Field Tubes and Bisurfaces in the Electromagnetism

Radi I. Khrapko*
Moscow Aviation Institute, 125993, Moscow, Russia

Abstract

Faraday's field lines are not enough for an adequate graphical representation of electromagnetic fields. It is necessary to use bisurfaces. The bisurfaces and field tubes, replacing the field lines, permit to represent evidently, for example, how electric current creates magnetic field, and electric field produces scalar potential field.

A conception of differential forms and contravariant tensor densities is used. We say that an exterior derivative of the form or a divergence of the density result in boundaries of the geometric quantities. The integration of the quantity by the Biot-Savarat formula results in a new quantity. We name the quantity a generation. Generating from a generation yields zero. So, generations are sterile as well as boundaries are closed. A conjugation is considered. The conjugation converts a closed quantity to a sterile quantity and back. The conjugation differs from the Hodge operation. The conjugation does not imply a dualization. Chains of field and an analog of Hodge decomposition theorem are considered

PACS numbers: 01.40.Fk, 03.50.De, 02.40.-k

1 Introduction

We consider stationary electromagnetic fields in vacuum.

1. As is orthodox, electrical charges produce electric field **E**. This phenomenon is usually depicted as follows [1, p. 670], [2, Fig. 4-12]

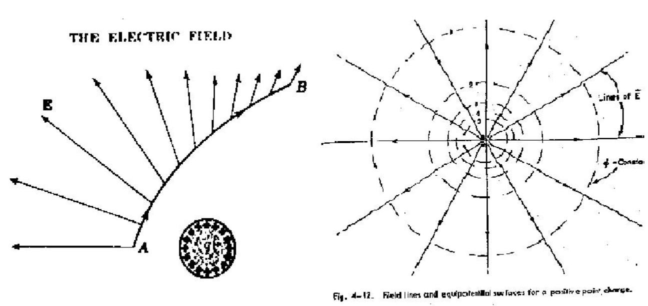

Figure 1: Electric strength **E** (from [1, 2])

In Fig. 1 I have reproduced the presentation of the electric field from the famous books [1, 2].

*Email addresses: khrapko_ri@mai.ru, khrapko_ri@hotmail.com

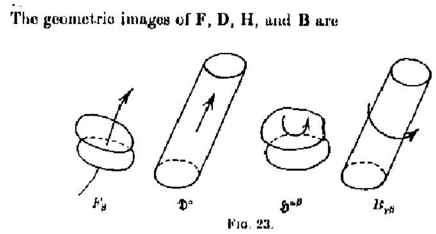

Figure 2: Images (**F** is electric strength) [3]

But the statement and the Figure seem to be incorrect. The point is that electric strength **E** is a covariant vector due to $\mathbf{E} = -\mathrm{grad}\phi$, or $E_i = -\partial_i\phi$, and covariant vectors are depicted by bisurface elements rather than by arrows [3, p. 133], [4].

This interpretation of a covariant vector was well known for a long time. For example, professor J. A. Schouten delivered lectures on this subject before the war at Delft, and after the war at Amsterdam (see the book [3] which was grown from the lectures; it is a classical monograph). The interpretation was used also by J. Napolitano and R. Lichtenstein [4]. They wrote, “We refer to this pictorial representation of a covector as a ‘lasagna’ vector” (lasagna is a type of Italian food made with flat pieces of pasta, meat or vegetables, and cheese). J. A. Schouten depicted a covector as a pair of flat elements. I have reproduced Schouten’s Fig. 23 here in Fig. 2. I name the pair of flat elements a bisurface element. Therefore, pairs of surfaces, i.e. bisurfaces, must depict a covector field. A covariant vector field $\mathbf{E}(x)$ is depicted by bisurfaces which are tangent to the elements rather than by lines.

In contrast to **E**, electric displacement **D** is not a gradient. **D** has another nature. Electric displacement **D** is a (contravariant) vector density. It agrees with $\mathrm{div}\mathbf{D} = \rho$, or $\partial_i D^i = \rho$. Field tubes depict contravariant vector densities. This fact is represented at Schouten’s Fig. 23 and in my Fig. 2. So, electric charges produce the field $\mathbf{D}(x)$ rather than **E**. The **D**-feld is depicted by tubes. Electric charge density ρ is a source of **D**. **D**-tubes are emerged from ρ (Fig. 5). We will make the statement more precise in Sec. 2. In Sec. 3 we show that **E** is a source of electrical scalar potential ϕ, that **E** produces ϕ.

2. As is orthodox, electric currents produce magnetic vector field. This phenomenon is depicted in Figure 3 [5, p.288], [2, Fig. 13-7]

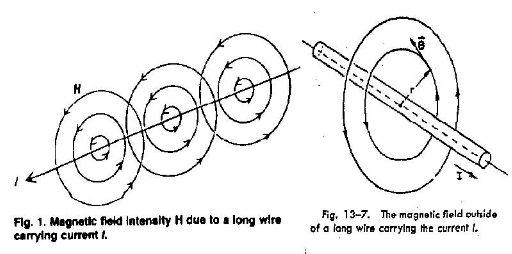

Figure 3: Magnetic field (from [5, 2])

I have reproduced the presentation of the magnetic field from the authoritative books [5, 2] in Fig 3. But the Figures seem to be incorrect.

