to Le Sage already presented in the Introduction. Since the trace-torsion includes the Maxwell fields, we have unified space-time geometries with Brownian motions and the theory of electromagnetism. While the metric must be positive-definite to be able to apply the Hodge decomposition, and in particular we can take the flat Euclidean metric for a start, it is known from Hehl's work that the metric can be deduced from the constitutive relations in the theory of electromagnetism, so it can be taken as derived from the mere existence of electromagnetic fields and the constitutive relations [31]. We have showed in [3] that the theory of electromagnetism in euclidean space is related to establishing that the universal time parameter, τ, is the basic time variable instead of the linear time of the observer, t, and that this transformation is related to a dissipative process. Furthermore, the role of torsion as an active field can be described in these terms [3].

6 RCW GEOMETRIES, BROWNIAN MOTIONS AND THE SCHROEDINGER EQUATION

We have seen in [3] that we can represent the space-time quantum geometries for the relativistic diffusion associated with the invariant distribution, so

[3] A word of caution. In principle, $-\delta\Pi/\rho$ and $A_{\rm harm}/\rho$ may not be the coexact and harmonic components of A/ρ respectively. If this would be the case, then we obtain that $d\ln\psi$ is g^{-1}-orthogonal to both $-\delta\Pi$ and $A_{\rm harm}$; furthermore $d\ln\psi \wedge A_{\rm harm} = 0$, so furthermore they are co-linear. This can only be for null $A_{\rm harm}$ or constant ρ, so that the normalization of the electromagnetic potentials is by a trivial constant. In the first case the invariant state has the sole function of determining the exact term of Q to be (up to a constant) $d\ln\psi$.

that $Q = \frac{1}{2} d\ln\rho$, with $\rho = \psi^2$ and $H(g, Q)$ has a self-adjoint extension for which we can construct the quantum geometry on state-space and still the stochastic extension of the Schroedinger equation defined by this operator on taking the analytical continuation on the time variable for the evolution parameter. In this section which follows the solution of the Schroedinger problem of interpolation by Nagasawa [13] interpreted in terms of the RCW geometries and the Hodge decomposition of the trace-torsion, we shall present the equivalence between RCW geometries, their Brownian motions and the Schroedinger equation [3]. We shall show in the next section that this extends to HM.

Thus, we shall now present the construction of non-relativistic quantum mechanics for the case that includes the full Hodge decomposition of the trace-torsion, so that $Q = Q(t, x) = d\ln f_t(x) + A(t, x)$ where $f(t, x) = f_t(x)$ is a function defined on the configuration manifold given by $[a, b] \times M$ (where M is provided with a metric, g), to be determined below, and $A(t, x)$ is the sum of the harmonic and co-closed terms of the Hodge decomposition of Q, which we shall write as $A(t, x) = A_t(x)$ as a time-dependent form on M. The scheme to determine f will be to manifest the time-reversal invariance of the Schroedinger representation in terms of a forward in time diffusion process and its time-reversed representation for the original equations for creation and annihilation diffusion processes produced by the electromagnetic potential term of the trace-torsion of a RCW connection whose explicit form we shall determine in the sequel. From now onwards, both the exterior differential and the divergence operator will act on the M manifold variables only, which we shall write, say, as $df_t(x)$ to signal that the exterior differential acts only on the x variables of M. We should remark that in this context, the time-variable t of non-relativistic theory and the evolution parameter τ, are identical [32]. Let

$$L = \frac{\partial}{\partial t} + \frac{1}{2}(\triangle_g + A(t, x).\nabla) = \frac{\partial}{\partial t} + H(g, A_t) \tag{35}$$

with

$$\delta \hat{A}_t = -\mathrm{div_g} A_t = 0. \tag{36}$$

Here, as above, $\hat{A}_t$ denotes the conjugate vector field to the one-form A_t. In this setting, we start with a background trace-torsion restricted to an electromagnetic potential. We think of this electromagnetic potential and the associated Brownian motion having its metric conjugate as its drift, as the background geometry of the vacuum, which we shall subsequently relate to a creation and annihilation of particles and the equation of creation and annihilation is given by the following equation.

Let $p(s, x; t, y)$ be the weak fundamental solution of

$$L\phi + c\phi = 0. \tag{37}$$

The interpretation of this equation as one of creation (whenever $c > 0$) and annihilation ($c < 0$) of particles is warranted by the Feynman-Kac representation for the solution of this equation [13]. Then $\phi = \phi(t, x)$ satisfies the equation

$$\phi(s, x) = \int_M p(s, x; t, y)\phi(t, y)dy, \tag{38}$$

where for the sake of simplicity, we shall write in the sequel $dy = \mathrm{vol_g(y)} = \sqrt{\mathrm{det(g)}}dy^1 \wedge \ldots \wedge dy^3$. Note that we can start for data with a given function $\phi(a, x)$, and with the knowledge of $p(s, x; a, y)$ we define $\phi(t, x) = \int_M p(t, x; a, y)dy$. Next we define

$$q(s, x; t, y) = \frac{1}{\phi(s, x)} p(s, x; t, y)\phi(t, y), \tag{39}$$

which is a transition probability density, i.e.

$$\int_M q(s, x; t, y)dy = 1, \tag{40}$$

while

$$\int_M p(s, x; t, y)dy \neq 1. \tag{41}$$

Having chosen the function $\phi(t, x)$ in terms of which we have defined the probability density $q(s, x; t, y)$ we shall further assume that we can choose a second bounded non-negative measurable function $\tilde{\phi}(a, x)$ on M such that

$$\int_M \phi(a, x)\tilde{\phi}(a, x)dx = 1, \tag{42}$$

We further extend it to $[a, b] \times M$ by defining

$$\breve{\phi}(t, y) = \int \tilde{\phi}(a, x)p(a, x; t, y)dx, \forall(t, y) \in [a, b] \times M, \tag{43}$$

where $p(s, x; t, y)$ is the fundamental solution of eq. (37).

Let $\{X_t \in M, \mathcal{Q}\}$ be the time-inhomogeneous diffusion process in M with the transition probability density $q(s, x; t, y)$ and a prescribed initial distribution density

$$\mu(a, x) = \breve{\phi}(t = a, x)\phi(t = a, x) \equiv \breve{\phi}_a(x)\phi_a(x). \tag{44}$$

The finite-dimensional distribution of the process $\{X_t \in M, t \in [a, b]\}$ with probability measure on the space of paths which we denote as Q; for $a = t_0 < t_1 < \ldots < t_n = b$, it is given by

$$\begin{aligned} E_Q[f(X_a, X_{t_1}, \ldots, X_{t_n - 1}, X_b)] \quad = \quad & \int_M dx_0 \mu(a, x_0) q(a, x_0; t_1, x_1) dx_1 \ldots \\ & q(t_1, x_1; t_2, x_2)dx_2 \ldots q(t_{n-1}, x_{n-1}, b, x_n)dx_n \\ & f(x_0, x_1, \ldots, x_{n-1}, x_n) := [\mu_a q >> \end{aligned} \tag{45}$$

which is the Kolmogorov forward in time (and thus time-irreversible) representation for the diffusion process with initial distribution $\mu_a(x_0) = \mu(a, x_0)$, which using eq. (37) can still be rewritten as

$$\int_M dx_0 \mu_a(x_0) \frac{1}{\phi_a(x_0)} p(a, x_0; t_1, x_1) \phi_{t_1}(x_1) dx_1 \frac{1}{\phi_{t_1}(x_1)} dx_1 p(t_1, x_1; t_2, x_2)$$
$$\phi_{t_2}(x_2) dx_2 \ldots \frac{1}{\phi(t_{n-1}, x_{n-1})} p(t_{n-1}, x_{n-1}; b, x_n) \phi_b(x_n) dx_n f(x_0, \ldots, x_n) \quad (46)$$

which in account of $\mu_a(x_0) = \breve{\phi}_a(x_0)\phi_a(x_0)$ and eq. (39) can be written in the time-reversible form

$$\int_M \breve{\phi}_a(x_0) dx_0 p(a, x_0; t_1, x_1) dx_1 p(t_1, x_1; t_2, x_2) dx_2 \ldots p(t_{n-1}, x_{n-1}; b, x_n)$$
$$\phi_b(x_n) dx_n f(x_0, \ldots, x_n) \quad (47)$$

which we write as

$$= [\breve{\phi}_a p >><< p\phi_b]. \quad (48)$$

This is the *formally* time-symmetric Schroedinger representation with the transition (but not probability) density p. Here, the formal time symmetry is seen in the fact that this equation can be read in any direction, preserving the physical sense of transition. This representation, in distinction with the Kolmogorov representation, does *not* have the Markov property.

We define the adjoint transition probability density $\tilde{q}(s, x; t, y)$ with the $\tilde{\phi}$-transformation

$$\breve{q}(s, x; t, y) = \breve{\phi}(s, x) p(s, x; t, y) \frac{1}{\breve{\phi}(t, y)} \quad (49)$$

which satisfies the Chapmann-Kolmogorov equation and the time-reversed normalization

$$\int_M dx \breve{q}(s, x; t, y) = 1. \quad (50)$$

We get

$$E_{\tilde{Q}}[f(X_a, X_{t_1}, \ldots, X_b)] = \int_M f(x_0, \ldots, x_n) \breve{q}(a, x_0; t_1, x_1) dx_1 \breve{q}(t_1, x_1; t_2, x_2) dx_2$$
$$\ldots \breve{q}(t_{n-1}, x_{n-1}; b, x_n) \breve{\phi}(b, x_n) \phi(b, x_n) dx_n, \quad (51)$$

which has a form non-invariant in time, i.e. reading from right to left, as

$$<< \breve{q} \breve{\phi}_b \phi_b] =<< \breve{q} \breve{\mu}_b], \quad (52)$$

which is the time-reversed representation for the final distribution $\mu_b(x) = \breve{\phi}_b(x)\phi_b(x)$. Now, starting from this last expression and rewriting it in a similar form that is in the forward process but now with $\breve{\phi}$ instead of ϕ, we get

$$\int_M dx_0 \breve{\phi}_a(x_0) p(a, x_0; t_1, x_1) \frac{1}{\breve{\phi}_{t_1}(x_1))} dx_1 \breve{\phi}(t_1, x_1) p(t_1, x_1; t_2, x_2) \frac{1}{\breve{\phi}_{t_2}(x_2)} dx_2$$
$$\ldots dx_{n-1} \breve{\phi}(t_{n-1}, x_{n-1}) p(t_{n-1}, x_{n-1}; b, x_n) \frac{1}{\breve{\phi}(b, x_n)} \breve{\phi}_b(x_n)\phi(b, x_n) dx_n f(x_0, \ldots, x_n) \quad (53)$$

which coincides with the time-reversible Schroedinger representation $[\breve{\phi}_a p >><< p\phi_b]$.

We therefore have three equivalent representations for the diffusion process: The forward in time Kolmogorov representation, the backward Kolmogorov representation, which are both naturally irreversible in time, and the time-reversible Schroedinger representation, so that we can write succintly,

$$[\mu_a q >>= [\breve{\phi}_a p >><< p\phi_b] =<< \breve{q}\mu_b], \text{with } \mu_a = \phi_a\breve{\phi}_a,\ \mu_b = \phi_b\breve{\phi}_b. \quad (54)$$

In addition of this formal identity,we have to establish the relations between the equations that have led to them. We first note, that in the Schroedinger representation, which is formally time-reversible, we have an interpolation of states between the initial data $\breve{\phi}_a(x)$ and the final data, $\phi_b(x)$. The information for this interpolation is given by a filtration of interpolation $\mathcal{F}^r_a \cup \mathcal{F}^s_b$, which is given in terms of the filtration for the forward Kolmogorov representation $\mathcal{F} = \mathcal{F}^t_a, t \in [a, b]$ which is used for prediction starting with the initial density $\phi_a\breve{\phi}_a = \mu_a$ and the filtration $\mathcal{F}^b_t$ for retrodiction for the time-reversed process with initial distribution μ_b.

