Submitted to Advanced Studies in Theoretical Physics

On the asymptotic transition to complexity in Quantum Chromodynamics

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Abstract

Quantum Chromodynamics (QCD) is a renormalizable gauge theory that successfully describes the fundamental interaction of quarks and gluons. The rich dynamical content of QCD is manifest, for example, in the spectroscopy of complex hadrons or the emergence of quark-gluon plasma. There is a fair amount of uncertainty regarding the behavior of perturbative QCD in the infrared and far ultraviolet regions. Our work explores these two domains of QCD using nonlinear dynamics and complexity theory. We find that local bifurcations of the renormalization flow destabilize asymptotic freedom and induce a steady transition to chaos in the far ultraviolet limit. We also conjecture that, in the infrared region, dissipative nonlinearity of the renormalization flow supplies a natural mechanism for confinement.

1. Introduction and motivation

As a building block of the Standard Model for particle physics, QCD is a successful gauge theory describing the coupling of quarks and gluons [1-3]. It has several defining features, namely: a) asymptotic freedom (the interaction becomes weaker at short distances and it can be determined from perturbation theory), b) around $200\ MeV$, confinement sets in and the particle spectrum consists exclusively of color neutral states,

c) QCD exhibits spontaneous chiral symmetry breaking due to non-vanishing quark masses [1-3], d) at high temperature or high density, QCD is conjectured to sustain phase transitions leading to quark-gluon plasma and the restoration of chiral symmetry [3]. Due to asymptotic freedom, perturbative QCD is reasonably effective in the high-energy limit but fails to provide accurate predictions in the infrared limit, where the theory becomes strongly coupled [1-4]. The infrared regime of QCD is a typical example where non-perturbative methods become compelling. Since closed-form solutions of field theory are, in general, difficult to extract and manage, lattice-based computations and numerical approximations are among the most frequently used techniques for investigation [2]. Less developed are methods based on nonlinear analysis and dynamical systems theory, whereby knowledge of explicit solutions is no longer critical. From this standpoint, it can be stated that nonlinear dynamics offers an attractive theoretical laboratory for probing the asymptotic dynamics of QCD. With regard to field theory in general, this is also true near any boundary of the stability region where randomness becomes the driving factor [5] and the emergence of bifurcations and complex behavior is a likely occurrence. It is in this region where traditional procedures are questionable and one usually appeals instead to alternative methods such as the ones provided by the renormalization group (RG) [6]. Starting from these considerations, our goal is to develop a first-order analysis of the RG flow near the boundary of the stability region. The sustained contribution of perturbations to the RG flow is modeled as follows: a) since QCD is asymptotically free, we assume that perturbations develop progressively but smoothly in the ultraviolet region, b) in contrast, because QCD becomes strongly coupled in the infrared, we assume that perturbations are best modeled here as random fluctuations of Levy type. We caution that our work has an introductory nature and does not claim to provide a comprehensive and rigorous coverage of the topic. As the contribution of fluctuations and nonlinearities becomes increasingly predominant in the asymptotic regime of QCD, a complete analysis needs to carefully account for a variety of factors that are deliberately left out in our derivation.

The paper is organized according to the following plan: section 2 examines the QCD dynamics in the far ultraviolet region; the impact of Levy noise on the mechanism of infrared confinement is outlined in section 3. The last section contains a brief summary of results. Appendix A includes a condensed presentation of RG equations in the context of perturbative QCD.

2. QCD dynamics in the far ultraviolet region

2.1 Perturbed RG flow equations

We start from the RG equations for coupling strength and quark masses [7]

$$\frac{d\alpha_s}{dt} \approx -b_0(n)\alpha_s^2 - b_1(n)\alpha_s^3$$
$$\frac{dm}{dt} \approx -m[c_0(n)\alpha_s + c_1(n)\alpha_s^2 + c_2(n)\alpha_s^3] + NP \tag{1}$$