The main aim of this paper is to demonstrate that the electric, magnetic fields, the fields of electric and magnetic potentials, i.e. all electromagnetic fields, are depicted by geometrical objects which are emerged from their sources just as tubes of electric displacement **D** are emerged from electric charges in Fig 5. To demonstrate this fact, we are forced to use the true geometrical interpretation of the electromagnetic fields which are either antisymmetric covariant tensor fields (ϕ, **E**, **A**, **B**), or fields of antisymmetric contravariant tensor densities (ρ, **j**, **D**, **H**). This interpretation is presented in Fig. 2.

We show that geometric images depicting a magnetic field, which is produced by electric currents, are emerged from the electric current tubes **j** (Fig. 8). These images must not make loops around the lines. You see, **D**-tubes do not make loops around a charge.

Note, electric current vector density **j** cannot produce a vector field, i.e. it cannot be a source of a vector field; and it cannot be a source of the magnetic induction **B** because **B** is a covariant bivector B_{jk} due to $\mathbf{B} = \mathrm{curl}\mathbf{A}$, or $B_{ik} = 2\partial_{[i}A_{k]}$. So, $\mathbf{j} = \mathrm{curl}\mathbf{B}$, or $j^i = \partial_k B_{ik}$ is nonsense.

As a matter of fact, the electric current vector density j^i produces magnetic strength bivector density H^{ik}. **j** is a source of **H**: $\mathbf{j} = \mathrm{curl}\mathbf{H}$, or $j^i = \partial_k H^{ik}$. But bivector density H^{ik} is depicted by a bisurface element [3] rather than a vector-arrow, and $\mathbf{H}(x)$-field is depicted by bisurfaces which are emerged from **j**-tubes (Fig. 8). We will make the statement more precise in Sec. 5.

3. As is orthodox, lines of the vector potential **A** make loops around **B**-lines as well as **B**-lines make loops around **j**-lines. This phenomenon is depicted in Figure 4 [2, Fig. 15-6].

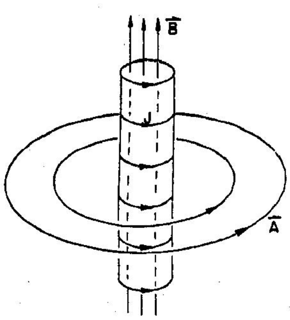

Fig. 15–6. The magnetic field and vector potential of a long solenoid.

Figure 4: Vector potential **A** (from [2])

But the statement and the Figure seem to be incorrect.

Since $\mathbf{B} = \mathrm{curl}\mathbf{A}$, or $B_{ik} = 2\partial_{[i}A_{k]}$, the vector potential **A** is a covariant vector, and **B** is a source of **A**, i.e. **B** produces **A**. So, geometric images depicting a field, which is produced by the **B** -tubes, must emerge from the tubes, rather than make loops around the tubes. **A**-field must be depicted by bisurfaces emerging from **B**-tubes. (Fig. 9). We will make the statement more precise in Sec. 6.

So, the four fields, E, D, H, B, differ from one another in physical sense and in geometrical representation even in vacuum. But the fields are *conjugate* in pairs (Sec. 9).

All electrimagnetic fields are fields of geometric quantities [3]. Geometric quantities are independent on a co-ordinate system. Because of this independence components of geometric quantities vary with the co-ordinate system. For example,

$$E_i = E_{i'}\partial_i^{i'}.$$

Here $\partial_i^{i'}$ is the matrix of the coordinate transformation $x'(x)$: $\partial_i^{i'} = \partial x^{i'}/\partial x^i$. We use marked indexes.

2 ρ generates $\mathbf{D}$

We use the verb *generate* instead of 'produce', 'create', 'set up'.

Scalar charge density $\rho_\wedge$ generates electric displacement $D^i_\wedge$ which is a vector density of weight $+1$ by the formula

$$\underset{\times}{D}{}^i_\wedge(x) = \int \frac{\rho_{\wedge'}(x')r^i_\wedge(x,x')dV^{\wedge'}}{4\pi r^3(x,x')}. \tag{1}$$

Gothic letters are usually applaid to denote tensor densities. We shall, instead, mark the density with the symbol 'wedge' at the level of bottom indices for a density of weight $+1$ and at the level of top indices for a density of weight -1. Volume element is a density of weight -1, $dV^\wedge$.

The primes mark a varying point x' in the integral (1) rather than another coordinate system.

We mark the generation by sign $\times$; generations mostly possess an important quality, see Sec. 7. The cross is the best notations for generations because geometrical images of the generations are emerged from their sources as well as in Figures 5, 8, 9, 10, 11.