We observe that q and $\breve{q}$ are in time-dependent duality with respect to the measure

$$\mu_t(x)dx = \breve{\phi}_t(x)\phi_t(x)dx, \quad (55)$$

since if we define the time-homogeneous semigroups

$$Q_{t-s}f(s, x) = \int q(s, x; t, y) f(t, y) dy, s < t \quad (56)$$
$$g\breve{Q}_{t-s}(t, y) = \int dx g(s, x)\breve{q}(s, x; t, y), s < t, \quad (57)$$

then

$$
\begin{aligned}
\int dx\mu_s(x)g(s,x)Q_{t-s}f(s,x) &= \int dxg(s,x)\phi_s(x)\breve{\phi}_s(x)\frac{1}{\phi_s(x)}p(s,x;t,y)\phi_t(y)f(t,y)dy \\
&= \int dxg(s,x)\breve{\phi}_s(x)p(s,x;t,y)\frac{1}{\breve{\phi}_t(y)}f(t,x)\breve{\phi}_t(y)\phi_t(y)dy \\
&= \int dxg(s,x)\breve{q}(s,x;t,y)f(t,y)\breve{\phi}_t(y)\phi \\
&= \int dxg(s,x)\breve{Q}_{t-s}(t,y)f(t,y)\mu_t(y)dy, \qquad (58)
\end{aligned}
$$

and thus we can write in a more succint expression in terms of weighted densities the expression:

$$< g, Q_{t-s}f >_{\mu_s} = < g\breve{Q}_{t-s}, f >_{\mu_t}, s < t. \qquad (59)$$

We shall now extend the state-space of the diffusion process to $[a,b] \times M$, to be able to transform the time-inhomogeneous processes into time-homogeneous processes, while the stochastic dynamics still takes place exclusively in M. This will allow us to define the duality of the processes to be with respect to $\mu_t(x)dtdx$ and to determine the form of the exact term of the trace-torsion, and ultimately, to establish the relation between the diffusion processes and Schroedinger equations, both for potential linear and non-linear in the wave-functions. If we define time-homogeneous semigroups of the processes on $\{(t, X_t) \in [a,b] \times M\}$ by

$$P_r f(s,x) = \begin{cases} Q_{s,s+r}f(s,x) & , \quad s \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad (60)$$

and

$$\breve{P}_r g(t,y) = \begin{cases} gQ_{t-r,t}(t,y) & , \quad r \geq 0 \\ 0 & , \quad \text{otherwise} \end{cases} \qquad (61)$$

then

$$
\begin{aligned}
< g, P_r f >_{\mu_t dtdx} &= \int_r^{b-r} ds < g, Q_{s,s+r}f > \mu_s \\
&= \int_b^{a+r} < g, Q_{t-r,t}f > \mu_{t-r}(x)dx \\
&= \int_{a+r}^{b} dt < g\breve{P}_{t-r}, f >_{\mu_t dx} = < \breve{P}_r g, f >_{\mu_t dtdx}, \qquad (62)
\end{aligned}
$$

which is the duality of $\{(t, X_t)\}$ with respect to the $\mu_t dtdx$ density. We remark here that we have an augmented density by integrating with respect to time t. Consequently, if in our spacetime case we define for $a_t(x), \breve{a}_t(x)$ time-dependent one-forms on M (to be determined later)

$$B\alpha : \;=\; \frac{\partial \alpha}{\partial t} + H(g, A_t + a_t)\alpha_t, \qquad (63)$$

$$B^0\mu : \quad = \quad -\frac{\partial \mu}{\partial t} + H(g, A_t + a_t)^\dagger \mu_t, \tag{64}$$

and its adjoint operators

$$\breve{B}\beta = -\frac{\partial \beta}{\partial t} - H(g, -A_t + \breve{a}_t)^\dagger \beta_t, \tag{65}$$

$$(\breve{B})^0 \mu_t = \frac{\partial \mu_t}{\partial t} - H(g, -A_t + \breve{a}_t)^\dagger \mu_t, \tag{66}$$

where by $H(g, -A_t + \breve{a}_t)^\dagger$ we mean the vol_g-adjoint of this operator defined in eq.(27); i.e. $H(g, -A_t + \breve{a}_t)^\dagger \mu_t = \frac{1}{2}\triangle_g \mu_t - \mathrm{div_g}(\mu_t(-A_t + \breve{a}_t))$. Now

$$\int_a^b dt \int 1_{D_t}[(B\alpha_t)\beta_t] \quad - \quad \alpha_t(\breve{B}\beta_t)]\mu_t(x)dx = \int_a^b dt \int 1_{D_t}\alpha_t\beta_t(B^0\mu_t)dx$$
$$- \quad \int_a^b \int 1_{D_t}\alpha_t g([a_t + \breve{a}_t] - d\mathrm{ln}\mu_t, d\beta_t)\mu_t dx, \tag{67}$$

for arbitrary α, β smooth compact supported functions defined on $[a, b] \times M$ which we have denoted as time-dependent functions α_t, β_t, where 1_{D_t} denotes the characteristic function of the set $D_t(x) := \{(t, x) : \mu_t(x) = \phi_t(x)\breve{\phi}_t(x) > 0\}$. Therefore, the duality of space-time processes

$$< B\alpha, \beta >_{\mu_t(x)dtdx} = < \alpha, \breve{B}\beta >_{\mu_t(x)dtdx}, \tag{68}$$

is equivalent to

$$a_t(x) + \breve{a}_t(x) \quad = \quad d \ \ln \ \mu_t(x) \equiv d \ \ln \ (\phi_t(x)\breve{\phi}_t(x)), \tag{69}$$
$$B^0\mu_t(x) = 0. \tag{70}$$

The latter equation being the Fokker-Planck equation for the diffusion with trace-torsion given by $a + A$, then the Fokker-Planck equation for the adjoint (time-reversed) process is valid, i.e.

$$(\breve{B})^0\mu_t(x) = 0. \tag{71}$$

Substracting eqs. (70) and (71) we get the final form of the duality condition

$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(A_t + \frac{a_t - \breve{a}_t}{2})\mu_t)] \quad = \quad 0, \text{ for } \mu_t(x) = \breve{\phi}_t(x)\phi_t(x). \tag{72}$$

Therefore, we can establish that the duality conditions of the diffusion equation in the Kolmogorov representation and its time reversed diffusion lead to the following conditions on the additional elements of the drift vector fields:

$$a_t(x) + \breve{a}_t(x) \quad = \quad d \ \ln \ \mu_t(x) \equiv d \ \ln \ (\phi_t(x)\breve{\phi}_t(x)), \tag{73}$$
$$\frac{\partial \mu}{\partial t} + \mathrm{div_g}[(A_t + \frac{a_t - \breve{a}_t}{2})\mu_t] \quad = \quad 0. \tag{74}$$

If we assume that $a_t - \breve{a}_t$ is an exact one-form, i.e., there exists a time-dependent differentiable function $S(t,x) = S_t(x)$ defined on $[a,b]\times M$ such that for $t \in [a,b]$,

$$a_t - \breve{a}_t = d \ln \frac{\phi_t(x)}{\breve{\phi}_t(x)} = 2dS_t \tag{75}$$

which together with

$$a_t + \breve{a}_t = d \ln \mu_t, \tag{76}$$

implies that on $D(t,x)$ we have

$$a_t = d \ln \phi_t, \tag{77}$$
$$\breve{a}_t = d \ln \breve{\phi}_t \tag{78}$$

Introduce now $R_t(x) = R(t,x) = \frac{1}{2}\ln\phi_t\breve{\phi}_t$ and $S_t(x) = S(t,x) = \frac{1}{2}\ln\frac{\phi_t}{\breve{\phi}_t}$, so that

$$a_t(x) = d(R_t(x) + S_t(x)), \tag{79}$$
$$\breve{a}_t(x) = d(R_t(x) - S_t(x)), \tag{80}$$

and the eq. (74) takes the form

$$\frac{\partial R}{\partial t} + \frac{1}{2}\triangle_g S_t + g(dS_t, dR_t) + g(A_t, dR_t) = 0, \tag{81}$$

where we have taken in account that $\mathrm{div_g} A_t = 0$.

Remarks. Note that the time-dependent function S on the 3-space manifold, is defined by eq. (75) up to addition of an arbitrary function of t, and when further below we shall take this function as defining the complex phase of the quantum Schroedinger wave, this will introduce the quantum-phase indetermination of the quantum evolution, just as we discussed already in the setting of geometry of the quantum state-space. In the other hand, this introduces as well the subject of the multivaluedness of the wave function, which by the way, leads to the Bohr-Sommerfeld quantization rules of QM established well before it was developed as an operator theory. It is noteworthy to remark that these quantization rules, later encountered in superfluidity and superconductivity, or still in the physics of defects of condensed matter physics, are of topological character. It has been proved by Kiehn that the Schroedinger wave equation contains the Navier-Stokes equations for a viscous compressible fluid in 2D, and that the probability density transforms into the enstrophy (i.e. the squared vorticity) of the viscous fluid obeying the Navier-Stokes equations [20]. Thus, one might expect that Navier-Stokes equations could also have multivalued solutions, namely in the 2D case of the already established relation, the vorticity reduces to a time-dependent function.

Therefore, together with the three different time-homogeneous representations $\{(t, X_t), t \in [a,b], X_t \in M\}$ of a time-inhomogeneous diffusion process

$\{X_t, Q)$ on M we have three equivalent dynamical descriptions. One description, with creation and killing described by the scalar field $c(t,x)$ and the diffusion equation describing it is given by a creation-destruction potential in the trace-torsion background given by an electromagnetic potential

$$\frac{\partial p}{\partial t} + H(g, A_t)(x)p + c(t,x)p = 0; \tag{82}$$

the second description has an additional trace-torsion $a(t,x)$, a 1-form on $R \times M$

$$\frac{\partial q}{\partial t} + H(g, A + a_t)q = 0. \tag{83}$$

while the third description is the adjoint time-reversed of the first representation given by $\tilde{\phi}$ satisfying the diffusion equation on the background of the reversed electromagnetic potential $-A$ in the vacuum, i.e.

$$-\frac{\partial \breve{\phi}}{\partial t} + H(g, -A_t)\breve{\phi} + c\breve{\phi} = 0. \tag{84}$$

The second representation for the full trace-torsion diffusion forward in time Kolmogorov representation, we need to adopt the description in terms of the fundamental solution q of

$$\frac{\partial q}{\partial t} + H(g, A_t + a_t)q = 0, \tag{85}$$

for which one must start with the initial distribution $\mu_a(x) = \breve{\phi}_a(x)\phi_a(x)$. This is a time t-irreversible representation in the real world, where q describes the real transition and μ_a gives the initial distribution. If in addition one traces the diffusion backwards with reversed time t, with $t \in [a,b]$ running backwards, one needs for this the final distribution $\mu_b(x) = \breve{\phi}_b(x)\phi_b(x)$ and the time t reversed probability density $\breve{q}(s,x;t,y)$ which is the fundamental solution of the equation

$$-\frac{\partial \breve{q}}{\partial t} + H(g, -A_t + \breve{a}_t)\hat{q} = 0, \tag{86}$$

with additional trace-torsion one-form on $R \times M$ given by $\tilde{a}$, where

$$\breve{a}_t + a_t = d\ln\mu_t(x), \text{ with } \mu_t = \phi_t\breve{\phi}_t. \tag{87}$$

where the diffusion process in the time-irreversible forward Kolmogorov representation is given by the Ito s.d.e

$$dX_t^i = \sigma_j^i(X_t)dW_t^j + (A + a)^i(t, X_t)dt, \tag{88}$$

and the backward representation for the diffusion process is given by

$$dX_t^i = \sigma_j^i(X_t)dW_t^j + (-A + \breve{a})^i(t, X_t)dt, \tag{89}$$

where $a, \breve{a}$ are given by the eqs. (79) and (80), and $(\sigma\sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$; see eq. (20).