Here, n stands for the number of quark flavors and

$$t = \ln(\frac{\Lambda}{\Lambda_0}) \tag{2}$$

represents the sliding scale, where the momentum cutoff Λ is normalized to an arbitrary reference value Λ_0 such as the strong interaction scale ($\Lambda_0 \cong 220\,MeV$). The non-perturbative term in the mass flow is denoted by NP and is typically presumed to vanish faster than any power of the coupling [8]. In the presence of generic perturbations, (1) becomes

$$\frac{d\alpha_s}{dt} \approx -b_0(n)\alpha_s^2 - b_1(n)\alpha_s^3 + \Phi(\alpha_s)$$
$$\frac{dm}{dt} \approx -m[c_0(n)\alpha_s + c_1(n)\alpha_s^2 + c_2(n)\alpha_s^3] + \Psi(m) \tag{3}$$

Let us assume that the two additive contributions may be expanded in power series of a small parameter that defines the perturbation amplitude ($\varepsilon \ll 1$)

$$\Phi(\alpha_s) = \Phi_0(\alpha_s) + \varepsilon\Phi_1(\alpha_s) + \varepsilon^2\Phi_2(\alpha_s) + ...$$
$$\Psi(m) = \Psi_0(m) + \varepsilon\Psi_1(m) + \varepsilon^2\Psi_2(m) + ... \tag{4}$$

For simplicity we take

$$\Phi_n(\alpha_s) = \Psi_n(m) = 0, \text{ if } n \neq 1$$
$$\Phi_1(\alpha_s) = \alpha_s^2, \quad \Psi_1(m) = -m \tag{5}$$

(3) is thereby well approximated by

$$\frac{d\alpha_s}{dt} \approx -[b_0(n) + \varepsilon]\alpha_s^2 - b_1(n)\alpha_s^3 \equiv -b_\varepsilon(n)\alpha_s^2 - b_1(n)\alpha_s^3 \tag{6a}$$

$$\frac{dm}{dt} \approx -m[\varepsilon + c_0(n)\alpha_s + c_1(n)\alpha_s^2 + c_2(n)\alpha_s^3] \tag{6b}$$

in which

$$b_\varepsilon(n) = b_0(n) + \varepsilon \tag{7}$$

2.2 Linear stability analysis

Apart from the trivial solution represented by the fixed point FP_0 = [$\alpha_s^* = 0$, $m^* = 0$], the non-trivial fixed point of (6a) is given by FP = [$\alpha_s^* = -(b_\varepsilon / b_1)$, $m^* = 0$]. Linearizing around

FP_0, we find that the Lyapunov exponent is vanishing regardless of the numerical value of coefficients b_0, b_1 and, implicitly, regardless of the number of flavors n. On the other hand, the Lyapunov exponents corresponding to FP are [9-10]

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \tag{8}$$

in which

$$\tau = -\frac{b_\varepsilon^2}{b_1} + \varepsilon + \frac{c_0 b_\varepsilon}{b_1} - \frac{c_1 b_\varepsilon^2}{b_1^2} + \frac{c_2 b_\varepsilon^3}{b_1^3}$$

$$\Delta = -\frac{b_\varepsilon^2}{b_1}(\varepsilon + \frac{c_0 b_\varepsilon}{b_1} - \frac{c_1 b_\varepsilon^2}{b_1^2} + \frac{c_2 b_\varepsilon^3}{b_1^3}) \tag{9a}$$

To streamline the analysis, we next assume that all terms and factors dependent on the "c" coefficients are negligible. Following the general guidelines of nonlinear analysis, we are interested in the so-called borderline cases (i.e. centers, non-isolated fixed points, degenerate nodes and stars). These are determined by the numerical value of the characteristic parameter [10]

$$R = \tau^2 - 4\Delta \tag{9b}$$

Fig. 1 graphs the variation of R as a function of n as R approaches zero (red = 0.1, blue = 0.05, black = 0.001). It confirms that the dynamics of the RG flow becomes bordeline as the number of quark flavors approaches $n_{cr} = 16$ and QCD reaches the point of losing its asymptotic freedom [11].

