Professor Soper's mistake

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Abstract

The impossibility to obtain the electrodynamics Maxwell tensor in the framework of the Lagrange formalism is confirmed by a professor Soper's mistake made on this way. Nevertheless, a heuristic method allows using the Lagrange energy-momentum and spin tensors for obtaining the classical spin tensor.

1. Desire

The modern classical theory of electromagnetic field uses free field Lagrangians, which are independent on coordinates explicitly, e.g. the canonical Lagrangian [1]

$$\underset{c}{\mathsf{L}} = -F_{\mu\nu}F^{\mu\nu}/4\,, \tag{1.1}$$

or the simple Lagrangian of a massless vector field [2]

$$\underset{v}{\mathsf{L}} = -\partial_\mu A_\nu \partial^\mu A^\nu /2\,, \tag{1.2}$$

or the Dirac-Fock-Podolsky Lagrangian [3]

$$\underset{D}{\mathsf{L}} = -F_{\mu\nu}F^{\mu\nu}/4 - \partial_\mu A^\mu \partial_\nu A^\nu /2\,. \tag{1.3}$$

Combining any free field Lagrangian with the interaction Lagrangian [1]

$$\underset{j}{\mathsf{L}} = -A_\mu j^\mu \tag{1.4}$$

of this field with its source, i.e. with the 4-current j^μ, physicists obtain field equations as the Euler-Lagrange equations

$$\frac{\partial \mathsf{L}}{\partial A_\mu} - \partial_\nu \frac{\partial \mathsf{L}}{\partial(\partial_\nu A_\mu)} = 0\,. \tag{1.5}$$

For example, the sum

$$\mathsf{L} = \underset{c}{\mathsf{L}} + \underset{j}{\mathsf{L}} \tag{1.6}$$

gives the Maxwell equations [1]

$$\partial_\nu F^{\mu\nu} = -j^\mu\,. \tag{1.7}$$

But, when so-called conserved quantities, namely energy-momentum tensor and angular momentum tensor are obtained, only a free field Lagrangian is used. For example, the canonical Lagrangian (1.1)

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gives the canonical pair of tensors: the canonical energy-momentum tensor $\underset{c}{T}^{\lambda\mu}$ and the canonical angular momentum tensor $\underset{c}{\mathsf{J}}^{\lambda\mu\nu}$:

$$\underset{c}{T}^{\lambda\mu} = \partial^{\lambda} A_{\alpha} \frac{\partial \underset{c}{\mathsf{L}}}{\partial(\partial_{\mu} A_{\alpha})} - g^{\lambda\mu} \underset{c}{\mathsf{L}} = -\partial^{\lambda} A_{\alpha} F^{\mu\alpha} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta}/4, \tag{1.8}$$

$$\underset{c}{\mathsf{J}}^{\lambda\mu\nu} = 2x^{[\lambda} \underset{c}{T}^{\mu]\nu} + \underset{c}{\Upsilon}^{\lambda\mu\nu}, \quad \underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}\delta^{\mu]}_{\alpha} \frac{\partial \underset{c}{\mathsf{L}}}{\partial(\partial_{\nu} A_{\alpha})} = -2A^{[\lambda} F^{\mu]\nu}, \tag{1.9}$$

where $\underset{c}{\Upsilon}^{\lambda\mu\nu}$ is the canonical spin tensor.

Unfortunately, the canonical energy-momentum tensor (1.8) differs considerably from the real energy-momentum tensor of electrodynamics, i.e. from the Maxwell tensor

$$T^{\lambda\mu} = -F^{\lambda}{}_{\alpha} F^{\mu\alpha} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta}/4. \tag{1.10}$$