Tensor densities differ from tensors: the transformation law of density involves the modulus of Jacobian. For example,

$$D^i_\wedge = D^{i'}_{\wedge'}\partial^i_{i'} \mid \Delta' \mid . \tag{2}$$

Here $\partial^i_{i'}$ is the matrix of the coordinate transformation: $\partial^i_{i'} = \partial x^i/\partial x^{i'}$, $\Delta' = \mathrm{Det}(\partial^{i'}_i)$ designates the determinant of the inverse matrix.

We say that $\underset{\times}{D}{}^i_\wedge$ is a *generation* from $\rho_\wedge$, or that $\rho_\wedge$ is a source of $\underset{\times}{D}{}^i_\wedge$. The symbol *dagger* † is used for a brief record of generating. For example:

$$\underset{\times}{D}{}^i_\wedge = \dagger^i \rho_\wedge. \tag{3}$$

The geometric image of a vector density of weight $+1$, $D^i_\wedge$, is a cylinder with an inner orientation [3]. Hence a vector density field is represented by tubes with an inner orientation.

The scalar density of weight $+1$, $\rho_\wedge(x)$, is represented by balls with orientation arrows which jut out.

Generating of $\underset{\times}{D}{}^i_\wedge$ from $\rho_\wedge$ is represented in Fig. 5

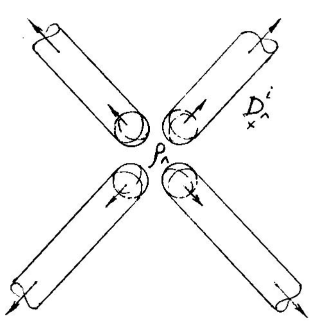

Figure 5: ρ generates $\mathbf{D}$

The orientation arrows of $\rho_\wedge$ and $D^i_\wedge$ are in concord.

It is clear that the balls are ends or *boundaries* of the tubes. This circumstance is expressed by the equation:

$$\rho_\wedge = \partial_i \underset{\times}{D}{}^i_\wedge. \tag{4}$$

We say that divergence of a vector density is a boundary of this density, and we name the density under the derivation a *filling* of the boundary. So $\rho_\wedge$ is the boundary of $\underset{\times}{D}{}^i_\wedge$, and $\underset{\times}{D}{}^i_\wedge$ is a filling of $\rho_\wedge$. We must replace Figure 1 by Figure 5.

It is important that the generation $\underset{\times}{D}{}^i_\wedge$ (1), (3) is determined uniquely, but the filling $\underset{\times}{D}{}^i_\wedge$ in (4) admits an addition of a divergenceless term $\underset{\bullet}{D}{}^i_\wedge$: $\partial_i \underset{\bullet}{D}{}^i_\wedge = 0$. We name divergenceless densities closed densities and mark them by bullet.

$$\rho_\wedge = \partial_i(\underset{\times}{D}{}^i_\wedge + \underset{\bullet}{D}{}^i_\wedge), \quad \underset{\times}{D}{}^i_\wedge = \dagger^i \rho_\wedge. \tag{5}$$

Note that the term ‘closed’ is inapplicable to $\rho_\wedge$ because a divergence of $\rho_\wedge$ do not exist.

3 E generates ϕ

As is orthodox , antisymmetric covariant tensors are called differential forms. In particular, an electric scalar potential ϕ is called a differential form of the degree 0 (or, simply, 0-form). A partial derivative of ϕ is called the exterior derivative of ϕ. It is the closed 1-form of potential electric strength $\underset{\bullet}{E}{}_i$

$$\underset{\bullet}{E}{}_i = \partial_i \phi, \quad \partial_{[k} \underset{\bullet}{E}{}_{i]} = 0 \tag{6}$$

(we do not use minus in this formula).

We say that exterior derivative of a form is a boundary of this form, and we name the form under the derivation a filling of the boundary. Note that a standard name of an exterior derivative is ‘exact form’. We say, electric strength $\underset{\bullet}{E}{}_i$ is a boundary of electric scalar potential ϕ, and ϕ is a filling of the boundary. This phenomenon is depicted in Figure 6

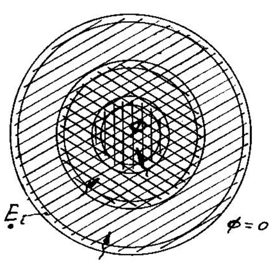

Figure 6: **E** generates ϕ

The geometric representation of a covariant vector is a biplane with an outer orientation [3, 4]. Hence a form $E_i(x)$ is represented by bisurfaces with an outer orientation, and the closed form $\underset{\bullet}{E}{}_i(x)$ is represented by closed bisurfaces. Hatching or darkening of the space represents a scalar field.

