On a Stochastic Model of Inflationary Cosmology

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Abstract

Observational evidence for the accelerated expansion of the Universe raises a fundamental challenge to standard cosmological models. It is generally presumed that acceleration of cosmic expansion emerges from an unknown physical component called dark energy whose contributions in negative pressure and energy density are substantial. One of the unsettled questions posed by the dark energy hypothesis relates to the magnitude of the cosmological constant: the observed vacuum energy density is exceedingly small as compared to predictions of quantum field physics. In this work we develop a derivation of the cosmological constant based on classical diffusion theory. Dynamics of the dark energy is modeled using the Langevin equation of a damped harmonic field in steady contact with a chaotic reservoir of vacuum fluctuations. The field evolves in the Friedmann-Robertson-Walker metric and dissipation arises as a result of expansion. The asymptotic limit of this process corresponds to setting the self-interaction gravity scale as the largest temperature of the reservoir. Predictions on vacuum energy density and cosmological constant are shown to be consistent with current experimental bounds.

1. Introduction

Recent years have witnessed accumulation of ample observational evidence in support of the accelerated cosmic expansion [1, 2]. Explaining the mechanism of accelerated expansion requires either the introduction of an unknown physical component called dark energy or basic revision of standard cosmological models [3]. The physics underlying the dark energy ansatz is not presently understood [3, 4] and several models have been developed. For example, it is hypothesized that the dark energy coincides with the vacuum energy of a fundamental self-interacting scalar field whose potential term generates the cosmological constant [5]. In some string based theories the dark energy is produced in the form of *dilatons* and *moduli* fields [4, 6]. Another class of models identifies the dark energy with a slowly varying self-interacting field called "*quintessence*" that is minimally coupled to classical gravity [7-9]. A model of dark energy based on stochastically quantized scalar fields is reported in [4], where a comprehensive list of relevant references is also included. Dynamics of the dark energy is extensively discussed in [5].

2. Langevin representation of vacuum dynamics

Let the cosmological vacuum be defined in the Friedmann-Robertson-Walker metric by a single homogeneous scalar field $\varphi(t)$. The field is embedded in the harmonic potential $V(\varphi)$ and undergoes large thermal fluctuations due to its permanent contact with a high-energy reservoir $\eta(t)$. We may describe this diffusion process using the classical Langevin equation

$$\ddot{\varphi} + \lambda_H \dot{\varphi} + V'(\varphi) = \eta(t) \tag{1}$$

Here, we assume a harmonic potential

$$V(\varphi) = \frac{1}{2}\omega^2 \varphi^2 \tag{2}$$

The damping coefficient is proportional to the Hubble parameter,

$$\lambda_H \equiv 3H \tag{3}$$

ω denotes a constant eigen-frequency and the statistics of the reservoir is assumed to be delta-correlated, i.e.

$$\langle \eta(t)\eta(t') \rangle = D\delta(t-t') \tag{4}$$

It is convenient for further analysis to turn (1) into a form that is independent of φ. We proceed to this end by introducing the concept of equivalent time constant. It is known that, in classical diffusion theory the evolution of field expectation satisfies

$$\frac{d\langle \varphi \rangle}{dt} = -(\tau)^{-1} \langle \varphi \rangle \tag{5}$$

Here τ is the time constant of the diffusive process described by (1). Alternatively, it can be stated that $\langle\varphi\rangle$ decays towards equilibrium through exponential relaxation

$$\langle\varphi(t)\rangle = \langle\varphi(0)\rangle \exp(-\frac{t}{\tau}) \tag{6}$$

By analogy with (5) and (6), we may define a time constant which is formally equivalent to τ and refers to the instantaneous realization of the field variable. Explicitly, the equivalent time constant is given in this picture by

$$\tau_{equiv} = \frac{\int_0^t \frac{d\varphi(s)}{ds} ds}{\dot{\varphi}(t)} \tag{7}$$

such that

$$\varphi(t) = -\tau_{equiv}\dot{\varphi}(t) \tag{8}$$

Direct substitution of (8) in (1) yields

$$\ddot{\varphi} + (\lambda_H - \lambda_{c,H})\dot{\varphi} = \eta(t) \tag{9}$$

where

$$\lambda_{c,H} = \omega^2 \tau_{equiv} \tag{10}$$

represents a correction to the Hubble parameter induced by $V(\varphi)$.