The difference between the Maxwell tensor and the canonical tensor is not equal to a divergence,

$$T^{\lambda\mu} - \underset{c}{T}^{\lambda\mu} = \partial_{\alpha} A^{\lambda} F^{\mu\alpha}, \tag{1.11}$$

and a divergence of the canonical tensor is somewhat senseless,

$$\partial_{\mu} \underset{c}{T}^{\lambda\mu} = -\partial^{\lambda} A_{\alpha} j^{\alpha}, \tag{1.12}$$

while the divergence of the Maxwell tensor is equal to a well-known expression for a 4-force acting on a field from sources of the field:

$$\partial_{\mu} T^{\lambda\mu} = -F^{\lambda\alpha} j_{\alpha}. \tag{1.13}$$

These cast doubt on an adequacy of the Lagrange formalism and cause a desire for obtaining the Maxwell tensor from the Lagrange formalism at any price to remove this doubt. This desire forced professor Soper to use the combined Lagrangian $\mathsf{L} = \underset{c}{\mathsf{L}} + \underset{j}{\mathsf{L}}$ (1.6), instead of the canonical Lagrangian $\underset{c}{\mathsf{L}}$ (1.1), in Eq. (1.8) to obtain an energy-momentum tensor notwithstanding the explicit dependence of the combined Lagrangian on coordinates.

So, Soper added the interaction Lagrangian $\underset{j}{\mathsf{L}} = -A_{\mu} j^{\mu}$ to the canonical Lagrangian $\underset{c}{\mathsf{L}}$. Because of the addend, he must obtain an addend to the canonical energy-momentum tensor (1.8) in a form of the transvection

$$\underset{j}{T}^{\lambda\mu} = \partial^{\lambda} A_{\alpha} \frac{\partial \underset{j}{\mathsf{L}}}{\partial(\partial_{\mu} A_{\alpha})} - g^{\lambda\mu} \underset{j}{\mathsf{L}} = -g^{\lambda\mu} \underset{j}{\mathsf{L}} = g^{\lambda\mu} A_{\alpha} j^{\alpha}. \tag{1.14}$$

But, Soper was dazzled by the desire, and he wrote a product (Soper's formula (8.3.8))

$$\underset{I}{T}^{\lambda\mu} = A^{\lambda} j^{\mu} \tag{1.15}$$

instead of (1.14) because the divergence of the product,

$$\partial_\mu \underset{I}{T}^{\lambda\mu} = \partial_\mu A^\lambda j^\mu , \tag{1.16}$$

made the divergence of the Soper's combined tensor

$$\underset{S}{T}^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} + \underset{I}{T}^{\lambda\mu} = -\partial^\lambda A_\alpha F^{\mu\alpha} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta} / 4 + A^\lambda j^\mu \tag{1.17}$$

equal to the true divergence, i.e. to the divergence of the Maxwell tensor:

$$\partial_\mu \underset{S}{T}^{\lambda\mu} = \partial_\mu (\underset{c}{T}^{\lambda\mu} + \underset{I}{T}^{\lambda\mu}) = -\partial^\lambda A^\alpha j_\alpha + \partial^\alpha A^\lambda j_\alpha = -F^{\lambda\alpha} j_\alpha . \tag{1.18}$$

Eq. (1.18) shows that the difference between the Soper's tensor (1.17) and the Maxwell tensor (1.10) is equal to a divergence of an antisymmetric quantity. Really,

$$\underset{S}{T}^{\lambda\mu} - T^{\lambda\mu} = -\partial_\alpha (A^\lambda F^{\mu\alpha}) . \tag{1.19}$$

Since such a difference between energy-momentum tensors considers by physicists as an inessential difference, Soper claims that he obtains the Maxwell tensor from Lagrange formalism.

Of course, this conclusion is incorrect because the interaction Lagrangian (1.4) used by Soper gives the expression (1.14) rather than the product (1.15) as an addend to the canonical energy-momentum tensor. Therefore, in reality the combined Lagrangian (1.6) $L = \underset{c}{L} + \underset{j}{L}$ gives not the Soper's tensor (1.17), but a sum

$$\underset{c}{T}^{\lambda\mu} + \underset{j}{T}^{\lambda\mu} , \tag{1.20}$$

which divergence

$$\partial_\mu (\underset{c}{T}^{\lambda\mu} + \underset{j}{T}^{\lambda\mu}) = A_\alpha \partial^\lambda j^\alpha \tag{1.21}$$

is senseless as well as the divergence (1.12) of the canonical energy-momentum tensor is. Besides, it is somewhat senseless to consider the expression (1.17), which is dependent on an electric current, as an energy-momentum tensor of electromagnetic field. Moreover, contrary to a common opinion, an addition of any terms to a true energy-momentum tensor is not admissible. So, in any case, the Soper's tensor (1.17) does not have anything common neither with the Maxwell tensor nor with common sense