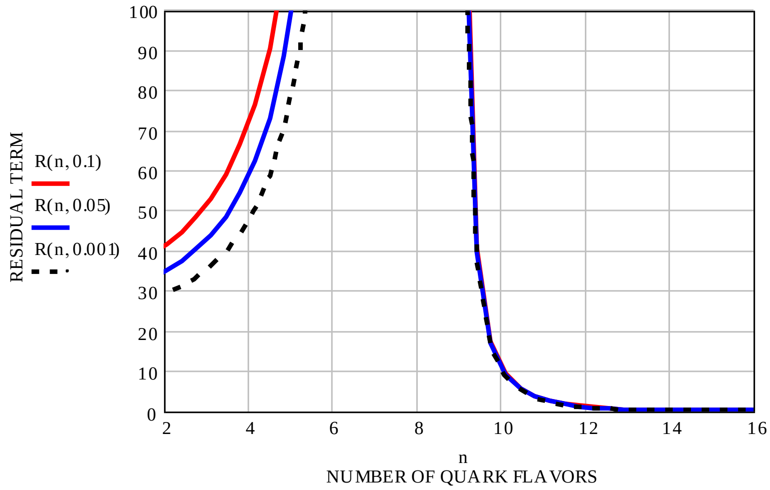

Fig. 1: Characteristic parameter ξ versus the number of quark flavors n

<u>2.3 Bifurcation of the fixed point</u>

We now wish to study the behavior of the nontrivial FP under the influence of a steadily increasing perturbation whose amplitude augments the noise terms previously considered ($\varepsilon \ll 1$). The origin of this perturbation may be related to thermal fluctuations (if the analysis is carried out at a high-temperature setting) or to the presence of a large number of high-order diagrams associated with the ultraviolet limit. To this end, let us add an infinite series of terms to the coupling flow equation (6a), that is

$$\frac{d\alpha_s}{dt} \approx -b_\varepsilon(n)\alpha_s^2 - b_1(n)\alpha_s^3 + \sum_{i=0}^{\infty}\kappa_i(n)\alpha_s^i \qquad (10)$$

We assume next that only the first two terms in the series (10) are non-vanishing and weakly dependent on n. Following the notation of [9], we obtain

$$\frac{d\alpha_s}{dt} \approx \kappa_1 + \alpha_s(\kappa_2 + \Lambda_1) + \Lambda_2\alpha_s^2 + \Lambda_3\alpha_s^3 \qquad (11)$$

Here, $\Lambda_{1,2,3}$ denote the so-called Lyapunov values which are respectively given by

$\Lambda_1 = -\frac{6b_\varepsilon^2}{b_1}$, $\Lambda_2 = b_\varepsilon$, $\Lambda_3 = -b_1$ .The FP is stable since $\Lambda_3 < 0$. The set of scalars $\kappa_{1,2}$ denote the governing parameters and measure the deviation of an arbitrary point in parameter space from origin $(\kappa_1 = 0, \kappa_2 = 0)$. Under these circumstances, the bifurcation curve has a cusp profile and is represented by [9-10]

$$\kappa_1 = \frac{\pm 2\kappa_2 \sqrt{|\kappa_2/(-b_1)|}}{3\sqrt{3}} + \ldots, \qquad \kappa_2(-b_1) < 0 \tag{12}$$

Fig. 2 plots the variation of the Lyapunov values as a function of n. Fig. 3 shows the emergence of a cusp bifurcation in the (κ_1, κ_2) plane when b_1 is computed at $n = 6$ (see (A2)). Depending on the location of the governing parameters in the (κ_1, κ_2) plane, the stable FP stays unchanged or splits into two or three equilibria. It follows that, near the non-trivial FP, irregular behavior of the coupling flow is likely to develop through a progressive cascade of cusp bifurcations.