You can see that the closed bisurfaces $\underset{\bullet}{E}{}_i(x)$ bound the hatching, i.e. ϕ-field. The cloused bisurfaces $E_i(x)$ are filled with the hatching, i.e. with the ϕ-field. The density of the hatching increases when the bisurfaces are crossed in the direction of the orientational arrows. It seems that the bisurfaces generate the hatching. It is possible to say that electric strength $\underset{\bullet}{E}{}_i$ generates electric scalar potential ϕ, that E_i is a source of ϕ. It is expressed by the integral formula

$$\phi = \int \frac{E_{i'}(x') r^{i'}_{\wedge'}(x, x') dV^{\wedge'}}{4\pi r^3(x, x')}, \quad \text{or} \quad \phi = \dagger^i E_i. \tag{7}$$

The formula determines the potential ϕ uniquely, even in the case of a nonpotential electric field, but the filling, ϕ, from (6) admits an addition of a constant, $\underset{\bullet}{\phi} = Const$,

$$\underset{\bullet}{E}_i = \partial_i(\phi + Const) \tag{8}$$

4 An example of the calculation of a generated potential

We apply Eq. (7) for a solution of the problem: "What potential is generated by a thin two-dimensional 'spherical' capacitor?"

So, in a two-dimensional (for simplicity) space there are two concentric circles between which a given radial electrical field $\mathbf{E}$ exists. It is necessary to find the potential ϕ in this space by the formula

$$\phi(x) = \int \frac{(\mathbf{Er})da}{2\pi r^2}, \tag{9}$$

where da is an element of the space (plane).

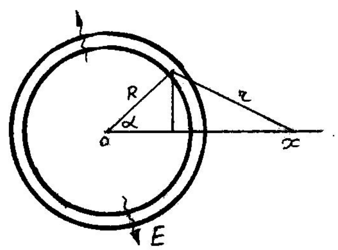

Figure 7: Two-dimensional capacitor

$$\mathbf{Er} = E_x r^x + E_y r^y = E\cos\alpha(x - R\cos\alpha) - E\sin\alpha R\sin\alpha = E(x\cos\alpha - R). \tag{10}$$

If we write δ for the small gap between the circles, then $da = R\delta d\alpha$.

In the sequel,

$$r^2 = R^2\sin^2\alpha + (x - R\cos\alpha)^2 = R^2 + x^2 - 2xR\cos\alpha. \tag{11}$$

Thus,

$$\phi = \frac{ER\delta}{2\pi x}\int_0^{2\pi}\frac{\cos\alpha + v/2}{u + v\cos\alpha}d\alpha, \quad u = R^2/x^2 + 1, \quad v = -2R/x. \tag{12}$$

Integrating yields [6]

$$\phi(x) = \frac{ER\delta}{x}\left(-\frac{x}{2R} + \frac{(R^2+x^2)x}{2R \mid R^2 - x^2 \mid} - \frac{Rx}{\mid R^2 - x^2 \mid}\right). \tag{13}$$

I.e. $\phi = 0$ on the outside of the circles, that is at $R < x$, and $\phi = -E\delta$ inside the circles, that is at $R > x$, just as expected.

5 j generates H

Electric current density $j^i_\wedge$ generates magnetic strength $H^{ik}_\wedge$ by the Biot-Savarat formula

$$\underset{\times}{H}{}^{ik}_{\wedge}(x) = 2\int\frac{j^{[i}_{\wedge'}(x')r^{k]}_{\wedge}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')}, \quad \text{or} \quad \underset{\times}{H}{}^{ik}_{\wedge} = 2\,\dagger^{[k} j^{i]}_{\wedge}. \tag{14}$$

$j^i_\wedge$ is a source of $\underset{\times}{H}{}^{ik}_\wedge$.

Vector density $j^i_\wedge$ is analogous to $D^i_\wedge$ and is represented by tubes with an inner orientation. But now the tubes may have no ends, i.e. they may be closed. The magnetic strength $H^{ik}_\wedge$, which is a bivector density of weight $+1$, is represented by bisurfaces with an inner orientation [3].

Generating of **H** from **j** is represented in Fig. 8. The $H^{ik}_\wedge$-bisurfaces are emerged from the closed $j^i_\wedge$ tubes, and their inner orientations are in concord.

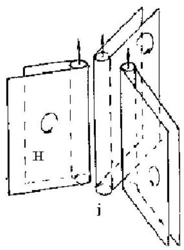

Figure 8: **j** generates **H**

We must replace Figure 3 by Figure 8.

Fig. 8 shows that the tubes are edges or boundaries of the bisurfaces. It seems that the tubes fill the space with the bisurfaces. This circumstance is expressed by the equation:

$$\underset{\bullet}{j}{}^i_\wedge = \partial_k H^{ik}_\wedge, \tag{15}$$

where

$$H^{ik}_\wedge = \underset{\times}{H}{}^{ik}_\wedge + \underset{\bullet}{H}{}^{ik}_\wedge \tag{16}$$

may contain a closed part $\underset{\bullet}{H}{}^{ik}_\wedge$: $\partial_k \underset{\bullet}{H}{}^{ik}_\wedge = 0$.

We say what $\underset{\bullet}{j}{}^i_\wedge$ is the boundary of $H^{ik}_\wedge$, and $H^{ik}_\wedge$ is a filling of $j^i_\wedge$.