2. Real way

Soper's casus as well as the history of theoretical physics show that, apparently, the Lagrange formalism cannot give the Maxwell tensor of electrodynamics. We must recognize that the canonical Lagrangian (1.1) leads to the canonical energy-momentum tensor (1.8), which differs from the Maxwell tensor (1.10); and the difference is (1.11). In other words, the fact of the matter is we must add "by hand" the term (1.11) to the canonical tensor for arriving to the Maxwell tensor.

Now let us consider spin. The Lagrange spin tensor $\underset{L}{Y}^{\lambda\mu\nu}$ of a free field is gained from the Lagrange formalism by the formula in Eq. (1.9). There is a sense in considering the Lagrange pair $\underset{L}{T}^{\lambda\mu}, \underset{L}{Y}^{\lambda\mu\nu}$. The canonical Lagrangian (1.1) gives the Lagrange pair

$$\underset{c}{T}^{\lambda\mu} = -\partial^{\lambda} A_{\alpha} F^{\mu\alpha} + g^{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta} / 4 , \qquad \underset{c}{Y}^{\lambda\mu\nu} = -2A^{[\lambda} \delta^{\mu]}_{\alpha} \frac{\partial \underset{c}{L}}{\partial(\partial_{\nu} A_{\alpha})} = -2A^{[\lambda} F^{\mu]\nu} , \tag{2.1}$$

and the simple Lagrangian of a vector field (1.2) gives the vector field pair:

$$\underset{v}{T}^{\mu\nu} = -\partial^{\mu} A_{\alpha} \partial^{\nu} A^{\alpha} + g^{\mu\nu} \partial_{\alpha} A_{\beta} \partial^{\alpha} A^{\beta} / 2 , \qquad \underset{v}{Y}^{\lambda\mu\nu} = 2A^{[\lambda} \partial^{|\nu|} A^{\mu]} . \tag{2.2}$$

We noted an interesting fact. Although the canonical tensors are not adequate, an expression for spin density, which enters an electromagnetic field from sources, according to the canonical tensors, i.e. the expression

$$\partial_{\nu} \underset{L}{Y}^{\lambda\mu\nu} - 2\underset{L}{T}^{[\lambda\mu]} , \tag{2.3}$$

is the same for the canonical pair and for the vector field pair:

$$\partial_{\nu} \underset{c}{Y}^{\lambda\mu\nu} - 2\underset{c}{T}^{[\lambda\mu]} = \partial_{\nu} \underset{v}{Y}^{\lambda\mu\nu} - 2\underset{v}{T}^{[\lambda\mu]} = 2A^{[\lambda} j^{\mu]} . \tag{2.4}$$

Expression (2.3) is a 4-spin density, which enters an electromagnetic field from external sources, because the total angular 4-momentum $J^{\lambda\mu}$, which goes out of a 4-volume Ω through its boundary V, can be presented as an integral of the sum of the orbital density

$$2r^{[\lambda} \partial_{\nu} T^{\mu]\nu} \tag{2.5}$$

and the spin density

$$\partial_{\nu} Y^{\lambda\mu\nu} - 2T^{[\lambda\mu]} , \tag{2.6}$$

which enters the field inside Ω from external sources placed inside Ω:

$$J^{\lambda\mu} = \oint_{V} (2r^{[\lambda} T^{\mu]\nu} + Y^{\lambda\mu\nu}) dV_{\nu} = \int_{\Omega} (-2T^{[\lambda\mu]} + 2r^{[\lambda} \partial_{\nu} T^{\mu]\nu} + \partial_{\nu} Y^{\lambda\mu\nu}) d\Omega . \tag{2.7}$$