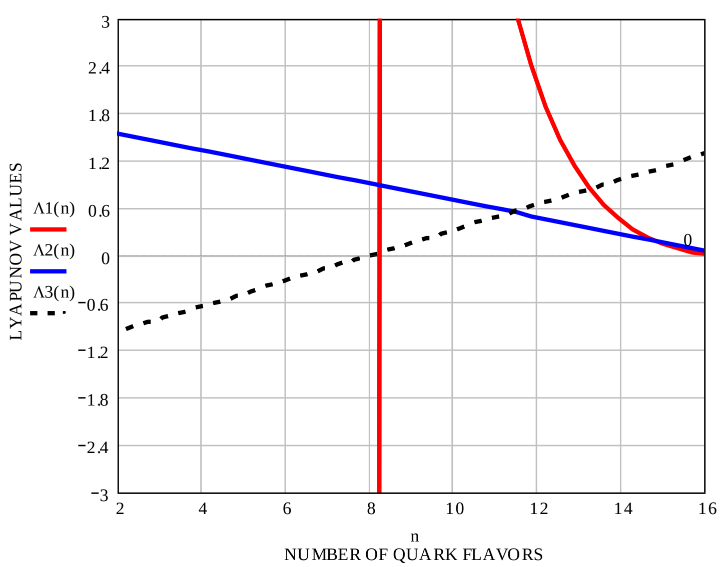

Fig. 2: Lyapunov values versus the number of quark flavors n

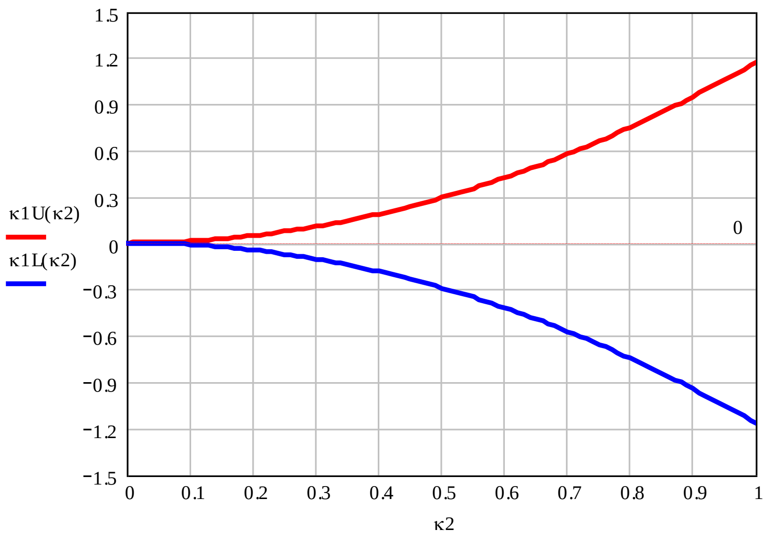

Fig 3: Cusp bifurcation in the (κ_1, κ_2) plane

3. Levy noise as possible mechanism for confinement

As noted in the first section, the infrared limit is characterized by large fluctuations of the RG flow induced by the strong-coupling regime of QCD. To preserve maximum generality of our approach and using arguments related to the ubiquity of Levy flights in stochastic transport processes [12], we model these fluctuations with the help of the generalized Langevin equation [13]

$$\frac{d\alpha_s}{dt} = \eta(\alpha_s)\alpha_s + \Gamma_\alpha(t) \tag{13}$$

Here, $\Gamma_\alpha(t)$ represents α-stable Levy noise and the dissipative non-linearity $\eta(\alpha_s)$ is given by

$$\eta(\alpha_s) = -b_1(n)\alpha_s^2 - b_1(n)\alpha_s^3 \tag{14}$$

Under these conditions, the asymptotic probability distribution function for $\alpha_s \gg 1$ is given by [13]

$$p(r,\alpha_s) \propto \frac{1}{b_r\left|\alpha_s\right|^{\sigma}} \tag{15}$$

in which $b_r > 0$ stands for the r^{th} order coefficient of (14) and

$$\sigma = 1 + r + \alpha \tag{16}$$

It follows that $\left\langle \alpha_s^2 \right\rangle$ stays finite if $r > r_{cr} = 2 - \alpha$, that is, $r > 0$ if $\alpha = 2$ and $r > 2$ if $\alpha = 0$. We conclude that the coupling flow driven by stable Levy noise remains confined if its expansion is taken at least to the second loop approximation. This ansatz suggests a plausible mechanism for confinement in the IR region of QCD: instead of reaching a regime of unbounded variations in interaction amplitude, higher order radiative corrections generated from $\eta(\alpha_s)$ dissipate the energy imparted by Levy noise. As a result, quarks and gluons form bounded states with a nearly-constant average coupling strength.