Following a customary procedure, we now normalize both field and time variables using their respective energy units. Let us note, for this purpose, that the non-equilibrium dynamics embodied in (1) entails two vastly different time scales: the *microscopic scale of vacuum fluctuations* generated by the reservoir, (kT) and the *cosmic scale of Universe expansion* dissipating the cumulative fluctuation energy over time intervals of order H^{-1}. It follows from this interpretation that the most natural normalization prescription is supplied by

$$\varphi^0 = \frac{\varphi}{kT}$$

$$t^0 = (\lambda_H - \lambda_{c,H})t \tag{11}$$

According to this prescription, the two dimensionless variables become comparable as normalized entities. They are both bounded by the unit interval and the uncertainty principle takes the minimal form:

$$\Delta\varphi^0 \Delta t^0 \approx 1 \tag{12}$$

The uppermost bound on the temperature developed by vacuum fluctuations may be fixed from basic relativistic arguments. Indeed, it can be seen from the minimal Einstein-Hilbert action for classical gravity [1, 3]

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}R \equiv \int d^4x\sqrt{-g}M_G^2 R \tag{13}$$

that $M_G = (16\pi G_N)^{-1/2}$ sets the natural energy scale of gravitational self-interaction. Contemporary field theory asserts that the deep ultraviolet regime of vacuum fluctuations is characterized by large coupling effects induced by gravitation above $\approx 100\, TeV$ [4, 5]. In light of these considerations we shall assume below that the asymptotic temperature of the thermal source is limited by the cutoff

$$kT_{\max} = M_G \tag{14}$$

(11) and (14) lead to the following parameterization of the field and its time rate

$$\varphi^0 = \frac{\varphi}{M_G} \tag{15a}$$

$$\dot{\varphi}^0 = \frac{\dot{\varphi}}{(\lambda_H - \lambda_{c,H})M_G} \tag{15b}$$

3. Connection to the Ornstein-Uhlenbeck process

It is instructive to reformulate the dynamics of the field using the overall energy density scale μ and a suitable scaling factor δ. Hence, let us define

$$\frac{1}{2}\frac{\lambda_{c,H}}{\tau_{equiv}}M_G^2 \equiv \mu^4 \tag{16}$$

and

$$\frac{1}{2}(\lambda_H - \lambda_{c,H})^2 M_G^2 \equiv \delta\mu^4 \tag{17}$$

The second moments for the field and its rate evolve according to the standard Ornstein-Uhlenbeck model [8]

$$\left\langle \varphi(t)^2 \right\rangle = \dot{\varphi}_0^2\{t\exp[-(\lambda_H - \lambda_{c,H})t]\}^2 + 2D\{t - 2t\exp[-(\lambda_H - \lambda_{c,H})t] + I[(\lambda_H - \lambda_{c,H})t]\}$$

$$\left\langle \dot{\varphi}(t)^2 \right\rangle = \varphi_0^2 \exp[-2(\lambda_H - \lambda_{c,H})t] + 2(\lambda_H - \lambda_{c,H})D\{1 - \exp[-2(\lambda_H - \lambda_{c,H})t]\} \tag{18}$$

where

$$I[(\lambda_H - \lambda_{c,H})t] = \int_0^\infty \exp 2[-(\lambda_H - \lambda_{c,H})s]ds \tag{19}$$

As the maximal field rate approaches unity at equilibrium, (18) leads to

$$\dot{\varphi}_{eq}^0 = 1 \rightarrow \frac{1}{2}\left\langle \dot{\varphi}(t)_{eq}^2 \right\rangle = \frac{1}{2}(\lambda_H - \lambda_{c,H})^2 M_G^2 \tag{20}$$

At the same time, the stationary regime of the Ornstein-Uhlenbeck model implies [8]

$$\frac{1}{2}\left\langle \dot{\varphi}(t)_{eq}^2 \right\rangle = (\lambda_H - \lambda_{c,H})D \tag{21}$$

Comparing the two relations yields the explicit form of the diffusion coefficient as

$$D = \frac{1}{2}(\lambda_H - \lambda_{c,H})M_G^2 \tag{22}$$

which stays constant as $\lambda_{c,H} - \lambda_H = O(\varepsilon)$ and $M_G \sim (\lambda_H - \lambda_{c,H})^{-1}$.