6 B generates A

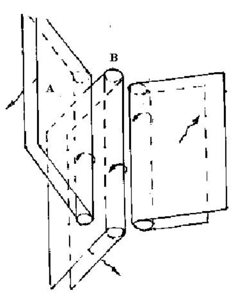

Figure 9: **B** generates **A**

Magnetic vector potential A_i is analogous to E_i and is represented by bisurfaces with an outer orientation. But now the bisurfaces have boundaries:

$$\underset{\bullet}{B}{}_{ij} = 2\partial_{[i}A_{j]}. \tag{17}$$

The magnetic induction B_{ij} is a covariant bivector and is represented by closed tubes with an *outer* orientation [3]. The tubes bound the bisurfaces, and their outer orientations are in concord. A_j is a filling of B_{ij}

On the other hand, the bisurfaces A_i are generated by the closed tubes B_{ij}. The bisurfaces are emerged from the tubes. B_{ij} is a source of A_i. It is expressed by the integral formula

$$\underset{\times}{A}_k = \int \frac{B_{i'k}(x')r^{i'}_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')}, \quad \text{or} \quad \underset{\times}{A}_k = \dagger^i B_{ik}. \tag{18}$$

We must replace Figure 4 by Figure 9.

This formula is analogous to Biot-Savarat law. It determines the potential $\underset{\times}{A}_k$ uniquely. The potential stands out against a background of all gauge equivalent vector potentials. But a filling A_j in (17) admits an addition of a closed term $\underset{\bullet}{A}_j$:

$$\underset{\bullet}{B}_{ij} = 2\partial_{[i}(\underset{\times}{A}_{j]} + \underset{\bullet}{A}_{j]}). \tag{19}$$

7 The generations are sterile

Here a problem arises. What shall we get if a generation will be used as a source of a generation? What shall we get if a generation will be substituted in the integral formula? For example, what are the values of the integrals

$$\int \frac{\underset{\times}{D}^{[i}_{\wedge'}(x')r^{k]}_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')}, \int \frac{\underset{\times}{H}^{[ij}_{\wedge'}(x')r^{k]}_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')}, \int \frac{\underset{\times}{A}_{i'}(x')r^{i'}_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')}, \tag{20}$$

where $\underset{\times}{D}$, $\underset{\times}{H}$, and $\underset{\times}{A}$ are from (1), (14), and (18)?

The question is a simple one: generating from a generation yields zero, $\dagger\dagger = 0$ (as well as $\partial\partial = 0$). We say that generations are *sterile*. For example,

$$\int \frac{\underset{\times}{D}^{[i}_{\wedge'}(x')r^{k]}_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')} = 0, \quad \text{or} \quad \dagger^{[k}\dagger^{i]}\rho = 0 \tag{21}$$

Indeed, substituting $\underset{\times}{D}$ from (1) into (21) yields

$$\int\int \frac{\rho_{\wedge''}(x'')r^{[i}_{1\wedge'}(x',x'')r^{k]}_{2\wedge}(x,x')dV^{\wedge''}dV^{\wedge'}}{4\pi r_1^3(x',x'')4\pi r_2^3(x,x')} = 0. \tag{22}$$

For the proof of Eq. (22) we fix the points x and x''. Then, because of symmetry of space, for each x' exists a $\bar{x}'$ such that the vector products $r_1^{[i}r_2^{k]}$ in the points x' and $\bar{x}'$ differ in sign only. So, integrating over dV' yields zero.

We mark sterile quanties by a sign $\times$

Note that ϕ from (7) cannot be used for the purpose of generating since the integral

$$\int \frac{\phi(x')r^i_{\wedge'}(x,x')dV^{\wedge'}}{4\pi r^3(x,x')} \tag{23}$$

is a vector but not a vector density. So, there is no sign $\times$ in Eq. (7).

It is important that [7, 8]

$$\dagger\partial\dagger = \dagger \qquad \text{and} \qquad \partial\dagger\partial = \partial. \tag{24}$$

It makes possible to decompose a field into sterile and closed parts. If the generation $\dagger$ is possible, the decomposition of a field j, for example, can be performed by the formula:

$$j = \underset{\times}{j} + \underset{\bullet}{j} + \underset{\times\bullet}{j} = \dagger\partial j + \partial\dagger j + \underset{\times\bullet}{j}. \tag{25}$$

This is an analog of Helmholtz's theorem or Hodge decomposition theorem [9, 10]. Here $\underset{\times\bullet}{j}$ is a field which is closed and sterile

8 Example of the decomposition

Consider a semi-infinite straight thin wire carrying an electric current I. Let the current density $j^i_\wedge$ be singular in the wire territory. Our aim is to decompose the density into sterile and closed parts, i.e. into irrotational and solenoidal parts [10],

$$j^i_\wedge = \underset{\times}{j}{}^i_\wedge + \underset{\bullet}{j}{}^i_\wedge = \dagger^i \partial_k j^k_\wedge + \partial_k 2\, \dagger^{[k} j^{i]}_\wedge. \tag{26}$$