It is instructive to note that this conjecture fits well various lattice studies and phenomenological theories of quark-antiquark ($q\bar{q}$) interaction, such as the Richardson or Cornell models [2]. For example, the Cornell model assumes that the long-range part of the static $q\bar{q}$ potential has the form

$$V(r) = br - \frac{a}{r} + V_0 \tag{17}$$

where a, b, V_0 are constants. The coefficient b is commonly referred to as the "string tension" by comparison with string theories of hadrons. The linear term of this potential (br) dominates the interaction at large distances where it models a color-flux tube of constant energy density.

It is also instructive to remark that, in a certain sense, the mechanism of confinement produced by Levy fluctuations is similar to the phenomenon of Anderson localization in which quantum waves become confined in random potentials [14].

We close this section with an evaluation on how statistical moments of coupling strengths and quark masses depend on the Levy parameter α. For this purpose, it is sufficient to solve (1) in closed-form and use the asymptotic probability distribution function (15) to determine the expectations and variances for coupling strength and mass. Fig. 4 plots the expectation of the coupling strength ($M1\alpha_s$) along with its variance ($Var\alpha_s$) as functions of the Levy index α, whereas Fig 5 plots the same behavior for the quark mass. The number of quark flavors is assumed to be $n=6$ in both cases. It is seen that the increase of variances with the Levy parameter is significantly faster than the corresponding increase of expectation values.

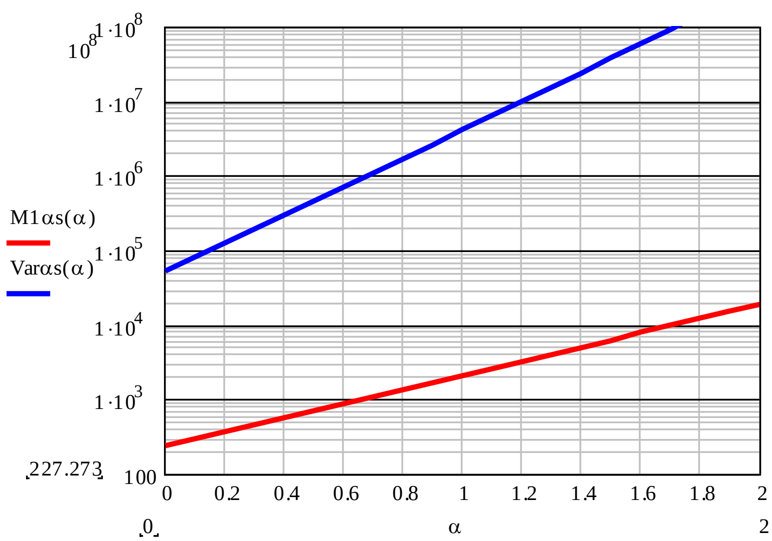

Fig. 4: QCD coupling strength and its variance versus α (log scale)

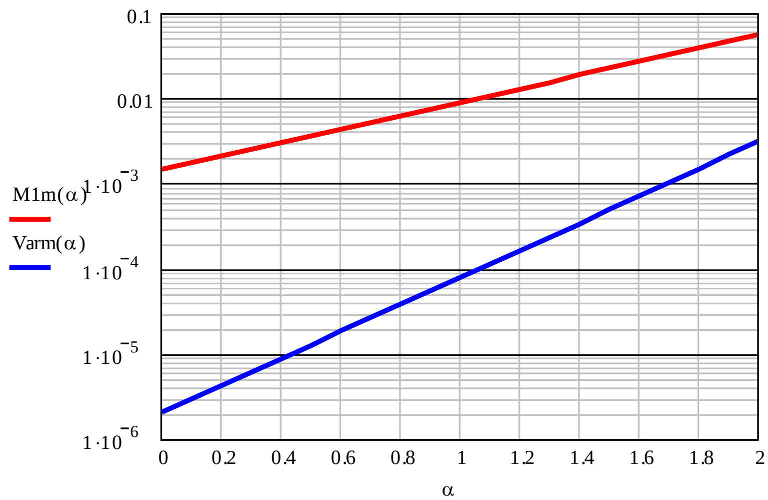

Fig. 5: Quark mass and its variance versus α (log scale)