We follow Schroedinger in pointing that ϕ and $\breve{\phi}$ separately satisfy the creation and killing equations, while in quantum mechanics ψ and $\bar{\psi}$ are the complex-valued counterparts of ϕ and $\breve{\phi}$, respectively, they are not arbitrary but

$$\phi\breve{\phi} = \psi\bar{\psi}. \tag{90}$$

Thus, in the following , this Born formula, once the equations for ψ are determined, will be a consequence of the constructions, and not an hypothesis on the random basis of non-relativistic mechanics.

Therefore, the equations of motion given by the Ito s.d.e.

$$dX_t^i = (\hat{A} + \mathrm{grad}\phi)^i(t, X_t)dt + \sigma_j^i(X_t)dW_t^j, \tag{91}$$

which are equivalent to

$$\frac{\partial u}{\partial t} + H(g, A_t + a_t)u = 0 \tag{92}$$

with $a_t = d\ln\phi_t = d(R_t + S_t)$, determines the motion of the ensemble of non-relativistic particles. Note that this equivalence requires only the Laplacian for the RCW connection with the forward trace-torsion full one-form

$$Q(t, x) = A_t(x) + d\ln\phi_t(x) = A_t(x) + d(R_t(x) + S_t(x)). \tag{93}$$

In distinction with Stochastic Mechanics due to Nelson, and contemporary ellaborations of this applied to astrophysics as the theory of Scale Relativity due to Nottale [10][12], we only need the form of the trace-torsion for the forward Kolmogorov representation, and this turns to be equivalent to the Schroedinger representation which interpolates in time-symmetric form between this forward process and its time dual with trace-torsion one-form given by $-A_t + \breve{a}_t(x) = -A_t(x) + d\ln\breve{\phi}_t(x) = -A_t(x) + d(R_t(x) - S_t(x))$.

Finally, let us how this is related to the Schroedinger equation. Consider now the Schroedinger equations for the complex-valued wave function ψ and its complex conjugate $\bar{\psi}$, i.e. introducing $i = \sqrt{-1}$, we write them in the form

$$\begin{aligned} i\frac{\partial\psi}{\partial t} + H(g, iA_t)\psi - V\psi &= 0 & (94)\\ -i\frac{\partial\bar{\psi}}{\partial t} + H(g, -iA_t)\bar{\psi} - V\bar{\psi} &= 0, & (95) \end{aligned}$$

which are identical to the usual forms. So, we have the imaginary factor appearing in the time t but also in the electromagnetic term of the RCW connection with trace-torsion given now by iA, which we confront with the diffusion equations generated by the RCW connection with trace-torsion A, i.e. the system

$$\frac{\partial\phi}{\partial t} + H(g, A_t)\phi + c\phi = 0, \qquad (96)$$

$$\frac{-\partial\breve{\phi}}{\partial t} + H(g, -A_t)\breve{\phi} + c\breve{\phi} = 0, \qquad (97)$$

and the diffusion equations determined by both the RCW connections with trace-torsion $A + a$ and $-A + \breve{a}$, i.e.

$$\frac{\partial q}{\partial t} + H(g, A_t + a_t)q = 0, \qquad (98)$$

$$\frac{-\partial\breve{q}}{\partial t} + H(g, -A_t + \breve{a}_t)\breve{q} = 0, \qquad (99)$$

which are equivalent to the single equation

$$\frac{\partial q}{\partial t} + H(g, A_t + d\ln\phi_t)q = 0. \qquad (100)$$

If we introduce a complex structure on the two-dimensional real-space with coordinates (R, S), i.e. we consider

$$\psi = e^{R+iS}, \psi = e^{R-iS}, \qquad (101)$$

viz a viz $\phi = e^{R+S}, \breve{\phi} = e^{R-S}$, with $\psi\bar{\psi} = \phi\breve{\phi}$, then for a wave-function differentiable in t and twice-differentiable in the space variables, then, ψ satisfies the Schroedinger equation if and only if (R, S) satisfy the difference between the Fokker-Planck equations , i.e.

$$\frac{\partial R}{\partial t} + g(dS_t + A_t, dR_t) + \frac{1}{2}\triangle_g S_t = 0, \qquad (102)$$

and

$$V = -\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t - A_t, dS_t). \qquad (103)$$

which follows from substituting ψ in the Schroedinger equation and further dividing by ψ and taking the real part and imaginary parts, to obtain the former and latter equations, respectively.

Conversely, if we take the coordinate space given by $(\phi, \breve{\phi})$, both non-negative functions, and consider the domain $D = D(s, x) = \{(s, x) : 0 < \breve{\phi}(s, x)\phi(s, x)\} \subset [a, b] \times M$ and define $R = \frac{1}{2}\ln\phi\breve{\phi}, S = \frac{1}{2}\ln\frac{\phi}{\breve{\phi}}$, with R, S having the same differentiabilty properties that previously ψ, then $\phi = e^{R+S}$ satisfies in D the equation

$$\frac{\partial\phi}{\partial t} + H(g, A_t)\phi + c\phi = 0, \qquad (104)$$

if and only if

$$
\begin{aligned}
-c &= [-\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t) - g(A_t, dS_t)] \\
&+ [\frac{\partial R}{\partial t} + H(g, dR_t)S_t + g(A_t, dR_t)] + [2\frac{\partial S}{\partial t} + g(dS_t + 2A_t, dS_t)] \quad (105)
\end{aligned}
$$

while $\breve{\phi} = e^{R-S}$ satisfies in D the equation

$$
-\frac{\partial \breve{\phi}}{\partial t} + H(g, -A_t)\breve{\phi} + c\breve{\phi} = 0, \quad (106)
$$

if and only if

$$
\begin{aligned}
-c &= [-\frac{\partial S}{\partial t} + H(g, dR_t)R_t - \frac{1}{2}g(dS_t, dS_t) - g(A_t, dS_t)] \\
&- [\frac{\partial R}{\partial t} + H(g, dR_t)S_t + g(A_t, dR_t)] + [2\frac{\partial S}{\partial t} + g(dS_t + 2A_t, dS_t)] \quad (107)
\end{aligned}
$$

Notice that $\phi, \breve{\phi}$ can be both negative or positive. So if we define $\psi = e^{R+iS}$, it then defines in weak form the Schroedinger equation in D with

$$
V = -c - 2\frac{\partial S}{\partial t} - g(dS_t, dS_t) - 2g(A_t, dS_t). \quad (108)
$$

We note that from eq. (108) follows that we can choose S in a way such that either c is independent of S and thus V is a potential which is non-linear in the sense that it depends on the phase of the wave function ψ and thus the Schroedinger equation with this choice becomes non-linear dependent of ψ, or conversely, we can make the alternative choice of c depending non-linearly on S, and thus the creation-annihilation of particles in the diffusion equation is non-linear, and consequently the Schroedinger equation has a potential V which does not depend on ψ. In the case that V is such that the spectrum of $H(g, A + a)$ is discrete, we know already we can represent the Schroedinger equation in state-space and further study the related stochastic Schroedinger equation as described above. Finally, we have presented a construction in which by using two scalar diffusing processes $\phi, \breve{\phi}$ we have been able to subsume them into a single forward in time process with additional trace-torsion given by $a_t = d\ln\phi_t\breve{\phi}_t$.

With respect to the issue of nonlinearity of the Schroedinger equation, one could argue that the former case means that the superposition principle of QM is broken, but then one observes that precisely due to the fact that the wave function depends on the phase, the superposition principle is invalid from the fact that we are dealing with complex-valued wave functions, and when dealing with the Schroedinger or Heisenberg evolutions in state-space, the complex factor has been quotiented [3].

In particular, nonlinear Schroedinger equations will appear in the Lie-isotopic extensions of the linear Schroedinger equation of QM, due to Santilli [14-19]. As explained in the Introduction, all the mathematical and conventional physical theories (Special and General Relativity, QM, RQM, etc.) can be isotopically lifted by applying a non-unitary transformation that produces a generalized unit. In the case of QM this construction yields a linear theory in the isotopically lifted mathematical structures (this property is called isolinearity [16]), but when translated to the original structures, the Schroedinger equation becomes non-linear. Conversely, we shall see that a non-linear Schroedinger equation can be turned into the iso-linear iso-Schroedinger equation by taking the non-linear terms of the potential into the isotopic generalized unit. Hence it follows that it is possible, in principle, to present HM as a theory of diffusion processes. This will be the subject of our next sections below.

Returning to the issue of the nonlinearity of the potential function V in Quantum Mechanics, the usual form is the known logarithmic expression $V = -b(\ln|\psi|^2)\psi$ introduced by Bialnicky-Birula and Mycielski [23]. Its importance in such diverse fields as quantum optics, superconductivity, atomic and molecular physics cannot be disregarded. Soliton solutions of nonlinear Schroedinger equations may have a role central to molecular biology, in which the DNA structure may be associated with a superconductive state. With regards as the relation between geometries, Brownian motions and the linear and Schroedinger equations, there is an alternative line of research which stems from two principles, which are interwoven. The first one is that all physical fields have to be construed in terms of scale fields starting from the fields appearing in the Einstein lambda transformations,of which, the Schroedinger wave function is an elementary example as shown here [2,23], and when further associated to the idea of a fractal spacetime, this has lead to Nottale's theory of Scale Relativity [10,26]. Nottale's theory starts from this fractal structure to construct a covariant derivative operator in terms of the forward and backward stochastic derivatives introduced by Nelson in his theory of stochastic mechanics [8]. Working with these stochastic derivatives, the basic operator of Nottale's theory, can be written in terms of our RCW laplacian operators of the form $\frac{\partial}{\partial\tau} + H_0(i\mathcal{D}g, \mathcal{V})$ where $\mathcal{D}$ is diffusion constant (equal to $\frac{\hbar}{2m}$ in nonrelativistic quantum mechanics), and $\mathcal{V}$ is a complex differentiable velocity field, our complex drift appearing after introducing the imaginary unit $i = \sqrt{-1}$; see [10]. In the present conception, this fundamental operator in terms of which Nottale constructs his theory, does not require for its introduction to assume that space-time has a fractal structure a priori, from which stochastic derivatives backward and forward to express the time asymmetry construct the dynamics of fields. We rather assume that at a fundamental scale which is generally associated with the Planck scale, we can represent space-time as a continuous media in which what really matters are the defects in it, and thus torsion has a fundamental role since it defines these dislocations. The fractal structure of spacetime arises from the association between the RCW laplacian operators which as coincides with Nottale's covariant derivative operator, and the Brownian motions which alternatively, can be seen as constructing the space-time geometry. So there is no place as to the discussion of what goes first, at least in the conception in the present work. Remarkably, the flow of these Brownian motions under general analytical conditions, define for every trial Wiener path, an active diffeomorphism of space-time. But this primeval role of the Brownian motions and fractal structures, stems from our making the choice -arbitrary, inasmuch as the other choice is arbitrary- as the fundamental structure instead of choosing the assumption of having a RCW covariant derivative with a trace-torsion field defined on a continuous model of space-time. In some sense the possibility of choosing as primeval the Brownian motions for starting the construction of the theoretical framework, is interesting in regards that they can be constructed as continous limits of discrete jumps, as every basic book in probability presents [46], and thus instead of positing a continuous space-time, we can think from the very beginning in a discrete spacetime, and construct a theory of physics in these terms [4]. In this case, instead of working with the field of the real number or its complex or biquaternion extensions, one can take a p-adic field, such as the one defined by the Mersenne prime number $2^{127} - 1$ which is approximately equal to the square of the ratio between the Planck mass and the proton mass [28]. In fact, a theory of physics in terms of discrete structures associated to the Mersenne prime numbers hierarchy, has been constructed in a program developed by P. Noyes, T. Bastin, P. Kilmister and others; see [47]. A remarkable unified theory of physics, biology and consciousness, in terms of p-adic field theory, has been elaborated by M.Pitkanen [51].