We have step by step:

$$\partial_k j^k_\wedge = -\dot{\rho}_\wedge = I\delta_\wedge(0), \tag{27}$$

$$\underset{\times}{j}{}^i_\wedge = \dagger^i \partial_k j^k_\wedge = \dagger^i I\delta_\wedge(0) = \int \frac{I\delta_{\wedge'}(0')r^i_\wedge(x,x')dV^{\wedge'}}{4\pi r^3(x,x')} = \frac{Ir^i_\wedge(x)}{4\pi r^3(x)}. \tag{28}$$

$$2\,\dagger^{[k} j^{i]}_\wedge = \underset{\times}{H}{}^{ik}_\wedge = 2\int \frac{j^{[i}_{\wedge'}(x')r^{k]}_\wedge(x,x')dV^{\wedge'}}{4\pi r^3(x,x')} = 2\int_{z=0}^{z=\infty} \frac{I\,dl^{[i}r^{k]}_\wedge}{4\pi r^3}$$

$$= \{\underset{\times}{H}{}^{yz}_\wedge = \underset{\times}{H}{}^*_x = -\frac{I}{4\pi}\left(\frac{y}{R^2}+\frac{yz}{R^3}\right),\ \underset{\times}{H}{}^{zx}_\wedge = \underset{\times}{H}{}^*_y = \frac{I}{4\pi}\left(\frac{x}{R^2}+\frac{xz}{R^3}\right)\}, \quad R=\sqrt{x^2+y^2}. \tag{29}$$

$$\underset{\bullet}{j}{}^i_\wedge = \partial_k 2\,\dagger^{[k} j^{i]}_\wedge = \text{Curl}\,\mathbf{H} = \partial_k \underset{\times}{H}{}^{ik}_\wedge = -\frac{Ir^i_\wedge(x)}{4\pi r^3(x)} + I\cdot(\text{semiaxis } z). \tag{30}$$

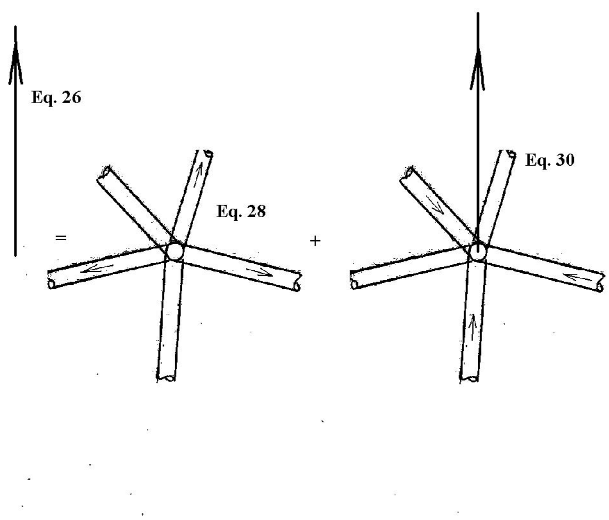

Figure 10: The decomposition

The fact is that Curl $\mathbf{H}$ is singular at the semiaxis z. Hence, the closed part of $j^i_\wedge$ (30) consists of radial tubes coming up to the point $z = 0$ and the semiaxis z going away to infinity.

9 The conjugation

In a metric space there is a relations between contra- and covariant tensors of the same valence (with the same number of indices). For example, the metric tensor g_{ij} associates a tensor X^{ij} with the tensor $X_{mn} = X^{ij}g_{im}g_{jn}$. This process is called the lowering of indices. In this case the kernel character is preserved.

In the electromagnetism a slightly different process is used. We call this process the *conjugation*. The conjugation establishes a one-to-one correspondence between forms and contravariant tensor densities. This process uses the metric tensor densities $g^{\wedge}_{ij} = g_{ij}/\sqrt{g}_{\wedge}$ or $g^{ij}_{\wedge} = g^{ij}\sqrt{g}_{\wedge}$. It appears that the electromagnetic fields are conjugated in pairs:

$$E_i = D^j_{\wedge} g^{\wedge}_{ij}, \quad D^j_{\wedge} = E_i g^{ij}_{\wedge}, \quad B_{ik} = H^{jl}_{\wedge} g^{\wedge}_{ij} g_{kl}, \quad H^{jl}_{\wedge} = B_{ik} g^{ij}_{\wedge} g^{kl}. \tag{31}$$

Conjugating of electromagnetic fields mostly changes the kernel characters. For brevity, we designate conjugating by the star $\star$.

$$E_i = \star(D^j_{\wedge}), \quad D^j_{\wedge} = \star(E_i), \quad B_{ik} = \star(H^{jl}_{\wedge}), \quad H^{jl}_{\wedge} = \star(B_{ik}). \tag{32}$$

Our star operator is involute: $\star\star = 1$. It differs from Hodge operator [11, 12, 9, 10] mainly because the Hodge operator, $*$, implies a dualisation. For example,

$$*(E_i) = E_i g^{ij}_{\wedge} \epsilon^{\wedge}_{jkl} = D_{kl}. \tag{33}$$

Here $\epsilon^{\wedge}_{jkl}$ is the absolute antisimmetric tensor density and D_{kl} is a covariant antisymmetric tensor.