Well, our idea about spin is the following. We know that we must add the quantity (1.11) to the canonical tensor (1.8) to convert it into the true energy-momentum electrodynamics tensor (the Maxwell tensor). The quantity (1.11) is designated as $\underset{c}{t}^{\lambda\mu}$:

$$T^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} + \underset{c}{t}^{\lambda\mu} = \underset{c}{T}^{\lambda\mu} + \partial_{\alpha} A^{\lambda} F^{\mu\alpha} . \tag{2.8}$$

Using this, we suggest to add a quantity $\underset{c}{s}^{\lambda\mu\nu}$ to the canonical spin tensor $\underset{c}{Y}^{\lambda\mu\nu}$ to convert $\underset{c}{Y}^{\lambda\mu\nu}$ into a unknown true electrodynamics spin tensor $Y^{\lambda\mu\nu}$:

$$Y^{\lambda\mu\nu} = \underset{c}{Y}^{\lambda\mu\nu} + \underset{c}{s}^{\lambda\mu\nu} ;$$

$\underset{c}{s}^{\lambda\mu\nu}$ together with $\underset{c}{t}^{\lambda\mu}$ must not contain sources of spin, i.e. must satisfy the equation

$$\partial_\nu \underset{c}{s}^{\lambda\mu\nu} - 2\underset{c}{t}^{[\lambda\mu]} = 0\,, \text{ i.e. } \partial_\nu \underset{c}{s}^{\lambda\mu\nu} - 2\partial_\alpha A^{[\lambda} F^{\mu]\alpha} = 0\,. \tag{2.9}$$

In other words, we correct the canonical spin tensor $\underset{c}{\Upsilon}^{\lambda\mu\nu}$ by $\underset{c}{s}^{\lambda\mu\nu}$ just as we correct the canonical energy-momentum tensor $\underset{c}{T}^{\lambda\mu}$ by $\underset{c}{t}^{\lambda\mu}$.

A simple expression

$$\underset{c}{s}^{\lambda\mu\nu} = 2A^{[\lambda}\partial^{\mu]}A^\nu \tag{2.10}$$

satisfies Eq. (2.9). So, the suggested electrodynamics spin tensor is

$$\Upsilon^{\lambda\mu\nu} = \underset{c}{\Upsilon}^{\lambda\mu\nu} + \underset{c}{s}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu} + 2A^{[\lambda}\partial^{\mu]}A^\nu = 2A^{[\lambda}\partial^{|\nu|}A^{\mu]}\,. \tag{2.11}$$

This expression can be also obtained immediately from the vector field pair (2.2). Because of the symmetry of the energy-momentum tensor of vector field $\underset{v}{T}^{\lambda\mu}$, the addend $\underset{v}{t}^{\lambda\mu}$ is also symmetric, $\underset{v}{t}^{[\lambda\mu]} = 0$. (We mean $T^{\lambda\mu} = \underset{v}{T}^{\lambda\mu} + \underset{v}{t}^{\lambda\mu}$). Therefore Eq. (2.9) for a vector field takes the form

$$\partial_\nu \underset{v}{s}^{\lambda\mu\nu} = 0 \tag{2.12}$$

with an evident solution $\underset{v}{s}^{\lambda\mu\nu} = 0$. Therefore the suggested electrodynamics spin tensor $\Upsilon^{\lambda\mu\nu}$ coincides with the vector field spin tensor $\underset{v}{\Upsilon}^{\lambda\mu\nu}$.

The result (2.11) was submitted to "JETP Letters" on May 12, 1998, and to "JETP" on Jan. 27 1999, but the submissions were rejected as unsuitable. From then the spin tensor (2.11) was rejected more than 300 times by all scientific journals of the world, including the arXiv. The only exception is "Measurement Techniques", which is free of standard theorists [10, 11].

It is remarkable that two referees referred to Soper when rejecting my papers. They are: an Adjudicator of "JMO" (5 April 2005) and a Referee of "PLA" (21 Oct 2002). And they ignored my objections.

The expression (2.11) was obtained heuristically. It is not final one. Its improvement and applications are presented in [5 – 15] and at the site http://www.mai.ru/projects/mai_works/. Absorption of a circularly polarized beam is calculated there, the result of the classical Beth experiment is explained, and a radiation of a rotating electrical dipole is considered in these works.

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