We would like to comment that Castro, Mahecha and Rodriguez [25], following the Nottale constructions have derived the nonlinear Schroedinger equation and associated it to a Brownian motion with a complex diffusion constant. Futhermore, working with Weyl connections (which are to be distinguished from the present work's RCW connections) in that they are not integrable and they have zero torsion (they can be introduced in terms of the reduced set of Einstein lambda transformations when one does not posit the tetrad or cotetrad fields as fundamental and the invariance of the Riemann-Cartan connection), they have derived the relativistic quantum potential in terms of the difference between the Weyl curvature of this connection and the Riemannian curvature, while in the present theory, we have associated above the relativistic quantum potential with the Riemannian curvature, which is more closely related with the idea of Brownian motion in spacetime (without additional internal degrees of freedom as the Weyl connections introduce) as being the generator of gravitation and all fundamental fields.

[4]Prof. Shan Gao,has initiated a program of construction of quantum mechanics as random discontinuous motions in discrete spacetime, in his recent work *Quantum Motion, Unveiling the Mysterious Quantum World*, Arima Publ., Suffolk (U.K.), 2006.

7 Isotopic Geometries and Isotopic Gauge Theories

We shall present next the essential elements of HM to relate them to torsion fields and the Brownian motions that we have presented already.

7.1 Lie-Isotopic Gauge Theory

In this section, we want to place in evidence the role the kinematical role that torsion has in the gauge theories for an isotopic Lie group. For a start in the introduction to the mathematical aspects of HM, in this section we shall explicit the fundamental kinematical role that torsion has in Lie-isotopic gauge theories, by starting at the level of the Lie algebras and their Lie-isotopic liftings.

Let G be a Lie group, $\mathbf{g}$ its Lie algebra and ξ its enveloping algebra.Thus, ξ has an associative product, which for reasons we shall denote its product as $A \times B$. We shall now onwards explicit the usual product and composition through the sign $\times$ to distinguish with the isotopic product we shall introduce next. Given an invertible and hermitean operator T, we shall introduce in ξ a generalized isotopic product $A \hat{\times} B = A \times T \times B$ with an isotopic unit $\hat{I} = T^{-1}$ such that $A \hat{\times} \hat{I} = \hat{I} \hat{\times} A = A$. Consequently, the usual definitions of hermitean conjugate, $A^\dagger$, and inverse, A^{-1} , of an operator A must be replaced by the following isotopic generalizations: The T-hermitean conjugate, $A^{\hat{\dagger}}$,

$$A^{\hat{\dagger}} = T^\dagger \times A^\dagger \times \hat{I}, \tag{109}$$

and the T-inverse, $\hat{A}^{-1}$,

$$\hat{A}^{-1} = \hat{I} \times A^{-1} \times \hat{I}. \tag{110}$$

Furthermore, the T-isotope, $\hat{\mathrm{e}}(A)$, of the exponential operator e^A, of an operator A, is defined as

$$\hat{\mathrm{e}}^A = \hat{I} \times \mathrm{e}^{T \times A} = \mathrm{e}^{A \times T} \times \hat{I}. \tag{111}$$

Suppose we have a field theory invariant under the action of the compact gauge group G,

$$\Psi' = U \times \Psi \tag{112}$$

with

$$U = I \times \mathrm{e}^{-i \times \theta^k \times X_k} = \mathrm{e}^{-i \times \theta^k \times X_k} \times I, \tag{113}$$

and θ^k is a set of real functions and X_k are generators of the algebra $\mathbf{g}$ satisfying

$$[X_i, X_j] = X_i \times X_j - X_j \times X_i = i \times c^k_{ij} \times X_k, \tag{114}$$

with the numbers c^k_{ij} being the structure constants, i.e. the coefficients of the torsion tensor of the canonical connection of the manifold given by G [49]. The infinitesimal form of the transformation is

$$\delta\Psi = -i \times \epsilon^k \times X_k \times \Psi, \tag{115}$$

where the ϵ^k are the infinitesimal parameters corresponding to θ^k. We remark that the representation matrices of the transformations are unitary: $U^\dagger \times U = I, [U^\dagger, U] = 0$, and the invariant of the theory is $\Psi^\dagger \times \Psi = \Psi'^\dagger \times \Psi'$.

The Santilli-Lie-isotopic lift of G, $\hat{G}$, is represented by the transformation given by

$$\Psi' = \hat{U} \hat{\times} \Psi, \tag{116}$$

where

$$\hat{U} = \hat{I} \times \mathrm{e}^{-i \times \theta^k \hat{\times} X_k} = \mathrm{e}^{-i \times \theta^k \hat{\times} X_k} \times \hat{I}, \tag{117}$$

with θ^k a set of real functions and X_k are generators of the algebra $\mathbf{g}$ are as before. We compute the isotopic hermitean conjugate of $\hat{U}$,

$$\hat{U}^{\hat{\dagger}} = T^\dagger \times (T^{-1})^\dagger \times \mathrm{e}^{i\theta^k \hat{\times} X_k} \times T^{-1} = \mathrm{e}^{i \times \theta^k \hat{\times} X_k} \times \hat{I}, \tag{118}$$

and its isotopic inverse

$$\hat{U}^{-1} = T^{-1} \times T \times \mathrm{e}^{i \times \theta^k \hat{\times} X_k} \times T^{-1} = \mathrm{e}^{i \times \theta^k \hat{\times} X_k} \times \hat{I}, \tag{119}$$

which proves that $\hat{U}$ is a T-unitary operator:

$$\hat{U}^{\hat{\dagger}} = \hat{U}^{-\hat{1}}. \tag{120}$$

We observe that if T is a positive-definite (alternatively, negative-definite), then $\hat{G}$ is locally isomorphic to G. Furthermore, the isotopic condition of hermiticity coincides with the usual one when defining an isotopic Hilbert space by the isotopic inner product

$$(A \hat{;} B) = (A, T \times B) \times \hat{I} \in \hat{C}, \tag{121}$$

where $\hat{C}$ is the Santilli iso-field of complex numbers , i.e. $\hat{C} = C \times \hat{I}$; we shall return to define this isofield in the next section. The infinitesimal form of the Lie-isotopic transformation follows from eqs. (116, 117) is given by

$$\hat{I} \approx -i \times \epsilon^k \hat{\times} X_k, \tag{122}$$

and

$$\delta\Psi = -i \times X_k \hat{\times} \epsilon^k \times \Psi. \tag{123}$$

The construction of a gauge theory for $\hat{G}$, i.e. a Lie-Santilli-isotopic theory for G [33], starts with T defining the isotopic unit, being locally dependent. In fact T can depend on the base manifold M as well as the tangent manifold and its higher orders. Since $\hat{U}$ is a T-unitary operator, from eq. (120) we have

$$\hat{U}^{\hat{\dagger}} \hat{\times} \hat{U} = I, \tag{124}$$

so that we can construct the following invariant

$$\Psi \hat{\times} \Psi^\dagger = \Psi' \hat{\times} \Psi'^\dagger = \Psi \hat{\times} \hat{U} \hat{\times} \hat{U}^{\hat{\dagger}} \hat{\times} \Psi^\dagger. \tag{125}$$

Finally, to preserve invariance under local isotopic transformations, i.e. $\hat{U} = \hat{U}(x)$, so that $\theta = \theta(x)$ and/or $T = T(x)$, we introduce -as in ordinary gauge theory for the group G- the isotopic covariant derivative

$$\hat{D}_\mu = (\partial_u - i \times e \times A^k_\mu \hat{\times} X_k) \times \hat{I}, \tag{126}$$

and we define the transformation rule for it as

$$\hat{D}'_\mu = \hat{U} \hat{\times} \hat{D}_\mu \hat{\times} \hat{U}^{-\hat{1}}, \tag{127}$$

or still,

$$\hat{D}'_\mu \hat{\times} \hat{U} \hat{\times} \Psi = \hat{U} \hat{\times} \hat{D}_\mu \hat{\times} \Psi. \tag{128}$$

Factorising $\hat{D} = \hat{D}_\mu \times \hat{I}$ and $\hat{U} = U^* \times \hat{I}$, where

$$\hat{D}_\mu = \partial_\mu - ieA^k_\mu \hat{\times} X_k, \tag{129}$$

and

$$U^* = \mathrm{e}^{-i \times \theta^k \hat{\times} X_k}, \tag{130}$$

we obtain from eq. (126)

$$A'^k_\mu \hat{\times} X_k = U^* \times A^k_\mu \hat{\times} X_k \times U^{*-1} - \frac{i}{e} \times (\partial_\mu U^*) \times U^{*-1}. \tag{131}$$

and we recognize in this expression the isotopic lifting of the gauge transformation fo the connection one-form $A = A^k_\mu X_k dx^k$. If we develop U^* as

$$U^* \approx I - i \times \epsilon^k \hat{\times} X_k, \tag{132}$$

and

$$U^{*-1} \approx I + i \times \epsilon^k \hat{\times} X_k, \tag{133}$$

then to first order in ϵ, we have

$$\delta A^k_\mu \times T \times X_k = -\frac{1}{\epsilon} \times \partial_\mu(\epsilon^k \hat{\times} X_k) + i \times [A^k_\mu \hat{\times} X_\mu, \epsilon^m \hat{\times} X_m], \tag{134}$$

which still is equal to

$$\frac{-1}{e} \times (\partial_\mu \epsilon^k \times T) \times X_k + i \times A^k_\mu \times \epsilon^m \times T \times [X_k \hat{,} X_m], \tag{135}$$

where we have introduced the isotopic commutator

$$[X_k \hat{,} X_m] = X_k \times T \times X_m - X_m \times T \times X_k, \tag{136}$$

where we note that we can read from this eqs. (135, 136) the modification of the structure constants of of $\mathbf{g}$, i.e. the coefficients of the torsion tensor of the canonical connection on the group-manifold G, to *local* dependent structure constants depending on T, the Lie-isotopic unit defining the gauge theory of $\hat{G}$. This is the essential geometrical *kinematical* character of a Lie-isotopic gauge theory, the appearance of the modification of the torsion tensor by the modification of the infinitesimal symmetries through the generalized units. On further gauging, appears a torsion field of the form of the logarithmic differential of the isotopic unit. Thus, we can apply this to the construction of a gauge theory for $\hat{G}$ constructed on an arbitrary space-time manifold, which can be provided with an Euclidean or Minkowski metrics, or still, in a more general Santilli-Lie-isotopic setting in which the metric is iso-Euclidean or iso-Minkowski, i.e. the isotopic lifts of the Euclidean and Minkowski metric respectively; more of this below. But in this sense, the Lie-Santilli-isotopic theory was constructed to account for the deformation of the usual group structures. This feature was overlooked in the original works of Santilli, and in subsequent developments by other researchers, but first treated in [29] with the then current lack of understanding of the need of introducing the lift of the whole mathematical structures in the theory, which was achieved in 1997; see [16] for a complete list of references and a history of these developments.