It is remarkable that conjugating transforms sterile fields to closed fields and back [7, 8]. For example,

$$\underset{\bullet}{E}_i = \star(\underset{\times}{D}{}^j_{\wedge}), \quad \underset{\times}{D}{}^j_{\wedge} = \star(\underset{\bullet}{E}_i), \quad \underset{\times}{H}{}^{jl}_{\wedge} = \star(\underset{\bullet}{B}_{ik}), \quad \underset{\bullet}{B}_{ik} = \star(\underset{\times}{H}{}^{jl}_{\wedge}), \quad \underset{\bullet}{A}{}^j_{\wedge} = \star(\underset{\times}{A}_i) \tag{34}$$

(by tradition, the kernel character A is preserved).

As a sample of a proof we presented here the equality

$$\partial_{[j} g_{k]i} \underset{\times}{D}{}^i_{\wedge} = \int \partial_{[j} \frac{r^i_{\wedge}(x, x')}{r^3(x, x')} g_{k]i} \frac{\rho_{\wedge'}(x') dV^{\wedge'}}{4\pi} = 0. \tag{35}$$

It holds by virtue of the simple identity $\partial_{[j}(g_{k]i} r^i / r^3) = 0$.

Figure 11 represents a geometrical sense of the conjugation.

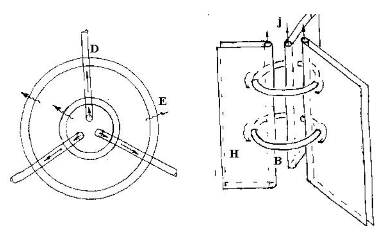

Figure 11: The conjugation

The conjugation changes a geometric representation of fields essentially: closed bisurfaces of $\underset{\bullet}{\mathbf{E}}$ are converted into radial $\underset{\times}{\mathbf{D}}$-tubes and back, radial bisurfaces of $\underset{\times}{\mathbf{H}}$ are converted into ring-shaped $\underset{\bullet}{\mathbf{B}}$-tubes and back.

The conjugation changes the orientation of a geometric quantity.

The conjugation permits to differentiate the field repeatedly. For example (without indexes),

$$\rho = \partial \star \partial \phi, \quad j = \partial \star \partial A. \tag{36}$$

These chains of fields may be depicted as follows,

$$\rho_{\wedge} \vdash \underset{\times}{D}{}^j_{\wedge}, \ \underset{\bullet}{E}_i \vdash \phi, \qquad \underset{\bullet}{j}{}^i_{\wedge} \vdash \underset{\times}{H}{}^{ik}_{\wedge}, \ \underset{\bullet}{B}_{jl} \vdash \underset{\times}{A}_l. \tag{37}$$

The simbol $\vdash$ means that, e.g., $\rho_\wedge$ is a boundary of $\underset{\times}{D}{}^j_\wedge$ or $\underset{\times}{A}_l$ is a filling of $\underset{\bullet}{B}_{jl}$

These chains may be extended in both directions. In this case new fields will appear. For example,

$$\mathcal{B} = \partial\star\partial\star\partial\star\partial\mathcal{H}, \qquad \ldots \underset{\bullet}{\mathcal{B}}_{ki} \vdash \underset{\times}{j}_i,\ \underset{\bullet}{j}{}^i_\wedge \vdash \underset{\times}{H}{}^{ik}_\wedge,\ \underset{\bullet}{B}_{jl} \vdash \underset{\times}{A}_l,\ \underset{\bullet}{A}{}^l_\wedge \vdash \underset{\times}{\mathcal{H}}{}^{lk}_\wedge \ldots \tag{38}$$

The same chains may be created by the use of generating. For example, instead of (38) we can get

$$\dagger\star\dagger\star\dagger\star\dagger\mathcal{B} = \mathcal{H}, \qquad \ldots \underset{\bullet}{\mathcal{B}}_{ki} \to \underset{\times}{j}_i,\ \underset{\bullet}{j}{}^i_\wedge \to \underset{\times}{H}{}^{ik}_\wedge,\ \underset{\bullet}{B}_{jl} \to \underset{\times}{A}_l,\ \underset{\bullet}{A}{}^l_\wedge \to \underset{\times}{\mathcal{H}}{}^{lk}_\wedge \ldots \tag{39}$$

The simbol $\to$ means here that, e.g., $\underset{\bullet}{\mathcal{B}}_{ki}$ generats $\underset{\times}{j}_i$, or $\underset{\bullet}{A}{}^l_\wedge$ is a source of $\underset{\times}{\mathcal{H}}{}^{lk}_\wedge$.