4. Summary and discussion

Using the analytical tools provided by nonlinear dynamics and complexity theory, we have examined the effect of the renormalization flow in the asymptotic regions of QCD. It was found that the steady addition of perturbations destabilizes asymptotic freedom and induces transition to chaos in the far ultraviolet limit. At the other end of the energy scale, dissipative nonlinearity of the renormalization flow provides a plausible mechanism for confinement.

In section 1 we pointed out the introductory nature of our treatment. QCD is a rich and complex theory that has the potential of exhibiting a large spectrum of behaviors. It is apparent that the outcome of the set of coupled nonlinear equations describing the RG flow depends strongly on how the model is formulated and how the boundary conditions are set. To be specific,

1) there are two dependent control parameters of the RG flow: the momentum scale Λ (or, equivalently, the dimensional regularization parameter $\varepsilon = 4 - d$ [1, 6]) and the

number of fermion flavors $n = n(\Lambda) = n(t)$. Obviously, a simplified setting is to assume $n = n(\Lambda) = n(t)$ is a slowly varying function and carry the analysis with a single control parameter defining the energy scale at which the physics is probed.

2) the dimensionality of the flow plays a critical role: a planar system of equations (such as the one for coupling and masses) does not lead to deterministic chaos. In contrast, the 3D system containing the flow of fields and mixing angles leads to a much richer spectrum of behaviors, including deterministic chaos.

3) addition of statistical perturbations leads to systems of coupled stochastic nonlinear equations. These have, in general, a complex array of possible dynamical patterns. In this case all parameters (fields, masses, mixing angles, correlation functions) become random variables and their behavior needs to be formulated in terms of probability distribution functions. The ability to formulate the correct noise model is critically important. For convenience, we have limited the discussion to the generic case of Levy noise.

4) finally, the presence of long-range interactions in space and time (extended spatial coupling, time-memory, delayed interactions) yields a problem with coupled multiple time-scales. The proper way to deal with this setting is to use the tools offered by fractional calculus and fractional dynamics [15-22] or, equivalently, with the formalism of non-extensive statistical physics. [12].

Appendix A

Within the framework of perturbative QCD, the flow of the effective coupling strength with the sliding energy scale μ is governed by the beta-function:

$$\mu \frac{\partial \alpha_s}{\partial \mu} = \beta(\alpha_s) \tag{A1}$$

where

$$\beta(\alpha_s) = -b_0\alpha_s^2 - b_1\alpha_s^3 - b_2\alpha_s^4 + O(\alpha_s^5)$$

$$b_0 = \frac{11 - \frac{2}{3}n}{2\pi}, \qquad b_1 = \frac{51 - \frac{19}{3}n}{4\pi^2}, \qquad b_2 = \frac{2857 - \frac{5033}{9}n + \frac{325}{27}n^2}{64\pi^3} \tag{A2}$$

and n is the effective number of quark flavors [7]. Likewise, the scale dependence of a running quark mass $m(\mu)$ is represented by:

$$\mu\frac{dm(\mu)}{d\mu} = -\gamma(\alpha_s)m(\mu) \tag{A4}$$

in which

$$\gamma(\alpha_s) = c_0\alpha_s + c_1\alpha_s^2 + c_2\alpha_s^3 + O(\alpha_s^4)$$

$$c_0 = \frac{2}{\pi}, \qquad c_1 = \frac{\frac{101}{12} - \frac{5}{18}n}{\pi^2}, \qquad c_2 = \frac{1}{32\pi^3}[1249 - (\frac{2216}{27} + \frac{160}{3}\varsigma(3))n - \frac{140}{81}n^2] \tag{A5}$$

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