Finally, the isotopic Yang-Mills field strength curvature $\hat{F}_{\mu\nu}$ are defined as follows:

$$\hat{F}_{\mu\nu}\hat{\times}\Psi = \frac{i}{e} \times [\hat{D}_\mu, \hat{D}_\nu]\hat{\times}\Psi = \frac{i}{e} \times (\hat{D}_\mu\hat{\times}\hat{D}_\nu - \hat{D}_\nu\hat{\times}\hat{D}_\mu)\hat{\times}\Psi. \tag{137}$$

It can be checked straightforwardly that it transforms covariantly under the isotopic gauge transformation, since from and we get

$$\hat{F}'^{k}_{\mu\nu} = \hat{U}\hat{\times}\hat{F}_{\mu\nu}\hat{\times}\hat{U}^{-1}. \tag{138}$$

Introducing eq. (127) into eq. (137), we get the expression

$$\begin{aligned} \hat{F}^i_{\mu\nu}\hat{\times}X_i &= (\partial_\mu A^k_\nu - \partial_\nu A^k_\mu)\hat{\times}X_k + A^k_\alpha \times (\delta^\alpha_\beta \times \partial_\mu T - \delta^\alpha_\mu \times \partial_\nu T) \times X_k \\ &- i \times e A^k_\mu \times A^m_\nu \times T \times [X_k \hat{,} X_m]. \end{aligned} \tag{139}$$

We shall now make an important assumption on T, namely that it lies in the center of $\mathbf{g}$, so that T commutes with any element of $\mathbf{g}$, i.e. $[T, X_k] = 0$ for any X_k. This is satisfied by isotopic units such as the one defined in eq. (163) below. Then, from eqs. (127, 136, 139) we get

$$\hat{D}_\mu\hat{\times}\Psi = (\partial_\mu - i \times e \times A^k_\mu \times T \times X_k) \times \Psi, \tag{140}$$

$$\delta A^i_\mu = \frac{-1}{e} \times T^{-1} \times \partial_\mu(\epsilon^i \times T) + c^i_{jk} \times \epsilon^j \times T \times A^k_\mu, \tag{141}$$

$$\hat{F}^i_{\mu\nu} = \nabla_\mu A^i_\nu - \nabla_\nu A^i_\mu + e \times T \times c^i_{jk} \times A^j_\mu \times A^k_\nu, \tag{142}$$

where

$$\nabla_\mu A^i_\nu = \partial_\mu A^i_\nu - A^i_\alpha \times \Gamma^\alpha_{\mu\nu}, \tag{143}$$

and

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} \times (\delta^{\alpha}_{\mu} \times \partial_{\nu} T - \delta^{\alpha}_{\nu} \times \partial_{\mu} T) \times T^{-1}. \tag{144}$$

Therefore,

$$\hat{F}^{i}_{\mu\nu} = \partial_{\mu} A^{i}_{\nu} - \partial_{\nu} A^{i}_{\mu} + A^{i}_{\alpha} \times T^{\alpha}_{\nu\mu} + e \times T \times c^{i}_{jk} \times A^{j}_{\mu} \times A^{k}_{\nu}, \tag{145}$$

where we have written the skew-symmetric tensor

$$T^{\alpha}_{\mu\nu} = (\delta^{\alpha}_{\mu} \partial_{\nu} T - \delta^{\alpha}_{\nu} \partial_{\mu} T) \times T^{-1}. \tag{146}$$

If in particular, we take

$$T = \psi(x) \times I, \tag{147}$$

and then

$$\hat{I} = \psi(x)^{-1} \times I, \tag{148}$$

we then obtain, under natural normalisation, a torsion tensor with trace given by $d\ln\psi$. Notice that in contrast with the gauge theory for G, from eqs. $(146-148)$ we can conclude that the gauge theory for the isotopic group $\hat{G}$ has infinitesimal transformations of the form $\epsilon' = \epsilon^{i} \times T$ and an effective coupling which is no longer constant, given by $e' = e \times T$.

Up to here, we have presented a construction which is independent of the metric g defined on M. In the first term in eq. (143) we can write instead of the usual derivative for A, on introducing a metric g, the covariant derivative with respect to a Levi-Civita connection defined by g, and thus the covariant derivative ∇ is a RCW connection defined by a metric g and trace-torsion $d\ln\psi$. This is nothing else than the equations introduced by Hojman et al for an abelian Lie-group in modifying the principle of minimal coupling to include torsion, further introduced to the present case of a non-abelian group by Mukku and Sayed [50]. In the former case the last r.h.s. element vanishes completely, and we are left with the same expression obtained for Maxwell's equations when we extend the minimal coupling principle to account for the torsion of the manifold. Furthermore, an identical expression was obtained when we presented above the relation between the Hodge decomposition of the trace-torsion and its relation to an invariant state. In distinction with another class of isotopic units we shall present below (see eq. (163)), the choice in eq. (148) does not have the non-linear and non-local characteristics we shall present below for the usual generalized isotopic units of HM.

8 Santilli-Lie isotopic Hilbert space

In this section we shall present briefly the core of HM [16] (for a complete list of references, the reader is urged to consult this work, as well as to the history of the developments until the theory reached the present form.

There is in Santilli's theory a canonical way of introducing the Lie-isotopic unit, with respect to which the complete theory is based upon. The prescription is to introduce an arbitrary non-unitary operator U and to substitute the unit I by the isotopic unit

$$\hat{I} = U \times I \times U^{\dagger} \neq I. \tag{149}$$

The usual Hilbert space of quantum mechanics, is denoted by $\mathcal{H} = \{|\Phi >, |\Psi >:< \Phi|\Psi > \in C(c,+,\times), < \Psi, \Psi >= 1\}$, where $C(c,+,\times)$ denotes the field of complex numbers with the usual addition and multiplication. The evolution equation in the Lie-Santilli theory of an observable is given by the equation

$$i\frac{dA}{dt} = [A\hat{,}H] = A \times T \times H - H \times T \times A, \tag{150}$$

so that

$$A(t) = e^{i\times H\times T\times t} \times A(0) \times e^{-i\times t\times T\times H}. \tag{151}$$

The problem with this quantum evolution is that it is non-unitary over the Hilbert space $\mathcal{H}$ over the field $C(c,+,\times)$. We have the following fundamental result, known as the López Lemma.

Theorem 1. All possible non-unitary deformations of quantum mechanics computed on a conventional Hilbert space $\mathcal{H}$ over the field $C(c,+,\times)$ have the following aspects:

i Lack of invariance of the unit, and consequently the lack of applicability to measurements. ii Lack of preservation of the Hermiticity in time, and consequently the lack of unambiguous observables. iii Lack of invariant eigenfunctions and their transforms, and consequently the lack of invariant numerical predictions.

Proof. The unit of quantum mechanics, I, of the enveloping algebra ξ verifies $I \times A = A \times I = A$, $\forall A \in \xi$. If we take a non-unitary transformation $I \mapsto \hat{I} = U \times I \times U^{\dagger} \neq I$, then

$$i\frac{dI}{dt} = [I\hat{,}H] = I \times T \times H - H \times T \times I \neq 0. \tag{152}$$

Under a non-unitary transformation, the associative modular action of the Schrodinger representation $H \times |\psi >$, for H an hermitean operator for $t = 0$ becomes

$$U \times H \times |\Psi >= U \times H \times U^{\dagger} \times (U \times U^{\dagger})^{-1} U|\Psi >= \hat{H} \times T^{-1} \times U|\Psi > \tag{153}$$

where the operator $\hat{H} := U \times H \times U^{\dagger}$, and $\hat{\Psi} = U \times |\Psi >$. We note that by definition $\hat{T} = (U \times U^{\dagger})^{-1} = \hat{T}^{\dagger}$. The initial condition of hermiticity of H on $\mathcal{H}$, i.e. $< \Psi| \times (H \times |\Psi >) = (< \Psi| \times H^{\dagger}) \times |\Psi >$, when applied to a Hilbert space with states of the form $|\hat{\Psi} >= U \times |\Psi >$, requires the action of the transformed operator $U \times H \times |\Psi >= \hat{H} \times T^{-1} \times |\hat{\Psi} >$ yet with the conventional inner product, then

$$< \hat{\Psi}| \times (\hat{H} \times \hat{T} \times |\hat{\Psi} >) =< \hat{\Psi}| \times \hat{T} \times \hat{H}^{\dagger} \times |\hat{\Psi} >, \tag{154}$$

i.e.

$$\hat{H}^\dagger = \hat{T}^{-1} \times \hat{H} \times \hat{T} \neq \hat{H}, \tag{155}$$

so that hermiticity is not preserved under non-unitary transformations formulated on the conventional Hilbert space $\mathcal{H}$ over the field $C(c, +, \times)$, due to the fact that $[\hat{T}, \hat{H}] \neq H$.

The general situation of non-unitary deformations computed on general Hilbert spaces will be addressed below. As a corollary of the Theorem 1, on $(\mathcal{H}, C(c, +, \times))$ non-unitary quantum deformations do not give invariant probabilities, nor posses unique invariant physical laws. While in QM unitary time evolution implies causality, in its non-unitary deformations there is a violation of causality.

So let us proceed to present the solution to this problem provided by Santilli, the construction of a non-unitary image of quantum mechanics. As we said already, the first element is given by the introduction of the generalized unit

$$\hat{I} = U \times I \times U^\dagger = \hat{I}^\dagger \neq I, \tag{156}$$

so that

$$\hat{T} = (U \times I \times U^\dagger)^{-1} = \hat{T}^\dagger. \tag{157}$$

The product in the generalized enveloping algebra $\hat{\xi}$ is given by elements of the form $U \times A \times B \times U^\dagger = \hat{A} \times \hat{T} \times \hat{B} := \hat{A} \hat{\times} \hat{B}$ for $\hat{A} = U \times A \times U^\dagger$ and $B = U \times B \times B^\dagger$. For a Hilbert space $(\mathcal{H}, <>, C(c, +, \times))$ we introduce the Lie-Santilli isotopic Hilbert space $\hat{\mathcal{H}}$ of elements of the form $|\hat{\psi} >= U \times |\psi >$ and $< \hat{\phi}| =< \phi| \times U^\dagger$, with inner product given by transforming the original $\mathcal{H}$ inner product by the non-unitary transformation

$$< \Phi, \Psi > \rightarrow < \Phi| \times U^\dagger \times U^{\dagger -1} \times U^{-1} \times U|\Psi > = < \hat{\Phi}| \times \hat{T} \times |\Psi > \equiv < \hat{\Phi}| \hat{\times} |\hat{\Psi} > . \tag{158}$$

The generalized enveloping algebra $\hat{\xi}$ is still associative

$$(\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{C} = \hat{A} \hat{\times} (\hat{B} \hat{\times} \hat{C}), \tag{159}$$

with identity given by $\hat{I}$, since $\hat{I} \hat{\times} \hat{A} = \hat{A} \hat{\times} \hat{I} = \hat{A}$ We know already that the modified Lie algebra is given by $[\hat{A} \hat{,} \hat{B}] = \hat{A} \times T \times \hat{B} - \hat{B} \times T \times \hat{A}$, which is isomorphic to the original one if $\hat{I}$ is positive definite.

Now let us see how the problem of hermiticity in the non-unitary frame is obtained. We have,

$$< \Psi| \times \hat{T} \times (H \times \hat{T} \times |\Psi >) = (< \hat{\Psi}| \times \hat{T} \times H^\dagger) \times |\hat{\Psi} >, \tag{160}$$

which yields

$$\hat{H}^\dagger = \hat{T}^{-1} \times \hat{T} \times H^\dagger \times \hat{T} \times T^{-1}. \tag{161}$$

Thus, starting with an hermitean operator H at $t = 0$, then $\hat{H} = U \times H \times U^\dagger$ remains hermitean under non-unitary transformations. But we note that the hermiticity is not computed in $(\mathcal{H}; <\ |\ >, C(c, +, \times))$ but on $(\hat{\mathcal{H}}, < |\hat{\times}| >, \hat{C}(\hat{c}, \hat{+}, \hat{\times}))$, where $\hat{C}(,\hat{c}, +, \hat{\times})$ is the Santilli-Lie isotopic lift of $C(c, +, \times)$ with elements of the form $\hat{c} = c \times \hat{I}$, where $\hat{I}$ not necessarily belongs to C, the summation is defined by $\hat{c}_1 + \hat{c}_2 = (c_1 + c_2) \times \hat{I}$ and the product is $\hat{c}_1 \hat{\times} \hat{c}_2 = \hat{c}_1 \times \hat{T} \times \hat{c}_2 = (c_1 \times c_2) \times \hat{I}$. Then $\hat{I} = \hat{T}^{-1}$ is the left and right multiplicative unit in $\hat{C}(\hat{c}, +, \hat{\times})$. We further have that $\hat{0} = 0$ satisfies $\hat{c} + \hat{0} = \hat{c}$. Furthermore, $\hat{c}^2 = \hat{c} \hat{\times} \hat{c} = \hat{c} \times \hat{T} \times \hat{c}$, $\hat{c}^{\frac{\hat{1}}{2}} = c^{\frac{1}{2}} \times \hat{I}^{\frac{1}{2}}$. The quotient is defined by $\hat{a} /^{\hat{\times}} \hat{b} = (\frac{a}{b} \times \hat{I})$, and $|\hat{c}| = |c| \times \hat{I}$ and finally for an arbitrary Q, $\hat{c} \hat{\times} Q = c \times \hat{I} \times T \times Q = c \times I \times Q = c \times Q$.