A pair, e.g., $(D^j_\wedge,\ E_i)$ is regular if the first and the second element of the pair can be a source and a filling. A pair $(\rho_\wedge,\ \phi)$ is degenerate because $\partial\rho_\wedge$ and $\dagger\phi$ are nonsense. A regular pair $(\underset{\times}{D}{}^j_\wedge,\ \underset{\bullet}{E}_i)$ is a pure pair. A regular pair $(\underset{\bullet}{D}{}^j_\wedge,\ \underset{\times}{E}_i)$ is a pure pair, complementary to the previous pair. If all admissible operation (the differentiation and the integration) on both elements of a pair yield zero, the pair is an endpair, because the chain of fields ends at the pair. For example,

$$0 \vdash \underset{\bullet}{\rho},\ \underset{\times}{\rho}_\wedge = 2 \vdash \underset{\times}{E}{}^i_\wedge,\ \underset{\bullet}{E}_i = \{x, y\} \vdash \phi,\ \phi_\wedge = (x^2+y^2)/2 \vdash \underset{\times}{D}{}^j_\wedge = \{x^3/6, y^3/6\}, \ldots \tag{40}$$

Here $\times$ means a conjugate closure, and kernel characters are preserved when conjugating.

A regular endpair generates two chains of fields. For example:

$$0 \vdash \underset{\times\bullet}{j}{}^i_\wedge,\ \underset{\times\bullet}{j}_i = \{y, x\} \vdash \phi,\ \phi_\wedge = xy \vdash \underset{\times}{A}{}^j_\wedge,\ \underset{\bullet}{A}_j = \{x^2y/4 + y^3/12, xy^2/4 + x^3/12\} \vdash \ldots \tag{41}$$

$$0 \vdash \underset{\times\bullet}{j}_i,\ \underset{\times\bullet}{j}{}^i_\wedge = \{y, x\} \vdash \underset{\times}{\Pi}{}^{xy}_\wedge,\ \underset{\bullet}{\Pi}_{xy} = -x^2/2 + y^2/2 \vdash \underset{\times}{A}_j,\ \underset{\bullet}{A}{}^j_\wedge = \{-y^3/6, -x^3/6\} \vdash \ldots \tag{42}$$

The conjugation makes it possible to express the operator $\nabla^2 = g^{ij}\partial_i\partial_j$ in terms of the exterior derivatives and the divergence. It appears that [7, 8]

$$\nabla^2 \overset{p}{\omega} = (-1)^p(\star\partial\star\partial - \partial\star\partial\star)\overset{p}{\omega}, \qquad \nabla^2 \overset{p}{\alpha}_\wedge = (-1)^{p+1}(\star\partial\star\partial - \partial\star\partial\star)\overset{p}{\alpha}_\wedge. \tag{43}$$

Here $\overset{p}{\omega}$ and $\overset{p}{\alpha}_\wedge$ designate a form of the degree p and a contravariant density of valence p, respectively. For example ($p = 1$),

$$\nabla^2 \underset{\times}{A}_i = -\star\partial\star\partial\underset{\times}{A}_i = -\underset{\times}{j}_i, \quad \nabla^2\underset{\times}{A}{}^i_\wedge = \star\partial\star\partial\underset{\times}{A}{}^i_\wedge = \underset{\times}{j}{}^i_\wedge, \quad \nabla^2\underset{\bullet}{A}{}^i_\wedge = -\partial\star\partial\star\underset{\bullet}{A}{}^i_\wedge = -\underset{\bullet}{j}{}^i_\wedge. \tag{44}$$

Eqs. (43) differ from a standard definition of the Laplace-dePham operator $\Delta = d\delta + \delta d$ [13, p. 153], [12] because of using the star operator $\star$ instead of the Hodge asterisk operator $*$.

The symbol $\vdash$ separates a filling from its boundary in the expressions (41), (42). We will say that the operator ∇^2 convert an element of a pure pair into its "second boundary", sometimes changing the sign. If the second boundaries of two mutually complementary pairs are an endpair, the sum of the complementary pair is a harmonic pair. For example, the use of $\underset{\bullet}{A}_j$ and $\underset{\times}{A}_j$ from (41), (42) gives

$$\nabla^2 A_j = \nabla^2(\underset{\bullet}{A}_j + \underset{\times}{A}_j) = \nabla^2(\{x^2y/4 + y^3/12, xy^2/4 + x^3/12\} + \{-y^3/6, -x^3/6\})$$

$$= \nabla^2\{x^2y/4 - y^3/12, xy^2/4 - x^3/12\} = \{0, 0\}. \tag{45}$$

So, in spite of [14, 5.7-3], [15, (4.35)] a harmonic form can be decomposed into solenoidal and irrotational parts (if the constraint of a compact Riemannian manifold is removed).

Conclusion

It is not uncommon today for an education to ignore all but the simplest geometrical ideas, despite the fact that students are encouraged to develop mental 'pictures' and 'intuition' appropriate to physical phenomena. This paper aims to improve the situation.

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