An alternative expression for the same coefficient at equilibrium is

$$D = \frac{\left\langle \varphi(t)_{eq}^2 \right\rangle}{t_{eq}} \tag{23}$$

Comparing (22) to (23) we find

$$\frac{\lambda_H - \lambda_{c,H}}{\lambda_{c,H}} \tau_{equiv} = \frac{2}{\omega^2 t_{eq}} \tag{24}$$

This relation shows that the time of transition to equilibrium scales as $(\lambda_H - \lambda_{c,H})^{-1}$ and is dependent on both ω and τ_{equiv}.

4. Cosmological implications

The expectations of kinetic and potential energy terms associated with the dynamics of the field are respectively given by

$$\begin{aligned} \left\langle T_{kin} \right\rangle &= \frac{1}{2}\left\langle \dot{\varphi}^2 \right\rangle = \frac{1}{2}(\lambda_H - \lambda_{c,H})^2 M_G^2 \left\langle (\dot{\varphi}^0)^2 \right\rangle \\ \left\langle V(\varphi) \right\rangle &= \frac{1}{2}\omega^2 \left\langle \varphi^2 \right\rangle = \frac{1}{2}\frac{\lambda_{c,H}}{\tau_{equiv}} M_G^2 \left\langle (\varphi^0)^2 \right\rangle \end{aligned} \tag{25}$$

Expectations for the total energy density and pressure [6], expressed as dimensionless parameters, assume the form

$$\begin{aligned} \left\langle \rho^0 \right\rangle &\equiv \frac{\left\langle \rho \right\rangle}{\mu^4} = \left\langle T_{kin} \right\rangle + \left\langle V(\varphi) \right\rangle = \delta \left\langle (\dot{\varphi}^0)^2 \right\rangle + \left\langle (\varphi^0)^2 \right\rangle \\ \left\langle p^0 \right\rangle &\equiv \frac{\left\langle p \right\rangle}{\mu^4} = \left\langle T_{kin} \right\rangle - \left\langle V(\varphi) \right\rangle = \delta \left\langle (\dot{\varphi}^0)^2 \right\rangle - \left\langle (\varphi^0)^2 \right\rangle \end{aligned} \tag{26}$$

with the corresponding equation of state

$$w \equiv \frac{\langle p^0 \rangle}{\langle \rho^0 \rangle} \tag{27}$$

where, in general, $-1 < w < 1$ [4-6]. For the vacuum energy, we obtain from (16) to (27)

$$w = -1 \rightarrow \delta = 0 \rightarrow \lambda_H = \lambda_{c,H} \tag{28}$$

and

$$\langle V(\varphi) \rangle \rightarrow \frac{1}{2} \frac{\lambda_H}{\tau_{equiv}} M_G^2 \langle (\varphi^0)^2 \rangle \tag{29}$$

The damping term in (9) suggests that a reasonable choice for the time scale defined by τ_{equiv} has the same order of magnitude as λ_H^{-1}, i.e.

$$\tau_{equiv} = \lambda_{c,H}^{-1} = \lambda_H^{-1} = (3H)^{-1} \tag{30}$$

From (24) and (30) it follows that $t_{eq} \sim O(\varepsilon)^{-1}$. As $t \rightarrow t_{eq}$, $\langle \rho^0 \rangle$ approaches unity and represents a stationary solution of the diffusion equation. It should be noted that the asymptotic time limit $t \rightarrow t_{eq}$ along with $\lambda_H - \lambda_{c,H} \rightarrow O(\varepsilon)$ implies that the dimensionless time t^0 stays finite. Under these circumstances, (17) and (26) yield the following prediction for the expected vacuum energy density

$$\boxed{\langle \rho^0 \rangle = 1 \rightarrow \langle \rho_{\mathrm{v}} \rangle = \frac{9}{2} H^2 M_G^2} \tag{31}$$

It is known that the critical energy density is determined by the standard prescription of Big Bang cosmology [6]

$$\rho_c \equiv \frac{3H^2}{8\pi G_N} = 6H^2 M_G^2 \tag{32}$$

Comparing (31) and (32) yields the result

$$\boxed{\Omega_{\mathrm{v}} = \frac{\langle \rho_{\mathrm{v}} \rangle}{\rho_c} = .75}$$

$$\boxed{\Omega_m \approx 1 - \Omega_{\mathrm{v}} = .25} \tag{33}$$

which fits well the current bounds for the normalized vacuum and matter densities [4-6].

6. References

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