The modular action $\hat{H} \hat{\times} |\hat{\Psi} >= \hat{H} \times \hat{T} \times |\hat{\psi} >$ with $< \hat{\psi} | \hat{\times} | \psi > = < \hat{\psi} | \times T \times |\hat{\psi} >$ has for generalized unit $\hat{I} = \hat{T}^{-1}$, because it is the only object such that $\hat{I} \hat{\times} |\hat{\psi} > = |\hat{\psi} >$. Consequently, referral to $C(c, +, \times)$ with unit I is inconsistent, and then $\hat{\mathcal{H}}$ must be referred to $\hat{C}$ with basic unit $\hat{I}$. Now the iso-Hilbert invariant iso-inner product is defined by

$$< \hat{\Phi} | \hat{\Psi} >^{\hat{I}} = < \hat{\Phi} | \hat{\times} | \hat{\Psi} > \times \hat{I} = < \hat{\Phi} | \times \hat{T} \times |\hat{\Psi} > \times \hat{T}^{-1} \in \hat{C}. \tag{162}$$

It is important to remark that the transformation $H \times p|\psi > \mapsto \hat{H} \hat{\times} |\hat{\psi} >$ satisfies linearity on isospace over isofields. The recovery of linearity in isospace is achieved by the embedding of the nonlinear terms in the isounit. Furthermore, any nonlinear theory with a Hamiltonian operator $H(p, x, \psi, \ldots)$ can always be rewritten by factorizing the nonlinear terms, which can then be assumed as the isotopic element of the theory. Indeed, if we have

$$H(p, x, \psi, \ldots) \times |\hat{\psi} > = H_0(x, p) \times \hat{T}(x, p, \psi, \ldots) \times |\hat{\psi} > := \hat{H}_0(, p) \hat{\times} |\hat{\psi} >,$$

with $\hat{T} := H_0^{-1} \times H$. Here, H_0 is the maximal (Hermitean) operator representing the total energy. Thus, superposition in isospace can be achieved, which allows a consistent treatment of composite systems under nonlinear interactions.

It is about time to present a general class of generalized units that appear in HM for the characterization of the strong interactions. Namely,

$$\hat{I} = \mathrm{diag}(n_1^2, n_2^2, n_3^2, n_4^2) \times \exp(tN(\frac{\psi_\uparrow}{\hat{\psi}_\downarrow} + \frac{\partial \psi_\downarrow}{\partial \hat{\psi}_\downarrow} + \ldots) \times \int d^3x \psi_\uparrow^\dagger(x) \psi_\downarrow(x), \tag{163}$$

where the quantities n_1^2, n_2^2, n_3^2 represent the extended, non-spherical deformable shapes of the hadron, n_4^2 its density, the quantities $\frac{\psi_\uparrow}{\hat{\psi}_\downarrow} + \frac{\partial \psi_\downarrow}{\partial \hat{\psi}_\downarrow} + \ldots$ represent a typical non-linearity, and the integral in the exponent, represents a typical non-linearity due to the interpenetration and overlapping of the charge distributions. Notably a coupling of spin-up and spin-down particles is present in the generalized unit. Whenever the hadrons are perfectly spherical and rigid, then we can take the density $n_4^2 = 1$ and the parameters of deformations can also be set equal to 1; if furthermore, their distances is such as to be nor interpenetration, then the integrand is zero and the exponential term is equal to 1 and thus, in this situation, $\hat{I} = I$, and we are in the domain of applicability of QM, where the unit is given in terms of the torsion structure constants of the Lie algebra, and dynamically, from the gradient logarithm of the wave function. The present choice of the isotopic unit has lead to the first ever model of a Cooper model with explicit attractive force between the pair of identical electrons, with excellent agreement with experimental data [68].

8.1 Santilli-Lie Isotopies of the Differential Calculus and Metric Structures, and the Iso-Schroedinger Equation

To present the isotopic Schroedinger equation, we have to introduce the Santilli-Lie-isotopic differential calculus [15,16] and the isotopic lift of manifolds, the so-called isomanifolds, due to Tsagas and Sourlas [19]. We start by considering the manifold M to be a vector space with local coordinates, which for simplicity we shall from now consider them to be a *contravariant*[5] system $x = (x^i), i = 1, \ldots, n$, unit given by $I = \mathrm{diag}(1, \ldots, 1)$ and metric g defined on the tangent manifold with coordinates $v = (v^i)$, so that $v^2 = v^i \times g_{ij} \times v^j \in R$. We shall lift this structure to a vector space $\hat{M}$ provided with isocoordinates $\hat{x}$, isometric $\hat{G}$ and defined on the isonumber field $\hat{F}$, where F can be the real or complex numbers; we denote this isospace by $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. Let us describe the construction of this isospace $\hat{M}$ [16,19]:

The isocoordinates are introduced by the transformation

$$x \mapsto U \times x \times U^{\dagger} = x \times \hat{I} := \hat{x}. \tag{164}$$

To introduce the isometric $\hat{G}$ we start by considering the transformation

$$g \mapsto U \times g \times U^{\dagger} = \hat{I} \times g := \hat{g}. \tag{165}$$

Yet we notice that the matrix elements $\hat{g}^{ij} = (\hat{I} \times g)^{ij}$ belong to the number field F, not to $\hat{F}$, so the correct definition of the isometric is $\hat{G} = \hat{g} \times \hat{I}$. Thus we have a transformed $M(x, g, F)$ into the isospace $\hat{M}(\hat{x}, \hat{G}, \hat{F})$. Thus the projection on $\hat{M}(\hat{x}, \hat{g}, F)$ of the isometric in $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ is defined by a contravariant tensor, $\hat{g} = (\hat{g}^{ij})$ with components

$$\hat{g}^{ij} = (\hat{I} \times g)^{ij}. \tag{166}$$

We must remark that in distinction of the usual scale transformation on the metrics in the Einstein λ and the Weyl transformations, the scale factor is *not* $\hat{I} \times \hat{I}$ but $\hat{I}$. If we start with g being the Euclidean or Minkowski metrics, we obtain the iso-Euclidean and iso-Minkowski metric, the latter being the basis for the formulation of the isotopic lift of Special Relativity, in addition of the isotopic lift of the Lorentz group [15-19]; in the case we start with a general

[5] The definitions for a covariant set of coordinates and the corresponding isodifferential calculus differs from the covariant case, and since we shall be using only contravariant objects, we shall not present them in this article.

metric as in GR, we obtain the isotopic lift of GR [15,19]. We shall now proceed to identify the isotopic lift of the noise tensor σ which verifies eq. (20), i.e. $\sigma \times \sigma^\dagger = g$. The non-unitary transform of σ is given by

$$\sigma \mapsto U \times \sigma \times U^\dagger = \sigma \times \hat{I} := \hat{\sigma}. \tag{167}$$

Then,

$$\hat{\sigma} \hat{\times} \hat{\sigma} = \sigma \times \hat{I} \times \hat{T} \times (\sigma \times \hat{I})^\dagger = (\sigma \times \sigma^\dagger) \times \hat{I} = g \times \hat{I} = \hat{g}. \tag{168}$$

Thus the isotopic lift of noise tensor defined on $\hat{M}(\hat{x}, \hat{G}, \hat{R})$ is given by $\hat{\sigma} = \sigma \times \hat{I}$ which on projection to $M(x, g, R)$ (we stress here that we are taking a tensor with real components) we retrieve σ.

We know introduce the Santilli isodifferential introduced in 1996 (see [16,17]) for contravariant coordinates which are given by

$$\hat{d}\hat{x} = \hat{T} \times d\hat{x} = \hat{T} \times d(x \times \hat{I}), \tag{169}$$

so that when $\hat{I}$ does not depend on x we have that $\hat{d}\hat{x} = dx$. We now follow Kadeisvili [19] in introducing the isofunctions $\hat{f}(\hat{x})$ defined on $\hat{M}$ with values on $\hat{F}$. We take an F-valued function, f, defined on M, and we produce the non-unitary transformation

$$f(x) \mapsto U \times f(x) \times U^\dagger = f(x) \times \hat{I} = f(\hat{x} \times \hat{T}) \times \hat{I} \in \hat{F}, \tag{170}$$

so that the definition of an isofunction $\hat{f}(\hat{x})$ is given by

$$\hat{f}(\hat{x}) = f(\hat{T} \times \hat{x}) \times \hat{I}. \tag{171}$$

The isoderivative of isofunctions on contravariant coordinates are given by

$$\hat{\partial}\hat{f}(\hat{x}) = \hat{I} \times \frac{\partial f(\hat{x})}{\partial \hat{x}}, \tag{172}$$

so that

$$\hat{d}\hat{f}(\hat{x}) = \hat{\partial}\hat{f}(\hat{x}) \hat{\times} \hat{d}\hat{x} = \hat{I} \times \frac{\partial f(\hat{x})}{\partial \hat{x}} \hat{\times} d\hat{x} = \hat{I} \times \frac{\partial f(\hat{x})}{\partial \hat{x}} \times \hat{T} \times d(x \times \hat{I}). \tag{173}$$

It follows from the definitions that

$$\frac{\hat{\partial}}{\hat{\partial}\hat{x}^k} = \hat{I}^i_k \times \frac{\partial}{\partial x^i}. \tag{174}$$

We now introduce (for the first time, to our best knowledge) the isotopic gradient operator of the isometric $\hat{G}$ (the $\hat{G}$-gradient, for short), $\widehat{\mathrm{grad}}_{\hat{\mathrm{G}}}$ applied to the isotopic lift $\hat{f}(\hat{x})$ of a function $f(x)$ is defined by

$$\widehat{\mathrm{grad}}_{\hat{\mathrm{G}}} \hat{f}(\hat{x})(\hat{v}) = \hat{G}(\hat{d}\hat{f}(\hat{x}) \hat{;} \hat{v}), \tag{175}$$

for any vector field $\hat{v} \in T_{\hat{x}}(\hat{M})$, $\hat{x} \in \hat{M}$; we have denoted the inner product with $\hat{;}$ to stress that the inner product is taken with respect to the product in $\hat{F}$. It follows from the definition $\hat{G} = \hat{g} \hat{\times} \hat{I}$ and eq. (175) that the operator $\widehat{\mathrm{grad}}_{\hat{\mathrm{G}}} \hat{f}(\hat{x})$ is the vector field on the tangent manifold to $\hat{M}(\hat{x}, \hat{G}, \hat{F})$ defined by

$$\hat{G}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{\beta}} = \hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^{i}} \times \hat{I}. \tag{176}$$

Therefore, the projection on $\hat{M}(\hat{x}, \hat{g}, F)$ of the $\hat{G}$-gradient vector field of $\hat{f}(\hat{x})$ is the vector field with components

$$\hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}} = \hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial} \hat{f}(\hat{x})}{\hat{\partial} \hat{x}^{\alpha}}. \tag{177}$$

This will be of importance for the determination of the drift vector field of the diffusion linked with the Santilli- iso-Schroedinger equation. We finally define the isolaplacian as

$$\hat{\triangle}_{\hat{g}} = \hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial}}{\partial \hat{x}^{\alpha}} \hat{\times} \frac{\hat{\partial}}{\partial \hat{x}^{\beta}}. \tag{178}$$

This definition differs from the original one in [19], but is the one that will allow us to associate diffusions to the Santilli-iso-Schroedinger equation.

9 Diffusions and the Heisenberg Representation

Up to now we have set our theory in terms of the Schroedinger representation, since the original setting for this theory has to do with scale transformations as introduced by Einstein in his last work [6] which was recognized already by London that the wave function was related to the Weyl scale transformation [48], and these scale fields have turned to be in the non-relativistic case, nothing else than the wave function of Schroedinger equation, both in the linear and the non-linear cases. Historically the operator theory of QM was introduced before the Schroedinger equation, who later proved the equivalence of the two. The ensuing dispute and rejection by Heisenberg of Schroedinger's equation is a dramatic chapter of the history of QM [36]. It turns out to be the case that we can connect the Brownian motion approach to QM and the operator formalism due to Heisenberg and Jordan. We must remark at this stage that the isotopic and Lie-admissible extensions of the operator formalism of QM by Santilli, was the starting point and building approach to the constructions of several theories. Some of these approaches, initiated by Santilli early in 1967, had to be abandoned later for reasons of inconsistency as elaborated in [16] in which the theory reaches final maturity.

Let us define the position operator as usual and the momentum operator by

$$q^{k} = x^{k}, p_{\mathcal{D}k} = \sigma \times \frac{\partial}{\partial x^{k}}, \tag{179}$$

which we call the diffusion quantization rule (the subscript $\mathcal{D}$ denotes diffusion) since we have a representation different to the usual quantization rule

$$p_k = -i \times \frac{\partial}{\partial x^k}, \tag{180}$$

with $\sigma = (\sigma^\alpha_a)$ the diffusion tensor verifying eq. (20), i.e. $(\sigma \times \sigma^\dagger)^{\alpha\beta} = g^{\alpha\beta}$ and substitute into the Hamiltonian function)

$$H(p, q) = \frac{1}{2}\sum_{k=1}^{d}(p_k - A_k)^2 + \mathbf{v}(q) \tag{181}$$

this yields the formal generator of a diffusion semigroup in $C^2(R^d)$ or $L^2(R^d)$. This yields the formal generator of a diffusion semigroup in $C^2(R^d)$ or $L^2(R^d)$. Thus, an operator algebra on $C^2(R^n)$ or $L^2(R^n)$ together with the postulate of the commutation relation (instead of the usual commutator relation of quantum mechanics $[p, q] = i\hat{h}I$)

$$[p, q] = pq - qp = -\sigma I \tag{182}$$

this yields the diffusion equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\sum_{k=1}^{d}(\sigma\frac{\partial}{\partial x^\alpha} - \sigma A_\alpha)^2\phi + \mathbf{v}\phi = 0, \tag{183}$$

which coincides with the diffusion equation (100) provided that $\mathrm{div}A = 0$ and $c = \mathbf{v} + A^2$.

Thus, in this approach, the operator formalism and the quantization postulates, allow to deduce the diffusion equation. If we start from either the diffusion process or the RCW geometry, without any quantization conditions we already have the equations of motion of the quantum system which are non other than the original diffusion equations, or equivalently, the Schroedinger equations. We stress the fact that these arguments are valid for both cases relative to the choice of the potential function V, i.e. if it depends nonlinearly on the wave function ψ, or acts linearly by multiplication on it. Further below, we shall use this modification of the Heisenberg representation of QM by the previous Heisenberg type representation for diffusion processes, to give an account of the diffusion processes that are associated with HM. This treatment differs from our original (inconsistent with respect to HM) treatment of the relation between RCW geometries and diffusions presented in [29] in that it incorporates the isotopic lift of all structures.

Let us frame now isoquantization in terms of diffusion processes. Define isomomentum, $\hat{p}_{\mathcal{D}}$, by

$$\hat{p}_{\mathcal{D}k} = \hat{\sigma}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{x}^k}, \text{ with } \hat{\sigma} = \sigma \times \hat{I}, \tag{184}$$

so that the kinetic term of the iso-Hamiltonian is

$$\begin{aligned}\hat{p}_{\mathcal{D}} \hat{\times} \hat{p}_{\mathcal{D}}^{\dagger} &= \hat{\sigma} \hat{\times} \hat{\sigma}^{\dagger} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} \\ &= \hat{g} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}} = \hat{\triangle}_{\hat{g}} \end{aligned} \tag{185}$$

We finally check the consistency of the construction by checking that it can be achieved via the non-unitary transformation

$$\begin{aligned} p_{\mathcal{D}_j} \mapsto U \times p_{\mathcal{D}_j} \times U^{\dagger} &= U \times \sigma \times \frac{\partial}{\partial x^j} \times U^{\dagger} \\ &= \sigma \times \hat{I} \times T \times \hat{I} \times \frac{\partial}{\partial x^j} = \hat{\sigma} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^j} = \hat{p}_{\mathcal{D} j}. \end{aligned} \tag{186}$$

Note that we have achieve this isoquantization in terms of the following transformations: First we transformed

$$p = -i \times \frac{\partial}{\partial x} \rightarrow p_{\mathcal{D}} := \sigma \times \frac{\partial}{\partial x}, \tag{187}$$

to further produce its isotopic lift

$$\hat{p}_{\mathcal{D}} = \hat{\sigma} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}}. \tag{188}$$

When the original diffusion tensor σ is the identity I from which follows from eq. (20) that the original metric g is Euclidean, we reach compatibility of the diffusion quantization with the Santilli-iso-Heisenberg representation given by taking the non-unitary transformation on the canonical commutation relations

$$[\hat{q}^i \hat{,} \hat{p}_j] = \hat{i} \hat{\times} \hat{\delta}^i_j = i \times \delta^i_j \times \hat{I}, \tag{189}$$

together with

$$[\hat{r}^i \hat{,} \hat{r}^j] = [\hat{p}_i \hat{,} \hat{p}_j] = 0, \tag{190}$$

with the Santilli-iso-quantization rule [16,17]

$$\hat{p}_j = -\hat{\imath} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^j}. \tag{191}$$

Thus, from the quantization by the diffusion representation we retrieve the Santilli-iso-Heisenberg representation, with the difference that the diffusion noise tensor in the above construction need not be restricted to the identity.

Finally, we consider the isoHamiltonian operator

$$\hat{H} = \hat{1} \hat{/} (\hat{2} \hat{\times} \hat{m}) \hat{\times} \hat{p}^{\hat{2}} + \hat{V}_0(\hat{t}, \hat{x}) + \hat{V}_k(\hat{t}, \hat{v}) \hat{\times} \hat{v}^k, \tag{192}$$

where $\hat{p}$ may be taken to be either given by the Santilli isoquantization rule

$$\hat{p}_k \hat{\times} |\hat{\psi}> = -\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} x^k} \hat{\times} |\hat{\psi}>, \tag{193}$$

or by the diffusion representation $\hat{p}_{\mathcal{D}}$, $\hat{V}_0(\hat{t}, \hat{x})$ and $\hat{V}_k(\hat{t}, \hat{v})$ are potential isofunctions, the latter dependent on the isovelocities. Then the iso-Schroedinger equation (or Schroedinger-Santilli isoequation) [16,17] is

$$\hat{i} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{t}} |\hat{\psi}> = \hat{H} \hat{\times} |\psi> \ = \hat{H}(\hat{t}, \hat{x}, \hat{p}) \times \hat{T}(\hat{t}, \hat{x}, \hat{\psi}, \hat{\partial}\hat{\psi}, \ldots) \times |\hat{\psi}>, \tag{194}$$

where the wave isofunction $\hat{\psi}$ is an element in $(\hat{\mathcal{H}}, <|\hat{\times}|>, \hat{C}(\hat{c}, \hat{+}, \hat{\times}))$ satisfies

$$\hat{I} \hat{\times} |\hat{\psi}> = |\hat{\psi}> . \tag{195}$$

Solutions of this equation and the many important consequences for the strong interactions and Hadronic Chemistry, have been discussed in [17] and references therein; they were first obtained in 1978, while Prof. Santilli was affiliated to the Physics Department, Harvard University, with the sponsorship of the Department of Energy, US Government.

9.1 Hadronic Mechanics and Diffusion Processes

As we derived above, the components of drift vector field, projected on $\hat{M}(\hat{x}, \hat{g}, R)$ in the isotopic form of eq. (91) with $A \equiv 0$, is given by eq. (177) with $\hat{f} = \hat{\ln} \hat{\phi}$, so that

$$\hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial} \hat{\ln} \hat{\phi}(\hat{x})}{\hat{\partial} \hat{x}^\alpha}, \tag{196}$$

with $\hat{\phi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x}) + \hat{\mathcal{S}}(\hat{x})}$ the diffusion wave associated to the solution $\hat{\psi}(\hat{x}) = \hat{e}^{\hat{\mathcal{R}}(\hat{x}) + i\hat{\mathcal{S}}(\hat{x})}$ of the Santilli-iso-Schroedinger eq. (194), and its adjoint wave is $\check{\hat{\phi}}(x) = \hat{e}^{\hat{\mathcal{R}}(x) - \hat{\mathcal{S}}(x)}$. Therefore, the drift vector field is

$$\hat{g}^{\alpha\beta}(\hat{x}) \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^\alpha} (\hat{\mathcal{R}}_{\hat{t}} \hat{+} \mathcal{S}_{\hat{t}})(\hat{x}). \tag{197}$$

Finally, we shall write the isotopic lift of the stochastic differential equation for eq. (194). By applying the non-unitary transformation of eq. (91) with $A \equiv 0$, we obtain the iso-equation on $\hat{M}(\hat{x}, \hat{G}, \hat{R})$ for $\hat{X}(\tau)$ given by

$$d\hat{X}^i_{\hat{t}} = (\hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial} \hat{x}^\alpha} (\hat{\mathcal{R}}_{\hat{t}} \hat{+} \mathcal{S}_{\hat{t}}))(\hat{X}_{\hat{t}}) \hat{\times} d\hat{t} + \hat{\sigma}^i_j(\hat{X}_{\hat{t}}) \hat{\times} d\hat{W}^j_{\hat{t}}, \tag{198}$$

with $d\hat{W}_{\hat{t}} = \hat{W}(\hat{t}\hat{+}\hat{d}\hat{t})\hat{-}\hat{W}(\hat{t})$ the increment of a iso- Wiener process $\hat{W}_{\hat{t}} = (\hat{W}^1_{\hat{t}}, \ldots, \hat{W}^m_{\hat{t}})$ with isoaverage equal to $\hat{0}$ and isocovariance given by $\hat{\delta}^i_j \hat{\times} \hat{t}$; i.e.,

$$\hat{1}\hat{/}(\hat{4}\hat{\times}\hat{\pi}\hat{\times}\hat{t})^{\hat{m}\hat{/}\hat{2}} \hat{\int} \hat{w}_i \hat{\times} \hat{e}^{-\hat{w}^{\hat{2}}\hat{/}\hat{4}\times\hat{t}^{\hat{2}}} \hat{\times} \hat{d}\hat{w} = \hat{0}, \forall i = 1, \ldots, m \tag{199}$$

and

$$\hat{1}\hat{/}(\hat{4}\hat{\times}\hat{\pi}\hat{\times}\hat{t})^{\hat{m}\hat{/}\hat{2}} \hat{\int} \hat{w}_i \hat{\times} \hat{w}_j \hat{\times} \hat{e}^{-\hat{w}^{\hat{2}}\hat{/}\hat{4}\hat{\times}\hat{t}^{\hat{2}}} \hat{\times} \hat{d}\hat{w} = \hat{\delta}^i_j \hat{\times} \hat{t}, \forall i,j = 1, \ldots, m, \tag{200}$$

and $\hat{\int}$ denotes the isotopic integral defined by $\hat{\int} \hat{d}\hat{x} = (\int \hat{T} \times \hat{I} \times dx) \times \hat{I} = (\int dx) \times \hat{I} = \hat{x}$. Thus, formally at least, we have

$$\hat{X}_{\hat{t}} = \hat{X}_{\hat{0}} \hat{+} \hat{\int}_{\hat{0}}^{\hat{t}} (\hat{g}^{\alpha\beta} \hat{\times} \frac{\hat{\partial}}{\hat{\partial}\hat{x}^{\alpha}} (\hat{\mathcal{R}}_{\hat{s}} \hat{+} \hat{\mathcal{S}}_{\hat{s}}))(\hat{X}_{\hat{s}}) \hat{\times} \hat{d}\hat{s} + \hat{\int}_{\hat{0}}^{\hat{t}} \hat{\sigma}^i_j (\hat{X}_{\hat{s}}) \hat{\times} d\hat{W}^j_{\hat{s}}. \tag{201}$$

The integral in the first term of eq. (201) is an isotopic integral of a usual Riemann-Lebesgue integral, while the second one is the isotopic lift of a stochastic Itô integral; we shall not present here in detail the definition of this last term, which follows from the notions of convergence in the isofunctional analysis elaborated by Kadeisvili, and the usual definition of Itô stochastic integrals [8,13,70].

9.2 The Extension to The Many-body Case

Up to know we have presented the case of the Schroedinger equation for an ensemble of one-particle systems on space-time. Of course, our previous constructions are also valid for the case of an ensemble of interacting multiparticle systems, so that the dimension of the configuration space is $3d + 1$, for indistinguishable d particles; the general case follows with minor alterations. If we start by constructing the theory as we did for an ensemble of one-particle systems (Schroedinger's cloud of electrons), we can still pass trivially for the general case, by considering a diffusion in the product configuration manifold with coordinates $X_t = (X^1_t, \ldots, X^d) \in M^d$, where M^d is the d Cartesian product of three dimensional space with coordinates $X^i_t = (x^{1,i}_t, x^{2,i}_t, x^{3,i}_t) \in M$, for all $i = 1, \ldots, d$. The distribution of this is $\mu_t = E_Q \circ X^{-1}_t$, which is a probability density in M^d. To obtain the distribution of the system on the three-dimensional space M, we need the distribution of the system X_t: $\mathrm{U}^x_t := \frac{1}{d} \sum_{i=1}^d \delta_{x_i}$. which is the same as

$$U^x_t(B) = \frac{1}{d} \sum_{i=1}^d 1_B(X^i_t), \tag{202}$$

where $1_B(X_t^i)$ is the characteristic system for a measurable set B , equal to 1 if $X_t^i \in B$, for any $i = 1 \ldots, d$ and 0 otherwise. Then, the probability density for the interacting ensembles is given by

$$\mu_t^x(B) = E_Q[U_t^x(B)], \tag{203}$$

where E_Q is the mean taken with respect to the forward Kolmogorov representation presented above, is the probability distribution in the three-dimensional space; see [13]. Therefore, the geometrical-stochastic representation in actual space is constructable for a system of interacting ensembles of particles. Thus the criticism to the Schroedinger equation by the Copenhagen school, as to the unphysical character of the wave function since it was originally defined on a multiple-dimensional configuration space of interacting system of ensembles, is invalid [36].

10 Vorticity and Anomalous Phenomenae in Electrolytic Cells

We have already discussed in [3] that the Brownian motions produce rotational fields simply by considering the Hodge duality applied to the trace-torsion. In the other hand, we have seen that this encompasses the Brownian motions produced by the wave function of arbitrary quantum systems, and the case of viscous fluids, magnetized or not. These examples are independent of any scale, from the galactic to the quantum scales. In the galactic scales, according to Arp, this may explain the red-shift without introducing any big-bang hypothesis [53]. Thus, we have a modified form of Le Sage's kinetic theory producing universal fluctuations which have additionally rotational fields associated to them, and due to the universality of quantum wave functions, either obeying the rules of linear or non-linear QM, or still of HM, then it comes as no surprise that vortices and superconductivity (which is the case of the Rutherford-Santilli model of the neutron which is derived from the previous constructions and can be framed as we showed in terms of torsion field) appear as universal coherent structures; superconductivity is usually related to a non-linear Schroedinger equation with a Landau-Ginzburg potential, which is just an example of the Brownian motions related to torsion fields with further noise related to the metric. Furthermore, atoms and molecules have spin-spin interactions which will produce a contribution to the torsion field; we have seen already that the torsion geometry exists in the realm of Quantum Chemistry and Hadronic Chemistry.[6] This is the case of the compressed hydrogen atom model of the neutron in the Rutherford-Santilli

[6]A different approach due to Akimov and Shipov is claimed to be at the origin of these torsion fields [58] which stems from the torsion geometry of the vacuum through the teleparallel geometries we explored in [24], alike to the present approach, yet it is not vinculated nor to QM nor to fluid-dynamics, further claims to an hypothetical particle known as the phyton. Further experiments related to torsion fields have been presented in the Journal of New Energies, we

model, in which their is a spin alignment with opposite direction and magnetic moments for the electron and the proton. This produces the stable state which leads to fusion. This is *not* a cold fusion process since it appears to occur at temperatures of the order of $5,000$ degrees Celsius. Yet in electrochemical reactions, there are sources of torsion which are given by the the wave functions of the components involved, but furthermore the production of vortex structures. Gas bubbles appear after switching off the electrochemical potentials, and sonoluminescence have been observed at the Oak Ridge National Laboratory at the USA [59]. There is a surprising phenomenae of remnant heat that persists after death which could be produced by the vortex dynamics of the tip effect [62]. These experimental findings have been claimed to be observed in different laboratories across the world [57,59], have in some instances lead to a theoretical explanation in terms of torsion fields [60,61]. Superconductors of class II present also some surprising phenomenae such as low-frequency noise, history- dependent dynamic response, and memory direction, amplitude direction and frequency of the previously applied current [60]. If these findings can be reproduced systematically, we would have a new class of sources of energy, which stem from the zero point fluctuations.

11 On the Experimental Evidence of Space-time Fluctuations and Conclusions

We have shown that the equations of QM have an equivalent formulation as diffusion processes which themselves generate space-time geometries or alternatively are generated by them. We have extended these relations to the isotopic deformations introduced in HM. Thus, wave functions of elementary particles, atoms and molecules described by the Schroedinger and Santilli-iso-Schroedinger equations, generate torsion fields. This is an universal phenomenae since the applicability of these equations does not restrict to the microcospic realm, as already shown in the astrophysical theory due to Nottale [10]; this universality is associated with the fact that the Planck constant (or equivalently, the diffusion constant) is multivalued, or still, it is context dependent, inasmuch as the velocity of light has the same feature [17]. In the case of HM this can be seen transparently in the fact that the isotopic unit plays the role, upon quantization, of the Planck constant in QM, as we have already seen when we introduced the diffusion-Heisenberg representation and its isotopic lift, or furthermore, by its product with the noise tensor of the underlying Brownian motions.In the galactic scales, this may explain the red-shift without introducing a big-bang hypothesis [16,17]; an identical conclusion was reached by Arp in considering as a theoretical framework the Le Sage's model of a Universe filled with a gas

direct the reader to [61]. As we have shown in this article and [3] torsion fields appear in the most fundamental theories of physics.

of particles [52,53], in our theory, the zero-point fluctuations described by the Brownian motions defined by the wave functions, as well as by viscous fluids, spinor fields, or electromagnetic fields [2] (and which one can speculate as related to the so-called dark energy problem). A similar view has been proposed by Santilli in which the elementary constituents are the so-called aetherinos [71], while in Sidharth's work, they appear to be elementary quantized vortices related to quantum-mechanical Kerr-Newman black holes [63]. Thus, whether we examine the domains of linear or non-linear QM, or still the hyperdense domain of HM, then it appears that vortices and superconductivity (which is the case of the Rutherford-Santilli model of the neutron which is derived from the previous constructions) appear as universal coherent structures. Superconductivity is usually related to a non-linear Schroedinger equation with a Landau-Ginzburg potential, which is just an example of the Brownian motions related to torsion fields with further noise related to the metric. Furthermore, atoms and molecules have spin-spin interactions which will produce a contribution to the torsion field; we have seen already that the torsion geometry exists in the realm of Quantum Chemistry and ultimately in Hadronic Chemistry, since we can extend the construction to the many-body case. In distinction with the usual Coulomb potential in nuclear physics, the isotopic deformations of the nuclear symmetries yield attractive potentials such as the Hulten potential, which in the range of 10^{-13} cm. yields the usual potential [15,56] without the need of introducing any sort of parameters or extra potentials. In contrast with the ad-hoc postulates of randomness in the fusion models which are considered in the usual approaches [32,33], in the present work randomness is intrinsic to space-time itself or alternatively a by product of it, and in the case of HM, these geometries incorporate at a foundational level, a generalized unit which incorporates all the features of the fusion process itself: the non-canonical, non-local and non-linear overlapping of the wave functions of the ensembles which correspond to the separate ensembles under deformable collisions in which the particles lose their pointlike structure, or in a hypercondensed plasma state, where the dynamics of the process may have a random behavior; the domain of validity of this constructions are 10^{-13} cm., and outside this domain we find the quantum fluctuations associated to the Schroedinger equation. This extension has been possible essentially by taking in account the generalized isotopic units and the isotopic lifts of all necessary mathematical structures.

There are already experimental findings that may lead to validate the present view. In the last fifty years, a number of scientists at the Biophysics Institute of the Academy of Sciences of Russia, directed by S. Shnoll (and presently developed in a world net which includes Roger Nelson, Engineering Anomalies Research, Princeton University, B. Belousov, International Institute of Biophysics, Neuss (Germany), J. Wilker, Max-Planck Institute for Aeronomy, Lindau, and others), have carried out tens of thousands of different experiments of very different nature and energy scales (α decay, biochemical reactions, gravitational waves antenna, etc.) in different points of the globe, and carried out a software analysis of the observed histograms and their fluctuations, to find out an amazing fit which is repeated with regularity of 24 hours, 27 days and the duration of a sidereal year. Thus, the fine spectrum of these experiments reveal a non-random pattern. At points of Earth with the same local hour, these patterns are reproduced with the said periodicity. The only thing in common to these experiments is that they are occur in spacetime, and thus this leads to conclude that they stem from spacetime fluctuations, which may further be associated with cosmological fields. Furthermore, the histograms reveal a fractal structure. This fractal structure has been found to follow the pattern of the logarithmic Muller fractal, which is associated with the existence of a global scale for all structures in the Universe; see H. Muller [85]. This leads to reinforce the thesis of time as an active field. Furthermore, the space and time Brownian motions can exist, in principle, in the different space and time scales warranted by these global scales. This structure is interpreted as appearing from an interference phenomena related to the cosmological field; we recall that diffusion processes present interference phenomena alike to , say, the two-slit experiment. Measurements taken with collimators show fluctuations emerging from the rotation of the Earth around its axis or its circumsolar orbit, showing a sharp anisotropy of space. Furthermore,it is claimed that the spatial heterogeinity occurs in a scale of 10^{-13} cm., coincidently with the scale of the strong interactions [80].

As a closing remark we would like to recall that Planck himself proposed the existence of ensembles of random phase oscillators having the zero-point structure as the basis for quantum physics [71]. Thus, the apeiron would be related to the Brownian motions which we have presented in this work, and define the space and time geometries, or alternatively, are defined by them. So we are back to the idea due to Clifford, that there is no-thing but space and time configurations, instead of a separation between substratum and fields and particles appearing on it. Furthermore, what we perceive to be void, is the hyperdense source of actuality.

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